2: VECTORS, MATRICES, AND LINEAR COMBINATIONS

David Austin
Grand Valley State University

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CHAPTER OVERVIEW

2: Vectors, matrices, and linear combinations

We began our study of linear systems in Chapter 1 where we described linear systems in terms of augmented matrices, such as

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ -3 & 3 & -1 & 2 \\ 2 & 3 & 2 & -1 \end{array}\right] \tag{2.1}$$

In this chapter, we will uncover geometric information in a matrix like this, which will lead to an intuitive understanding of the insights we previously gained into the solutions of linear systems.

2.1: Vectors and Linear Combinations

It is a remarkable fact that algebra, which is about equations and their solutions, and geometry are intimately connected. For instance, the solution set of a linear equation in two unknowns, such as $2x+y=1$, can be represented graphically as a straight line. The aim of this section is to further this connection by introducing vectors, which will help us to apply geometric intuition to our thinking about linear systems.

Vectors

A *vector* is most simply thought of as a matrix with a single column. For instance,

$$\mathbf{v}=\begin{bmatrix}2\\1\end{bmatrix}, \mathbf{w}=\begin{bmatrix}-3\\1\\0\\2\end{bmatrix}$$

are both vectors. Since the vector $\mathbf{v}$ has two entries, we say that it is a two-dimensional vector; in the same way, the vector $\mathbf{w}$ is a four-dimensional vector. We denote the set of all m-dimensional vectors by $\mathbb{R}^m$. Consequently, if $\mathbf{u}$ is a 3-dimensional vector, we say that $\mathbf{u}$ is in $\mathbb{R}^3$.

While it can be difficult to visualize a four-dimensional vector, we can draw a simple picture describing the two-dimensional vector $\mathbf{v}$.

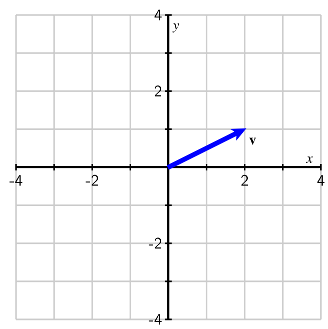


Figure 2.1.1: A graphical representation of the vector $\mathbf{v}=\begin{bmatrix}2\\1\end{bmatrix}$.

In this chapter, we will uncover geometric information in a matrix like this, which will lead to an intuitive understanding of the insights we previously gained into the solutions of linear systems.

We think of $\mathbf{v}$ as describing a walk we take in the plane where we move two units horizontally and one unit vertically. Though we allow ourselves to begin walking from any point in the plane, we will most frequently begin at the origin, in which case we arrive at the the point $(2,1)$, as shown in the figure.

There are two simple algebraic operations we can perform on vectors.

Scalar Multiplication

We multiply a vector $\mathbf{v}$ by a real number a by multiplying each of the components of $\mathbf{v}$ by a. For instance,

$$-3\begin{bmatrix}2\\-4\\1\end{bmatrix}=\begin{bmatrix}-6\\12\\-3\end{bmatrix}.$$

The real number a is called a *scalar*.

Vector Addition

We add two vectors of the same dimension by adding their components. For instance,

$$\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} -5 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}.$$

Preview Activity 2.1.1. Scalar Multiplication and Vector Addition.

Suppose that

$$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

1. Find expressions for the vectors

$$\begin{array}{cccc} \mathbf{v}, & 2\mathbf{v}, & -\mathbf{v}, & -2\mathbf{v}, \\ \mathbf{w}, & 2\mathbf{w}, & -\mathbf{w}, & -2\mathbf{w}. \end{array}$$

and sketch them below.

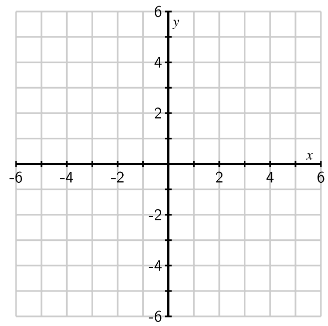


Figure 2.1.2: Sketch the vectors on this grid.

- What geometric effect does scalar multiplication have on a vector? Also, describe the effect multiplying by a negative scalar has.
- Sketch the vectors $\mathbf{v}, \mathbf{w}, \mathbf{v}+\mathbf{w}$ below.

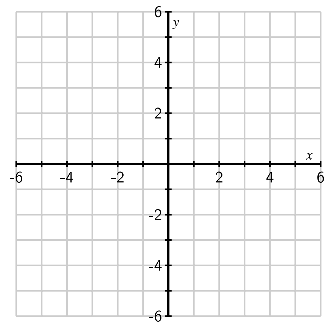


Figure 2.1.3: Sketch the vectors on this grid.

- Consider vectors that have the form $\mathbf{v}+a\mathbf{w}$ where a is any scalar. Sketch a few of these vectors when, say, $a=-2,-1,0,1,$ and 2. Give a geometric description of this set of vectors.

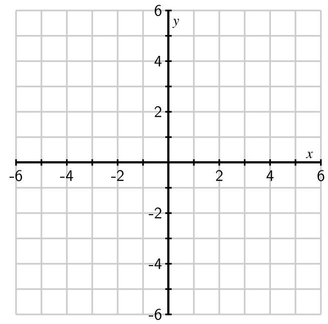


Figure 2.1.4: Sketch the vectors on this grid.

- If a and b are two scalars, then the vector

$$a\mathbf{v}+b\mathbf{w}$$

is called a linear combination of the vectors $\mathbf{v}$ and $\mathbf{w}$. Find the vector that is the linear combination when $a=-2$ and $b=1$.

- Can the vector $\begin{bmatrix}-31\\37\end{bmatrix}$ be represented as a linear combination of $\mathbf{v}$ and $\mathbf{w}$?

The preview activity demonstrates how we may interpret scalar multiplication and vector addition geometrically.

First, we see that scalar multiplication has the effect of stretching or compressing a vector. Multiplying by a negative scalar changes the direction of the vector. In either case, we see that scalar multiplying the vector $\mathbf{v}$ produces a new vector on the line defined by $\mathbf{v}$, as shown in Figure $2.1.5$.

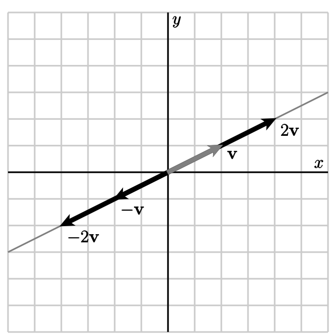


Figure $2.1.5$: Scalar multiples of the vector $\mathbf{v}$.

To understand the sum $\mathbf{v}+\mathbf{w}$, we imagine walking from the origin with the appropriate horizontal and vertical changes given by $\mathbf{v}$. From there, we continue our walk using the horizontal and vertical changes prescribed by $\mathbf{w}$, after which we arrive at the sum $\mathbf{v}+\mathbf{w}$. This is illustrated on the left of Figure $2.1.6$ where the tail of $\mathbf{w}$ is placed on the tip of $\mathbf{v}$.

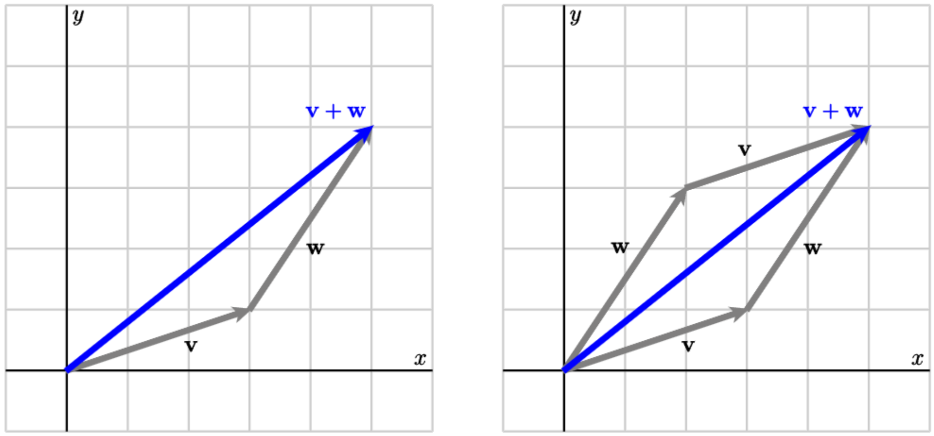


Figure 2.1.6: Vector addition as a simple walk in the plane is illustrated on the left. The vector sum represented as the diagonal of a parallelogram is on the right

Alternatively, we may construct the parallelogram with $\mathbf{v}$ and $\mathbf{w}$ as two sides. The sum is then the diagonal of the parallelogram, as illustrated on the right of Figure $2.1.6$.

We have now seen that the set of vectors having the form $a\mathbf{v}$ is a line. To form the set of vectors $a\mathbf{v}+\mathbf{w}$, we can begin with the vector $\mathbf{w}$ and add multiples of $\mathbf{v}$. Geometrically, this means that we begin from the tip of $\mathbf{w}$ and move in a direction parallel to $\mathbf{v}$. The effect is to translate the line $a\mathbf{v}$ by the vector $\mathbf{w}$, as shown in Figure $2.1.7$.

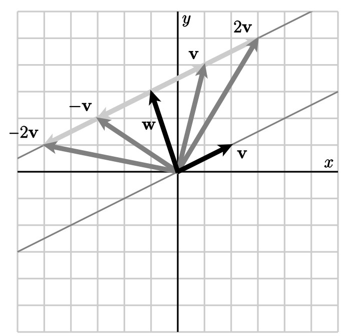


Figure $2.1.7$: The set of vectors $a\mathbf{v}+\mathbf{w}$ form a line.

At times, it will be useful for us to think of vectors and points interchangeably. That is, we may wish to think of the vector $\begin{bmatrix}2\\1\end{bmatrix}$ as describing the point $(2,1)$ and vice-versa. When we say that the vectors having the form $a\mathbf{v}+\mathbf{w}$ form a line, we really mean that the tips of the vectors all lie on the line passing through $\mathbf{w}$ and parallel to $\mathbf{v}$.

Observation 2.1.4.

Even though these vector operations are new, it is straightforward to check that some familiar properties hold.

Commutativity

$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

Distributivity

$a(\mathbf{v}+\mathbf{w})=a\mathbf{v}+a\mathbf{w}$.

Sage can perform scalar multiplication and vector addition. We define a vector using the `vector` command; then `*` and `+` denote scalar multiplication and vector addition.

```
v = vector([3,1])
w = vector([-1,2])
```

```
print (2*v)
print (v + w)
```

2.1.2 Linear combinations

Linear combinations, which we encountered in the preview activity, provide the link between vectors and linear systems. In particular, they will help us apply geometric intuition to problems involving linear systems.

Definition 2.1.5

The *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with scalars $c_1, c_2, \ldots, c_n$ is the vector

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are called the *weights* of the linear combination.

Activity 2.1.2.

In this activity, we will look at linear combinations of a pair of vectors,

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

with weights a and b.

The diagram below can be used to construct linear combinations whose weights a and b may be varied using the sliders at the top. The vectors $\mathbf{v}$ and $\mathbf{w}$ are drawn in gray while the linear combination

$$a\mathbf{v} + b\mathbf{w} \tag{2.1.1}$$

is in red.

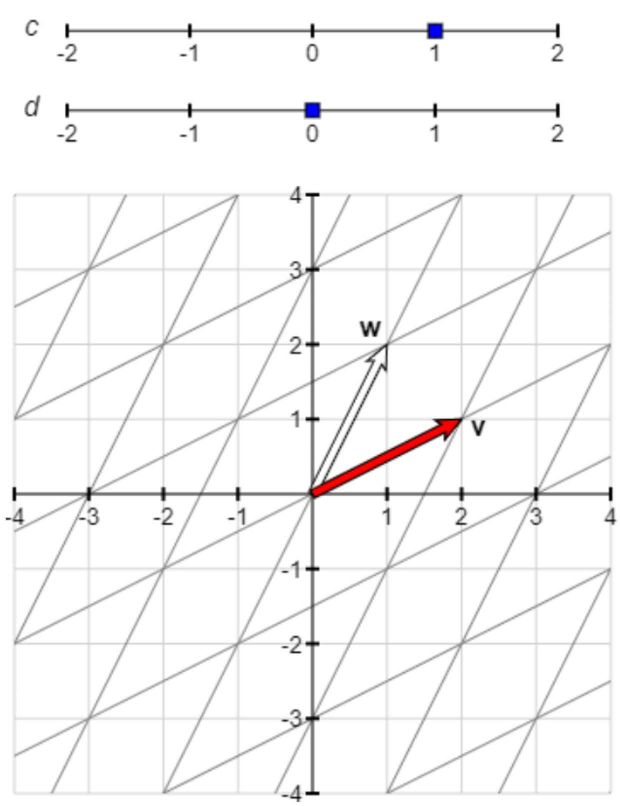


Figure 2.1.8: Linear combinations of vectors $\mathbf{v}$ and $\mathbf{w}$.

1. The weight b is initially set to 0. Explain what happens as you vary a with $b=0$? How is this related to scalar multiplication?
2. What is the linear combination of $\mathbf{v}$ and $\mathbf{w}$ when $a=1$ and $b=-2$? You may find this result using the diagram, but you should also verify it by computing the linear combination.
3. Describe the vectors that arise when the weight b is set to 1 and a is varied. How is this related to our investigations in the preview activity?
4. Can the vector $\begin{bmatrix}0\\0\end{bmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? If so, what are weights a and b?
5. Can the vector $\begin{bmatrix}3\\0\end{bmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? If so, what are weights a and b?
6. Verify the result from the previous part by algebraically finding the weights a and b that form the linear combination $\begin{bmatrix}3\\0\end{bmatrix}$.
7. Can the vector $\begin{bmatrix}1.3\\-1.7\end{bmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? What about the vector $\begin{bmatrix}15.2\\7.1\end{bmatrix}$?
8. Are there any two-dimensional vectors that cannot be expressed as linear combinations of $\mathbf{v}$ and $\mathbf{w}$?

This activity illustrates how linear combinations are constructed geometrically: the linear combination $a\mathbf{v}+b\mathbf{w}$ is found by walking along $\mathbf{v}$ a total of a times followed by walking along $\mathbf{w}$ a total of b times. When one of the weights is held constant while the other varies, the vector moves along a line.

Example 2.1.6

The previous activity also shows that questions about linear combinations lead naturally to linear systems. Let's ask how we can describe the vector $\mathbf{b}=\begin{bmatrix}-1\\4\end{bmatrix}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. We need to find weights a and b such that

$$a\begin{bmatrix}2\\1\end{bmatrix}+b\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$

$$\begin{bmatrix}2a\\a\end{bmatrix}+\begin{bmatrix}b\\2b\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$

$$\begin{bmatrix}2a+b\\a+2b\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$

Equating the components of the vectors on each side of the equation, we arrive at the linear system

$$\begin{aligned}2a+\ \ b&=-1\\ a+2b&=\ \ \ 4\end{aligned}$$

This means that $\mathbf{b}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ if this linear system is consistent.

To solve this linear system, we construct its corresponding augmented matrix and find its reduced row echelon form.

$$\left[\begin{array}{rr|r}2&1&-1\\1&2&4\end{array}\right]\sim\left[\begin{array}{rr|r}1&0&-2\\0&1&3\end{array}\right],$$

which tells us the weights $a=-2$ and $b=3$; that is,

$$-2\mathbf{v}+3\mathbf{w}=\mathbf{b}.$$

In fact, we know even more because the reduced row echelon matrix tells us that these are the only possible weights. Therefore, $\mathbf{b}$ may be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ in exactly one way.

This example demonstrates the connection between linear combinations and linear systems. Asking if a vector $\mathbf{b}$ is a linear combination of vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$ is the same as asking whether an associated linear system is consistent.

In fact, we may easily describe the linear system we obtain in terms of the vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{b}$. Notice that the augmented matrix we found was $\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 2 & 4 \end{array}\right]$. The first two columns of this matrix are $\mathbf{v}$ and $\mathbf{w}$ and the rightmost column is $\mathbf{b}$. As shorthand, we will write this augmented matrix replacing the columns with their vector representation:

$$\left[\begin{array}{cc|c} \mathbf{v} & \mathbf{w} & \mathbf{b} \end{array}\right].$$

This fact is generally true so we record it in the following proposition.

Proposition 2.1.7.

The vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if and only if the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n & \mathbf{b} \end{array}\right]$$

is consistent. A solution to this linear system gives weights $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{b}.$$

The next activity puts this proposition to use.

Activity 2.1.3. Linear combinations and linear systems.

1. Given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix},$$

   we ask if $\mathbf{b}$ can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Rephrase this question by writing a linear system for the weights c_1, c_2, and c_3 and use the Sage cell below to answer this question.

2. Consider the following linear system.

$$\begin{alignedat}{4} 3x_1 & {}+{} & 2x_2 & {}-{} & x_3 & {}={} & 4 \\ x_1 & & & {}+{} & 2x_3 & {}={} & 0 \\ -x_1 & {}-{} & x_2 & {}+{} & 3x_3 & {}={} & 1 \end{alignedat}$$

   Identify vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{b}$ and rephrase the question "Is this linear system consistent?" by asking "Can $\mathbf{b}$ be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?"

3. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

   Can $\mathbf{b}$ be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, can $\mathbf{b}$ be written as a linear combination of these vectors in more than one way?

4. Considering the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from the previous part, can we write every three-dimensional vector $\mathbf{b}$ as a linear combination of these vectors? Explain how the pivot positions of the matrix $\left[\begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}\right]$ help answer this question.
5. Now consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}.$$

Can $\mathbf{b}$ be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, can $\mathbf{b}$ be written as a linear combination of these vectors in more than one way?

6. Considering the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from the previous part, can we write every three-dimensional vector $\mathbf{b}$ as a linear combination of these vectors? Explain how the pivot positions of the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ help answer this question.

Example 2.1.13.

Consider the vectors $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, as shown in Figure 2.1.9.

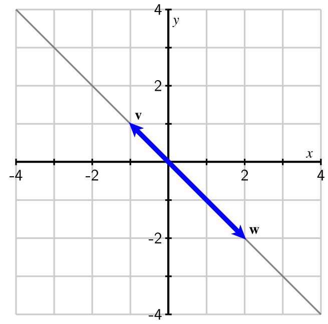


Figure 2.1.9: Vectors $\mathbf{v}$ and $\mathbf{w}$

These vectors appear to lie on the same line, a fact that becomes apparent once we notice that $\mathbf{w} = -2\mathbf{v}$. Intuitively, we think of the linear combination

$$c\mathbf{v} + d\mathbf{w} \tag{2.1.2}$$

as the result of walking c times in the $\mathbf{v}$ direction and d times in the $\mathbf{w}$ direction. With these vectors, we are always walking along the same line so it would seem that any linear combination of these vectors should lie on the same line. In addition, a vector that is not on the line, say $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, should be not be expressible as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

We can verify this by checking

$$\left[\begin{array}{rr|r} -1 & 2 & 3 \\ 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -2 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{2.1.3}$$

This shows that the associated linear system is inconsistent, which means that the vector $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ cannot be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

Notice that the reduced row echelon form of the coefficient matrix

$$\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \tag{2.1.4}$$

tells us to expect this. Since there is not a pivot position in the second row of the coefficient matrix $\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$, it is possible for a pivot position to appear in the rightmost column of the augmented matrix

$$\left[\begin{array}{cc|c} \mathbf{v} & \mathbf{w} & \mathbf{b} \end{array}\right] \tag{2.1.5}$$

for some choice of $\mathbf{b}$.

Summary

This section has introduced vectors, linear combinations, and their connection to linear systems.

- There are two operations we can perform with vectors: scalar multiplication and vector addition. Both of these operations have geometric meaning.
- Given a set of vectors and a set of scalars we call weights, we can create a linear combination using scalar multiplication and vector addition.

- A solution to the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{b} \end{array}\right]$$

is a set of weights that expressex $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

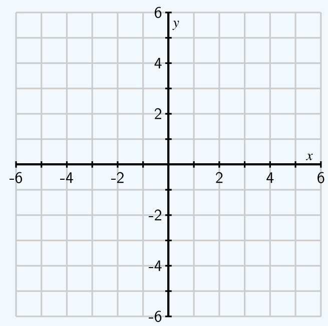
Exercises 2.1.4Exercises

? 1

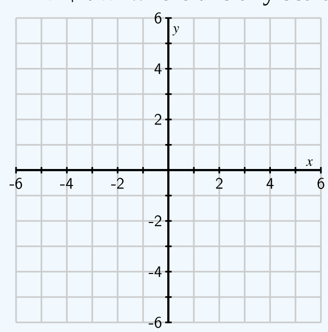
Consider the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

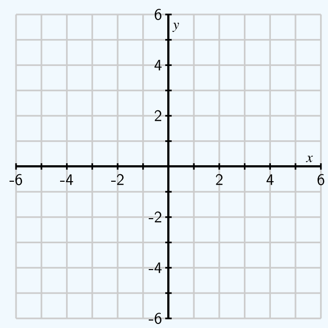
1. Sketch these vectors below.



2. Compute the vectors $-3\mathbf{v}$, $2\mathbf{w}$, $\mathbf{v}+\mathbf{w}$, and $\mathbf{v}-\mathbf{w}$ and add them into the sketch above.
3. Sketch below the set of vectors having the form $2\mathbf{v}+t\mathbf{w}$ where t is any scalar.

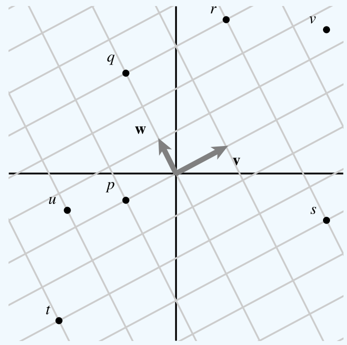


4. Sketch below the line $y = 3x - 2$. Then identify two vectors $\mathbf{v}$ and $\mathbf{w}$ so that this line is described by $\mathbf{v}+t\mathbf{w}$. Are there other choices for the vectors $\mathbf{v}$ and $\mathbf{w}$?



? 2

Shown below are two vectors $\mathbf{v}$ and $\mathbf{w}$



1. Express the labeled points as linear combinations of $\mathbf{v}$ and $\mathbf{w}$.
2. Sketch the line described parametrically as $-2\mathbf{v}+t\mathbf{w}$.

? 3

Consider the vectors

$$\mathbf{v}_1=\begin{bmatrix}2\\1\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}-1\\1\end{bmatrix}, \mathbf{v}_3=\begin{bmatrix}-2\\0\end{bmatrix}$$

1. Find the linear combination with weights $c_1=2$, $c_2=-3$, and $c_3=1$.
2. Can you write the vector $\mathbf{0}=\begin{bmatrix}0\\0\end{bmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, describe all the ways in which you can do so.
3. Can you write the vector $\mathbf{0}=\begin{bmatrix}0\\0\end{bmatrix}$ as a linear combination using just the first two vectors $\mathbf{v}_1$ $\mathbf{v}_2$? If so, describe all the ways in which you can do so.
4. Can you write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? If so, in how many ways?

? 4

Nutritional information about a breakfast cereal is printed on the box. For instance, one serving of Frosted Flakes has 111 calories, 140 milligrams of sodium, and 1.2 grams of protein. We may represent this as a vector

$$\begin{bmatrix}111\\140\\1.2\end{bmatrix}.$$

One serving of Cocoa Puffs has 120 calories, 105 milligrams of sodium, and 1.0 grams of protein.

1. Write the vector describing the nutritional content of Cocoa Puffs.
2. Suppose you eat a servings of Frosted Flakes and b servings of Cocoa Puffs. Use the language of vectors and linear combinations to express the total amount of calories, sodium, and protein you have consumed.
3. How many servings of each cereal have you eaten if you have consumed 342 calories, 385 milligrams of sodium, and 3.4 grams of protein.
4. Suppose your sister consumed 250 calories, 200 milligrams of sodium, and 4 grams of protein. What can you conclude about her breakfast?

? 5

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}.$$

1. Can you express the vector $\mathbf{b} = \begin{bmatrix} 10 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, describe all the ways in which you can do so.
2. Can you express the vector $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, describe all the ways in which you can do so.
3. Show that $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
4. Explain why any linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$,

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3,$$

can be rewritten as a linear combination of just $\mathbf{v}_1$ and $\mathbf{v}_2$.

? 6

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

For what value(s) of k, if any, can the vector $\begin{bmatrix} k \\ -2 \\ 5 \end{bmatrix}$ be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$?

? 7

Provide a justification for your response to the following statements or questions.

1. True of false: Given two vectors $\mathbf{v}$ and $\mathbf{w}$, the vector $2\mathbf{v}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.
2. True or false: Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a collection of m-dimensional vectors and that the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ has a pivot position in every row. If $\mathbf{b}$ is any m-dimensional vector, then $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
3. True or false: Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a collection of m-dimensional vectors and that the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ has a pivot position in every row and every column. If $\mathbf{b}$ is any m-dimensional vector, then $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in exactly one way.
4. True or false: It is possible to find two 3-dimensional vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that every 3-dimensional vector can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

? 8

A theme that will later unfold concerns the use of coordinate systems. We can identify the point (x, y) with the tip of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$, drawn emanating from the origin. We can then think of the usual Cartesian coordinate system in terms of linear combinations of the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

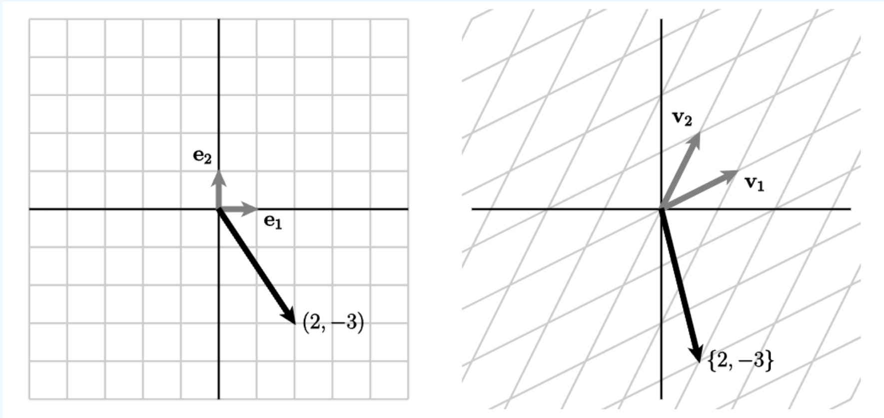


Figure 2.1.1: The usual Cartesian coordinate system, defined by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, is shown on the left along with the representation of the point $(2, -3)$. The right shows a nonstandard coordinate system defined by vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

The point $(2, -3)$ is identified with the vector

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2\mathbf{e}_1 - 3\mathbf{e}_2.$$

If we have vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we may define a new coordinate system, such that a point $\{x, y\}$ will correspond to the vector

$$x\mathbf{v}_1 + y\mathbf{v}_2.$$

For instance, the point $\{2, -3\}$ is shown on the right side of Figure 2.1.8

1. Write the point $\{2, -3\}$ in standard coordinates; that is, find x and y such that
$$(x, y) = \{2, -3\}.$$
2. Write the point $(2, -3)$ in the new coordinate system; that is, find a and b such that
$$\{a, b\} = (2, -3).$$
3. Convert a general point $\{a, b\}$, expressed in the new coordinate system, into standard Cartesian coordinates (x, y).
4. What is the general strategy for converting a point from standard Cartesian coordinates (x, y) to the new coordinates $\{a, b\}$? Actually implementing this strategy in general may take a bit of work so just describe the strategy. We will study this in more detail later.

2.2: Matrix multiplication and linear combinations

The previous section introduced vectors and linear combinations and demonstrated how they provide a means of thinking about linear systems geometrically. In particular, we saw that the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n & \mathbf{b} \end{array}\right]$$

is consistent.

Our goal in this section is to introduction matrix multiplication, another algebraic operation that connects linear systems and linear combinations.

2.2.1 Matrices

We first thought of a matrix as a rectangular array of numbers. When the number of rows is m and columns is n, we say that the dimensions of the matrix are $m \times n$. For instance, the matrix below is a 3×4 matrix:

$$\begin{bmatrix} 0 & 4 & -3 & 1 \\ 3 & -1 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix}.$$

We may also think of the columns of a matrix as a collection of vectors. For instance, the matrix above may be represented as

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

In this way, we see that our 3×4 matrix is the same as a collection of 4 vectors in $\mathbb{R}^3$.

This means that we may define scalar multiplication and matrix addition operations using the corresponding vector operations.

$$\begin{aligned} a\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} &= \begin{bmatrix} a\mathbf{v}_1 & a\mathbf{v}_2 & \ldots & a\mathbf{v}_n \end{bmatrix} \\ \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} &+ \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1 + \mathbf{w}_1 & \mathbf{v}_2 + \mathbf{w}_2 & \ldots & \mathbf{v}_n + \mathbf{w}_n \end{bmatrix}. \end{aligned}$$

Preview Activity 2.2.1. Matrix operations.

1. Compute the scalar multiple

$$-3\begin{bmatrix} 3 & 1 & 0 \\ -4 & 3 & -1 \end{bmatrix}.$$

2. Suppose that A and B are two matrices. What do we need to know about their dimensions before we can form the sum $A + B$?
3. Find the sum

$$\begin{bmatrix} 0 & -3 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -2 & 2 \\ 1 & 1 \end{bmatrix}.$$

4. The matrix I_n, which we call the *identity* matrix is the $n \times n$ matrix whose entries are zero except for the diagonal entries, which are 1. For instance,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we can form the sum $A + I_n$, what must be true about the matrix A?

5. Find the matrix $A - 2I_3$ where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -3 & 3 \\ -2 & 3 & 4 \end{bmatrix}.$$

As this preview activity shows, both of these operations are relatively straightforward. Some care, however, is required when adding matrices. Since we need the same number of vectors to add and since the vectors must be of the same dimension, two matrices must have the same dimensions as well if we wish to form their sum.

The identity matrix will play an important role at various points in our explorations. It is important to note that it is a square matrix, meaning it has an equal number of rows and columns, so any matrix added to it must be square as well. Though we wrote it as I_n in the activity, we will often just write I when the dimensions are clear.

2.2.2 Matrix-vector multiplication and linear combinations

A more important operation will be matrix multiplication as it allows us to compactly express linear systems. For now, we will work with the product of a matrix and vector, which we illustrate with an example.

Example 2.2.1

Suppose we have the matrix A and vector $\mathbf{x}$ as given below.

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Their product will be defined to be the linear combination of the columns of A using the components of $\mathbf{x}$ as weights. This means that

$$\begin{aligned} A\mathbf{x} = \begin{bmatrix} -2 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= 2 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}. \end{aligned}$$

Let's take note of the dimensions of the matrix and vectors. The two components of the vector $\mathbf{x}$ are weights used to form a linear combination of the columns of A. Since $\mathbf{x}$ has two components, A must have two columns. In other words, the number of columns of A must equal the dimension of the vector $\mathbf{x}$.

In the same way, the columns of A are 3-dimensional so any linear combination of them is 3-dimensional as well. Therefore, $A\mathbf{x}$ will be 3-dimensional.

We then see that if A is a 3×2 matrix, $\mathbf{x}$ must be a 2-dimensional vector and $A\mathbf{x}$ will be 3-dimensional.

More generally, we have the following definition.

Definition 2.2.2

The product of a matrix A by a vector $\mathbf{x}$ will be the linear combination of the columns of A using the components of $\mathbf{x}$ as weights.

If A is an $m \times n$ matrix, then $\mathbf{x}$ must be an n-dimensional vector, and the product $A\mathbf{x}$ will be an m-dimensional vector. If

$$A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,], \mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

then

$$A\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots c_n\mathbf{v}_n.$$

The next activity introduces some properties of matrix multiplication.

Activity 2.2.2. Matrix-vector multiplication.

1. Find the matrix product

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & -3 & -2 \\ -1 & -2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

2. Suppose that A is the matrix

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & -2 & 4 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}.$$

   If $A\mathbf{x}$ is defined, what is the dimension of the vector $\mathbf{x}$ and what is the dimension of $A\mathbf{x}$?

3. A vector whose entries are all zero is denoted by $\mathbf{0}$. If A is a matrix, what is the product $A\mathbf{0}$?
4. Suppose that $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the product $I\mathbf{x}$ and explain why I is called the identity matrix.
5. Suppose we write the matrix A in terms of its columns as

$$A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,].$$

   If the vector $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, what is the product $A\mathbf{e}_1$?

6. Suppose that

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

   Is there a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$?

Multiplication of a matrix A and a vector is defined as a linear combination of the columns of A. However, there is a shortcut for computing such a product. Let's look at our previous example and focus on the first row of the product.

$$\begin{bmatrix} -2 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} -2 \\ * \\ * \end{bmatrix} + 3\begin{bmatrix} 3 \\ * \\ * \end{bmatrix} = \begin{bmatrix} 2(-2) + 3(3) \\ * \\ * \end{bmatrix} = \begin{bmatrix} 5 \\ * \\ * \end{bmatrix}.$$

To find the first component of the product, we consider the first row of the matrix. We then multiply the first entry in that row by the first component of the vector, the second entry by the second component of the vector, and so on, and add the results. In this way, we see that the third component of the product would be obtained from the third row of the matrix by computing $2(3)+3(1)=9$.

You are encouraged to evaluate Item a using this shortcut and compare the result to what you found while completing the previous activity.

Activity 2.2.3.

In addition, Sage can find the product of a matrix and vector using the `*` operator. For example,

```
A = matrix(2,2,[1,2,2,1])
v = vector([3,-1])
A*v
```

1. Use Sage to evaluate the product Item a yet again.
2. In Sage, define the matrix and vectors

$$A=\begin{bmatrix}-2 & 0\\ 3 & 1\\ 4 & 2\end{bmatrix}, \mathbf{0}=\begin{bmatrix}0\\0\end{bmatrix}, \mathbf{v}=\begin{bmatrix}-2\\3\end{bmatrix}, \mathbf{w}=\begin{bmatrix}1\\2\end{bmatrix}.$$

3. What do you find when you evaluate $A\mathbf{0}$?
4. What do you find when you evaluate $A(3\mathbf{v})$ and $3(A\mathbf{v})$ and compare your results?
5. What do you find when you evaluate $A(\mathbf{v}+\mathbf{w})$ and $A\mathbf{v}+A\mathbf{w}$ and compare your results?
6. If $I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is the 3×3 identity matrix, what is the product IA?

This activity demonstrates several general properties satisfied by matrix multiplication that we record here.

Proposition 2.2.3. Linearity of matrix multiplication.

If A is a matrix, $\mathbf{v}$ and $\mathbf{w}$ vectors, and c a scalar, then

- $A\mathbf{0}=\mathbf{0}$.
- $A(c\mathbf{v})=cA\mathbf{v}$.
- $A(\mathbf{v}+\mathbf{w})=A\mathbf{v}+A\mathbf{w}$.

2.2.3 Matrix-vector multiplication and linear systems

So far, we have begun with a matrix A and a vector $\mathbf{x}$ and formed their product $A\mathbf{x}=\mathbf{b}$. We would now like to turn this around: beginning with a matrix A and a vector $\mathbf{b}$, we will ask if we can find a vector $\mathbf{x}$ such that $A\mathbf{x}=\mathbf{b}$. This will naturally lead back to linear systems.

To see the connection between the matrix equation $A\mathbf{x}=\mathbf{b}$ and linear systems, let's write the matrix A in terms of its columns $\mathbf{v}_i$ and $\mathbf{x}$ in terms of its components.

$$A=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots\mathbf{v}_n\end{bmatrix}, \mathbf{x}=\begin{bmatrix}c_1\\c_2\\ \vdots\\ c_n\end{bmatrix}.$$

We know that the matrix product $A\mathbf{x}$ forms a linear combination of the columns of A. Therefore, the equation $A\mathbf{x}=\mathbf{b}$ is merely a compact way of writing the equation for the weights c_i:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n=\mathbf{b}.$$

We have seen this equation before: Remember that Proposition 2.1.7 says that the solutions of this equation are the same as the solutions to the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{b} \end{array}\right].$$

This gives us three different ways of looking at the same solution space.

Proposition 2.2.4.

If $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \mathbf{v}_n\,]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then the following are equivalent.

- The vector $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$.
- The vector $\mathbf{b}$ is a linear combination of the columns of A with weights x_j:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{b}.$$

- The components of $\mathbf{x}$ form a solution to the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{b} \end{array}\right].$$

When the matrix $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,]$, we will frequently write

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{b} \end{array}\right] = \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$$

and say that we augment the matrix A by the vector $\mathbf{b}$.

We may think of $A\mathbf{x} = \mathbf{b}$ as merely giving a notationally compact way of writing a linear system. This form of the equation, however, will allow us to focus on important features of the system that determine its solution space.

Example 2.2.5

Describe the solution space of the equation

$$\begin{bmatrix} 2 & 0 & 2 \\ 4 & -1 & 6 \\ 1 & 3 & -5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix}$$

By Proposition 2.2.4, the solution space to this equation is the same as the equation

$$x_1 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix},$$

which is the same as the linear system corresponding to

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 4 & -1 & 6 & -5 \\ 1 & 3 & -5 & 15 \end{array}\right].$$

We will study the solutions to this linear system by finding the reduced row echelon form of the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 4 & -1 & 6 & -5 \\ 1 & 3 & -5 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This gives us the system of equations

$$\begin{alignedat}{3} x_1 & & {}+{} & x_3 & {}={} & 0 \\ & x_2 & {}-{} & 2x_3 & {}={} & 5 \end{alignedat}.$$

The variable x_3 is free so we may write the solution space parametrically as

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= 5+2x_3 \end{aligned}.$$

Since we originally asked to describe the solutions to the equation $A\mathbf{x}=\mathbf{b}$, we will express the solution in terms of the vector $\mathbf{x}$:

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -x_3 \\ 5+2x_3 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

This shows that the solutions $\mathbf{x}$ may be written in the form $\mathbf{v}+x_3\mathbf{w}$, for appropriate vectors $\mathbf{v}$ and $\mathbf{w}$. Geometrically, the solution space is a line in $\mathbb{R}^3$ through $\mathbf{v}$ moving parallel to $\mathbf{w}$.

Activity 2.2.4. The equation $A\mathbf{x}=\mathbf{b}$.

1. Consider the linear system

$$\begin{aligned} 2x+\ y-3z &= \ \ 4 \\ -x+2y+\ z &= \ \ 3 \\ 3x-\ y \qquad &= -4 \end{aligned}.$$

   Identify the matrix A and vector $\mathbf{b}$ to express this system in the form $A\mathbf{x}=\mathbf{b}$.

2. If A and $\mathbf{b}$ are as below, write the linear system corresponding to the equation $A\mathbf{x}=\mathbf{b}$.

$$A=\begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 6 \end{bmatrix}, \mathbf{b}=\begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

   and describe the solution space.

3. Describe the solution space of the equation

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & -3 & -2 \\ -1 & -2 & 6 & 1 \end{bmatrix}\mathbf{x}=\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}.$$

4. Suppose A is an $m\times n$ matrix. What can you guarantee about the solution space of the equation $A\mathbf{x}=\mathbf{0}$?

2.2.4 Matrix products

In this section, we have developed some algebraic operations on matrices with the aim of simplifying our description of linear systems. We will now introduce a final operation, the product of two matrices, that will become important when we study linear transformations in Section 2.5.

Given matrices A and B, we will form their product AB by first writing B in terms of its columns:

$$B=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \end{bmatrix}.$$

We then define

$$AB=\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \dots & A\mathbf{v}_p \end{bmatrix}.$$

✔ Example 2.2.6

Given the matrices

$$A=\begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -3 & 4 \\ 2 & 0 \end{bmatrix}, B=\begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -2 \end{bmatrix},$$

we have

$$AB = \left[A\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad A\begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad A\begin{pmatrix} 0 \\ -2 \end{pmatrix} \right] = \begin{bmatrix} -6 & 16 & -4 \\ 1 & 2 & -2 \\ 10 & -1 & -8 \\ -4 & 6 & 0 \end{bmatrix}.$$

It is important to note that we can only multiply matrices if the dimensions of the matrices are compatible. More specifically, when constructing the product AB, the matrix A multiplies the columns of B. Therefore, the number of columns of A must equal the number of rows of B. When this condition is met, the number of rows of AB is the number of rows of A, and the number of columns of AB is the number of columns of B.

Activity 2.2.5.

Consider the matrices

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -3 & 4 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ -2 & -1 \end{bmatrix}.$$

1. Suppose we want to form the product AB. Before computing, first explain how you know this product exists and then explain what the dimensions of the resulting matrix will be.
2. Compute the product AB.
3. Sage can multiply matrices using the `*` operator. Define the matrices A and B in the Sage cell below and check your work by computing AB.
4. Are you able to form the matrix product BA? If so, use the Sage cell above to find BA. Is it generally true that $AB = BA$?
5. Suppose we form the three matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} -1 & 3 \\ 4 & 3 \end{bmatrix}.$$

   Compare what happens when you compute $A(B+C)$ and $AB+AC$. State your finding as a general principle.
6. Compare the results of evaluating $A(BC)$ and $(AB)C$ and state your finding as a general principle.
7. When we are dealing with real numbers, we know if $a \neq 0$ and $ab = ac$, then $b = c$. Define matrices

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

   and compute AB and AC.

   If $AB = AC$, is it necessarily true that $B = C$?
8. Again, with real numbers, we know that if $ab = 0$, then either $a = 0$ or $b = 0$. Define

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$$

   and compute AB.

   If $AB = 0$, is it necessarily true that either $A = 0$ or $B = 0$?

This activity demonstrated some general properties about products of matrices, which mirror some properties about operations with real numbers.

Properties of Matrix-matrix Multiplication.

If A, B, and C are matrices such that the following operations are defined, it follows that

Associativity:

$A(BC) = (AB)C$.

Distributivity:

$A(B+C) = AB + AC$.

$(A+B)C = AC+BC$.

At the same time, there are a few properties that hold for real numbers that do not hold for matrices.

Things to be careful of.

The following properties hold for real numbers but not for matrices.

Commutativity:

It is *not* generally true that $AB = BA$.

Cancellation:

It is *not* generally true that $AB = AC$ implies that $B = C$.

Zero divisors:

It is *not* generally true that $AB = 0$ implies that either $A = 0$ or $B = 0$.

Summary

In this section, we have found an especially simple way to express linear systems using matrix multiplication.

- If A is an $m\times n$ matrix and $\mathbf{x}$ an n-dimensional vector, then $A\mathbf{x}$ is the linear combination of the columns of A using the components of $\mathbf{x}$ as weights. The vector $A\mathbf{x}$ is m-dimensional.
- The solution space to the equation $A\mathbf{x} = \mathbf{b}$ is the same as the solution space to the linear system corresponding to the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.
- If A is an $m\times n$ matrix and B is an $n\times p$ matrix, we can form the product AB, which is an $m\times p$ matrix whose columns are the products of A and the columns of B.

Exercises 2.2.6Exercises

? 1

Consider the system of linear equations

$$\begin{alignedat}{4} x & {}+{} & 2y & {}-{} & z & {}={} & 1 \\ 3x & {}+{} & 2y & {}+{} & 2z & {}={} & 7 \\ -x & & & {}+{} & 4z & {}={} & -3 \end{alignedat}.$$

1. Find the matrix A and vector $\mathbf{b}$ that expresses this linear system in the form $A\mathbf{x} = \mathbf{b}$.
2. Give a description of the solution space to the equation $A\mathbf{x} = \mathbf{b}$.

? 2

Suppose that A is a 135×2201 matrix. If $A\mathbf{x}$ is defined, what is the dimension of $\mathbf{x}$? What is the dimension of $A\mathbf{x}$?

? 3

Suppose that A is a 3×2 matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$; that is,

$$A = \left[\begin{array}{cc} \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right].$$

1. What is the dimension of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$?
2. What is the product $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$? What is the product $A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$? What is the product $A\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?
3. Suppose that

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix},\ A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

What is the matrix A?

? 4

Shown below are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Suppose that the matrix A is

$$A = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,].$$

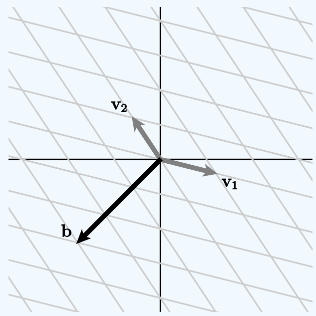


Figure 2.2.1: Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. that form the columns of the matrix $\mathbf{A}$

1. What are the dimensions of the matrix A?
2. On the plot above, indicate the vectors

$$A\begin{pmatrix}1\\0\end{pmatrix}, A\begin{pmatrix}2\\3\end{pmatrix}, A\begin{pmatrix}0\\-3\end{pmatrix}.$$

3. Find all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.
4. Find all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

? 5

Suppose that

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ -1 & -3 & 1 \end{bmatrix}.$$

1. Describe the solution space to the equation $A\mathbf{x} = \mathbf{0}$.
2. Find a 3×2 matrix B with no zero entries such that $AB = 0$.

? 6

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 4 & 6 \end{bmatrix}.$$

1. Find the product $A\mathbf{x}$ where

$$\mathbf{x} = \begin{pmatrix}1\\-2\\0\\2\end{pmatrix}.$$

2. Give a description of the vectors $\mathbf{x}$ such that

$$A\mathbf{x} = \begin{pmatrix} -1 \\ 15 \\ 17 \end{pmatrix}.$$

3. Find the reduced row echelon form of A and identify the pivot positions.
4. Can you find a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent?
5. For a general 3-dimensional vector $\mathbf{b}$, what can you say about the solution space of the equation $A\mathbf{x} = \mathbf{b}$?

? 7

The operations that we perform in Gaussian elimination can be accomplished using matrix multiplication. This observation is the basis of an important technique that we will investigate in a subsequent chapter.

Let's consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & 3 \end{bmatrix}.$$

1. Suppose that

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Verify that SA is the matrix that results when the second row of A is scaled by a factor of 7. What matrix S would scale the third row by -3?

2. Suppose that

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Verify that PA is the matrix that results from interchanging the first and second rows. What matrix P would interchange the first and third rows?

3. Suppose that

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Verify that L_1A is the matrix that results from multiplying the first row of A by -2 and adding it to the second row. What matrix L_2 would multiply the first row by 3 and add it to the third row?

4. When we performed Gaussian elimination, our first goal was to perform row operations that brought the matrix into a triangular form. For our matrix A, find the row operations needed to find a row equivalent matrix U in triangular form. By expressing these row operations in terms of matrix multiplication, find a matrix L such that $LA = U$.

? 8

In this exercise, you will construct the *inverse* of a matrix, a subject that we will investigate more fully in the next chapter. Suppose that A is the 2×2 matrix:

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

1. Find the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ such that the matrix $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ satisfies

$$AB = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. In general, it is not true that $AB = BA$. Check that it is true, however, for the specific A and B that appear in this problem.
3. Suppose that $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. What do you find when you evaluate $I\mathbf{x}$?
4. Suppose that we want to solve the equation $A\mathbf{x} = \mathbf{b}$. We know how to do this using Gaussian elimination; let's use our matrix B to find a different way:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ B(A\mathbf{x}) &= B\mathbf{b} \\ (BA)\mathbf{x} &= B\mathbf{b} \\ I\mathbf{x} &= B\mathbf{b} \\ \mathbf{x} &= B\mathbf{b} \end{aligned}.$$

In other words, the solution to the equation $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = B\mathbf{b}$.

Consider the equation $A\mathbf{x} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$. Find the solution in two different ways, first using Gaussian elimination and then as $\mathbf{x} = B\mathbf{b}$, and verify that you have found the same result.

? 9

Determine whether the following statements are true or false and provide a justification for your response.

1. If $A\mathbf{x}$ is defined, then the number of components of $\mathbf{x}$ equals the number of rows of A.
2. The solution space to the equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the solution space to the linear system whose augmented matrix is $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.
3. If a linear system of equations has 8 equations and 5 unknowns, then the dimensions of the matrix A in the corresponding equation $A\mathbf{x} = \mathbf{b}$ is 5×8.
4. If A has a pivot in every row, then every equation $A\mathbf{x} = \mathbf{b}$ is consistent.
5. If A is a 9×5 matrix, then $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$.

? 10

Suppose that A is an 4×4 matrix and that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for some vector $\mathbf{b}$.

1. What does this say about the pivots of the matrix A? Write the reduced row echelon form of A.
2. Can you find another vector $\mathbf{c}$ such that $A\mathbf{x} = \mathbf{c}$ is inconsistent?
3. What can you say about the solution space to the equation $A\mathbf{x} = \mathbf{0}$?
4. Suppose $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$. Explain why every four-dimensional vector can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ in exactly one way.

? 11

Define the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & -3 \\ 3 & 1 & 7 \end{bmatrix}.$$

1. Describe the solution space to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. What does this solution space represent geometrically?

2. Describe the solution space to the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$. What does this solution space represent geometrically and how does it compare to the previous solution space?
3. We will now explain the relationship between the previous two solution spaces. Suppose that $\mathbf{x}_h$ is a solution to the homogeneous equation; that is $A\mathbf{x}_h = \mathbf{0}$. We will also suppose that $\mathbf{x}_p$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$; that is, $A\mathbf{x}_p = \mathbf{b}$.

   Use the Linearity Principle expressed in Proposition 2.2.3 to explain why $\mathbf{x}_h + \mathbf{x}_p$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. You may do this by evaluating $A(\mathbf{x}_h + \mathbf{x}_p)$.

   That is, if we find one solution $\mathbf{x}_p$ to an equation $A\mathbf{x} = \mathbf{b}$, we may add any solution to the homogeneous equation to $\mathbf{x}_p$ and still have a solution to the equation $A\mathbf{x} = \mathbf{b}$. In other words, the solution space to the equation $A\mathbf{x} = \mathbf{b}$ is given by translating the solution space to the homogeneous equation by the vector $\mathbf{x}_p$.

? 12

Suppose that a city is starting a bicycle sharing program with bicycles at locations B and C. Bicycles that are rented at one location may be returned to either location at the end of the day. Over time, the city finds that 80% of bicycles rented at location B are returned to B with the other 20% returned to C. Similarly, 50% of bicycles rented at location C are returned to B and 50% to C.

To keep track of the bicycles, we form a vector

$$\mathbf{x}_k = \begin{pmatrix} B_k \\ C_k \end{pmatrix}$$

where B_k is the number of bicycles at location B at the beginning of day k and C_k is the number of bicycles at C. The information above tells us

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}.$$

1. Let's check that this makes sense.
   1. Suppose that there are 1000 bicycles at location B and none at C on day 1. This means we have $\mathbf{x}_1 = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}$. Find the number of bicycles at both locations on day 2 by evaluating $\mathbf{x}_2 = A\mathbf{x}_1$.
   2. Suppose that there are 1000 bicycles at location C and none at B on day 1. Form the vector $\mathbf{x}_1$ and determine the number of bicycles at the two locations the next day by finding $\mathbf{x}_2 = A\mathbf{x}_1$.
2. Suppose that one day there are 1050 bicycles at location B and 450 at location C. How many bicycles were there at each location the previous day?
3. Suppose that there are 500 bicycles at location B and 500 at location C on Monday. How many bicycles are there at the two locations on Tuesday? on Wednesday? on Thursday?

? 13

This problem is a continuation of the previous problem.

1. Let us define vectors

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

   Show that

$$A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = 0.3\mathbf{v}_2.$$

2. Suppose that $\mathbf{x}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ where c_2 and c_2 are scalars. Use the Linearity Principle expressed in Proposition 2.2.3 to explain why

$$\mathbf{x}_2 = A\mathbf{x}_1 = c_1\mathbf{v}_1 + 0.3c_2\mathbf{v}_2.$$

3. Continuing in this way, explain why

$$\begin{aligned} \mathbf{x}_3 = A\mathbf{x}_2 &= c_1\mathbf{v}_1 + 0.3^2c_2\mathbf{v}_2 \\ \mathbf{x}_4 = A\mathbf{x}_3 &= c_1\mathbf{v}_1 + 0.3^3c_2\mathbf{v}_2 \\ \mathbf{x}_5 = A\mathbf{x}_4 &= c_1\mathbf{v}_1 + 0.3^4c_2\mathbf{v}_2 \end{aligned}.$$

4. Suppose that there are initially 500 bicycles at location B and 500 at location C. Write the vector $\mathbf{x}_1$ and find the scalars c_1 and c_2 such that $\mathbf{x}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.
5. Use the previous part of this problem to determine $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$.
6. After a very long time, how are all the bicycles distributed?

2.3: The span of a set of vectors

Our work in this chapter enables us to rewrite a linear system in the form $A\mathbf{x} = \mathbf{b}$. Besides being a more compact way of expressing a linear system, this form allows us to think about linear systems geometrically since matrix multiplication is defined in terms of linear combinations of vectors.

We now return, in this and the next section, to the two fundamental questions asked in Question 1.4.2.

- *Existence:* Is there a solution to the equation $A\mathbf{x} = \mathbf{b}$?
- *Uniqueness:* If there is a solution to the equation $A\mathbf{x} = \mathbf{b}$, is it unique?

In this section, we focus on the existence question and introduce the concept of *span* to provide a framework for thinking about it geometrically.

Preview Activity 2.3.1. The existence of solutions.

1. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, what can we say about the pivots of the augmented matrix $\left[\begin{array}{r|r} A & \mathbf{b} \end{array}\right]$?
2. Consider the matrix A

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 2 & 2 \\ 1 & 1 & -3 \end{bmatrix}.$$

   If $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$, is the equation $A\mathbf{x} = \mathbf{b}$ consistent? If so, find a solution.
3. If $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$, is the equation $A\mathbf{x} = \mathbf{b}$ consistent? If so, find a solution.
4. Identify the pivot positions of A.
5. For our two choices of the vector $\mathbf{b}$, one equation $A\mathbf{x} = \mathbf{b}$ has a solution and the other does not. What feature of the pivot positions of the matrix A tells us to expect this?

2.3.1 The span of a set of vectors

In the preview activity, we considered a 3×3 matrix A and found that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for some vectors $\mathbf{b}$ in $\mathbb{R}^3$ and has no solution for others. We will introduce a concept called *span* that describes the vectors $\mathbf{b}$ for which there is a solution.

Since we would like to think about this concept geometrically, we will consider an $m \times n$ matrix A as being composed of n vectors in $\mathbb{R}^m$; that is,

$$A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n].$$

Remember that Proposition 2.2.4 says that the equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if we can express $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

Definition 2.3.1

The span of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is the set of all linear combinations of the vectors.

In other words, the span of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ consists of all the vectors $\mathbf{b}$ for which the equation

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n]\,\mathbf{x} = \mathbf{b}$$

is consistent.

The span of a set of vectors has an appealing geometric interpretation. Remember that we may think of a linear combination as a recipe for walking in $\mathbb{R}^m$. We first move a prescribed amount in the direction of $\mathbf{v}_1$, then a prescribed amount in the direction of $\mathbf{v}_2$, and so on. As the following activity will show, the span consists of all the places we can walk to.

Activity 2.3.2.

Let's look at two examples to develop some intuition for the concept of span.

a. First, we will consider the set of vectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

The diagram below can be used to construct linear combinations whose weights a and b may be varied using the sliders at the top. The vectors $\mathbf{v}$ and $\mathbf{w}$ are drawn in gray while the linear combination

$$a\mathbf{v} + b\mathbf{w} \tag{2.3.1}$$

is in red.

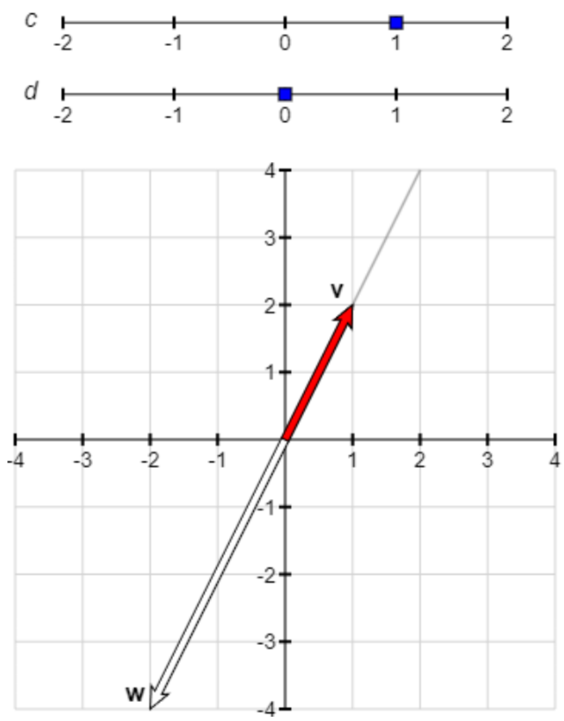


Figure 2.3.1: An interactive diagram for constructing linear combinations of the vectors $\mathbf{v}$ and $\mathbf{w}$.

1. What vector is the linear combination of $\mathbf{v}$ and $\mathbf{w}$ with weights:
   - $a = 2$ and $b = 0$?
   - $a = 1$ and $b = 1$?
   - $a = 0$ and $b = -1$?

2. Can the vector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? Is the vector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ in the span of $\mathbf{v}$ and $\mathbf{w}$?

3. Can the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? Is the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ in the span of $\mathbf{v}$ and $\mathbf{w}$?

4. Describe the set of vectors in the span of $\mathbf{v}$ and $\mathbf{w}$.

5. For what vectors $\mathbf{b}$ does the equation

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

have a solution?

b. We will now look at an example where

\begin{equation*} \mathbf v = \twovec{2}{1}, \mathbf w = \twovec{1}{2}\text{.} \end{equation*}

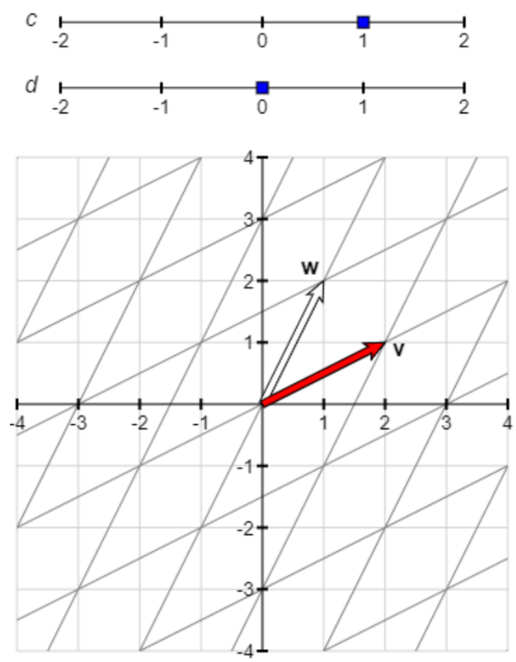


Figure 2.3.21: An interactive diagram for constructing linear combinations of the vectors $\mathbf{v}$ and $\mathbf{w}$.

In a similar way, the diagram below can be used to construct linear combinations $a\mathbf{v}+b\mathbf{w}$.

1. What vector is the linear combination of \(\mathbf v\) and \(\mathbf w\) with weights:

- $a=2$ and $b=0$?
- $a=1$ and \(b=1\text{?}\
- $a=0$ and $b=-1$?

2. Can the vector $\begin{pmatrix}-2\\2\end{pmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? Is the vector $\begin{pmatrix}-2\\2\end{pmatrix}$ in the span of $\mathbf{v}$ and $\mathbf{w}$?

3. Can the vector $\begin{pmatrix}3\\0\end{pmatrix}$ be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$? Is the vector $\begin{pmatrix}3\\0\end{pmatrix}$ in the span of $\mathbf{v}$ and $\mathbf{w}$?

4. Describe the set of vectors in the span of $\mathbf{v}$ and $\mathbf{w}$.

5. For what vectors $\mathbf{b}$ does the equation

$$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\mathbf{x}=\mathbf{b}$$

have a solution?

Let's consider the first example in the previous activity. Here, the vectors $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples of one another, which means that they lie on the same line. When we form linear combinations, we are allowed to walk only in the direction of $\mathbf{v}$ and $\mathbf{w}$, which means we are constrained to stay on this same line. Therefore, the span of $\mathbf{v}$ and $\mathbf{w}$ consists only of this line.

Figure 2.3.2.

With this choice of vectors $\mathbf{v}$ and $\mathbf{w}$, all linear combinations lie on the line shown. This line, therefore, is the span of the vectors $\mathbf{v}$ and $\mathbf{w}$.

We may see this algebraically since the vector $\mathbf{w} = -2\mathbf{v}$. Consequently, when we form a linear combination of $\mathbf{v}$ and $\mathbf{w}$, we see that

$$\begin{aligned} a\mathbf{v} + b\mathbf{w} &= a\mathbf{v} + b(-2\mathbf{v}) \\ &= (a-2b)\mathbf{v} \end{aligned}.$$

Therefore, any linear combination of $\mathbf{v}$ and $\mathbf{w}$ reduces to a scalar multiple of $\mathbf{v}$, and we have seen that the scalar multiples of a nonzero vector form a line.

In the second example, however, the vectors are not scalar multiples of one another, and we see that we can construct any vector in $\mathbb{R}^2$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

Figure 2.3.3.

With this choice of vectors $\mathbf{v}$ and $\mathbf{w}$, we are able to form any vector in $\mathbb{R}^2$ as a linear combination. Therefore, the span of the vectors $\mathbf{v}$ and $\mathbf{w}$ is the entire plane, $\mathbb{R}^2$.

Once again, we can see this algebraically. Asking if the vector $\mathbf{b}$ is in the span of $\mathbf{v}$ and $\mathbf{w}$ is the same as asking if the linear system

$$\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} \mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

is consistent.

The augmented matrix for this system is

$$\left[\begin{array}{rr|r} 2 & 1 & * \\ 1 & 2 & * \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & * \\ 0 & 1 & * \end{array}\right].$$

Since it is impossible to obtain a pivot in the rightmost column, we know that this system is consistent no matter what the vector $\mathbf{b}$ is. Therefore, every vector $\mathbf{b}$ in $\mathbb{R}^2$ is in the span of $\mathbf{v}$ and $\mathbf{w}$.

In this case, notice that the reduced row echelon form of the matrix

$$\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has a pivot in every row. When this happens, it is not possible for any augmented matrix to have a pivot in the rightmost column. Therefore, the linear system is consistent for every vector $\mathbf{b}$, which implies that the span of $\mathbf{v}$ and $\mathbf{w}$ is $\mathbb{R}^2$.

Notation 2.3.4.

We will denote the span of the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ by $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

2.3.2 Pivot positions and span

In the previous activity, we saw two examples, both of which considered two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$. In one example, the $Span\{\mathbf{v}, \mathbf{w}\}$ consisted of a line; in the other, the $Span\{\mathbf{v}, \mathbf{w} = \mathbb{R}^2\}$. We would like to be able to distinguish these two situations in a more algebraic fashion. After all, we will need to be able to deal with vectors in many more dimensions where we will not be able to draw pictures.

The key is found by looking at the pivot positions of the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots \mathbf{v}_n \end{bmatrix}$. In the first example, the matrix whose columns are $\mathbf{v}$ and $\mathbf{w}$ is

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix},$$

which has exactly one pivot position. We found the $Span\{\mathbf{v}, \mathbf{w}\}$ to be a line, in this case.

In the second example, this matrix is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which has two pivot positions. Here, we found $Span\{\mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$.

These examples point to the fact that the size of the span is related to the number of pivot positions. We will develop this idea more fully in Section 2.4 and Section 3.5. For now, however, we will examine the possibilities in $\mathbb{R}^3$.

Activity 2.3.3.

In this activity, we will look at the span of sets of vectors in $\mathbb{R}^3$.

1. Suppose $v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Give a written description of $Span\{v\}$ and a rough sketch of it below.

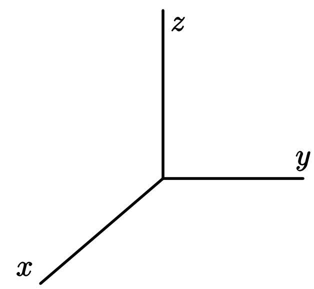


Figure 2.3.1: Copy and Paste Caption here. (Copyright; author via source)

2. Consider now the two vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Sketch the vectors below. Then give a written description of $Span\{\mathbf{e}_1, \mathbf{e}_2\}$ and a rough sketch of it below.

Let's now look at this algebraically by writing write $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Determine the conditions on b_1, b_2, and b_3 so that $\mathbf{b}$ is in $Span\{\mathbf{e}_1, \mathbf{e}_2\}$ by considering the linear system

$$[\,\mathbf{e}_1 \quad \mathbf{e}_2\,]\,\mathbf{x} = \mathbf{b}$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Explain how this relates to your sketch of $Span\{\mathbf{e}_1, \mathbf{e}_2\}$.

3. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

   1. Is the vector $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$?

2. Is the vector $\mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$?
3. Give a written description of $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.

4. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

Form the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and find its reduced row echelon form.

What does this tell you about $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

5. If a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ spans $\mathbb{R}^3$, what can you say about the pivots of the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$?
6. What is the smallest number of vectors such that $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^3$?

This activity shows us the types of sets that can appear as the span of a set of vectors in $\mathbb{R}^3$.

- First, with a single vector, all linear combinations are simply scalar multiples of that vector, which creates a line.

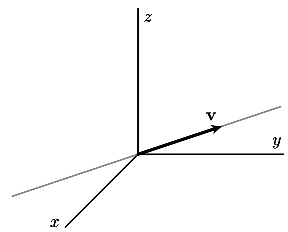


Figure 2.3.1:The span of a single nonzero vector is a line.

Notice that the matrix formed by this vector has one pivot, just as in our earlier example in $\mathbb{R}^2$.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- When we consider linear combinations of the vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

we must obtain vectors of the form

$$a\mathbf{e}_1 + b\mathbf{e}_2 = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

Since the third component is zero, these vectors form the plane $z = 0$.

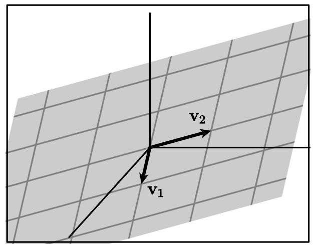


Figure 2.3.1: The span of these two vectors in $\mathbb{R}^3$ is a plane.

- Notice here that the matrix composed of the vectors has two pivot positions.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

- Similarly, the span of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},$$

  will form a plane.

  We saw one vector $\mathbf{b}$ that was not in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ and one that is.

  Once again, the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  has two pivot positions.

- Finally, we looked at a set of vectors whose matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -2 \\ -1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  has three pivot positions, one in every row. This is significant because it means that if we consider an augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & * \\ 1 & 2 & -2 & * \\ -1 & 1 & 4 & * \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right],$$

  there cannot be a pivot position in the rightmost column. This linear system is consistent for every vector $\mathbf{b}$, which tells us that $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$.

To summarize, we looked at the pivot positions in the matrix whose columns were the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We found that with

- one pivot position, the span was a line.
- two pivot positions, the span was a plane.
- three pivot positions, the span was $\mathbb{R}^3$.

Once again, we will develop these ideas more fully in the next and subsequent sections. However, we saw that, when considering vectors in $\mathbb{R}^3$, a pivot position in every row implied that the span of the vectors is $\mathbb{R}^3$. The same reasoning applies more generally.

> **Proposition 2.3.5.**
>
> Suppose we have vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$. Then $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^m$ if and only if the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ has a pivot position in every row.

This tells us something important about the number of vectors needed to span $\mathbb{R}^m$. Suppose we have n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that span $\mathbb{R}^m$. The proposition tells us that the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \ldots \mathbf{v}_n \end{bmatrix}$ has a pivot position in every row, such as in this reduced row echelon matrix.

$$\begin{bmatrix} 1 & 0 & * & 0 & * & 0 \\ 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since a matrix can have at most one pivot position in a column, there must be at least as many columns as there are rows, which implies that $n \geq m$.

For instance, if we have a set of vectors that span $\mathbb{R}^{632}$, there must be at least 632 vectors in the set.

> **Proposition 2.3.6.**
>
> If a set of vectors span $\mathbb{R}^m$, there must be at least m vectors in the set.

This makes sense intuitively. We have thought about a linear combination of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as the result of walking a certain distance in the direction of $\mathbf{v}_1$, followed by walking a certain distance in the direction of $\mathbf{v}_2$, and so on. If $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^m$, this means that we can walk to any point in $\mathbb{R}^m$ using the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. It makes sense that we would need at least m directions to give us the flexibilty needed to reach any point in $\mathbb{R}^m$.

> **Terminology**
>
> Because span is a concept that is connected to a set of vectors, we say, "The span of the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is ...." While it may be tempting to say, "The span of the matrix A is ...," we should instead say "The span of the columns of the matrix A is ...."

Summary

We defined the span of a set of vectors and developed some intuition for this concept through a series of examples.

- The span of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the set of linear combinations of the vectors. We denote the span by $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
- A vector $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ if an only if the linear system

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots \mathbf{v}_n \end{bmatrix} \mathbf{x} = \mathbf{b}$$

  is consistent.
- If the $m \times n$ matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots \mathbf{v}_n \end{bmatrix}$$

  has a pivot in every row, then the span of these vectors is $\mathbb{R}^m$; that is, $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^m$.
- Any set of vectors that spans $\mathbb{R}^m$ must have at least m vectors.

Exercises 2.3.4Exercises

? 1

In this exercise, we will consider the span of some sets of two- and three-dimensional vectors.

1. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

   1. Is $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$?
   2. Give a written description of $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.
2. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}.$$

   1. Is the vector $\mathbf{b} = \begin{pmatrix} -10 \\ -1 \\ 5 \end{pmatrix}$ in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
   2. Is the vector $\mathbf{v}_3$ in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
   3. Is the vector $\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
   4. Give a written description of $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

? 2

Provide a justification for your response to the following questions.

1. Suppose you have a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Can you guarantee that $\mathbf{0}$ is in $Span\{\mathbf{v}_1\ \mathbf{v}_2, \ldots, \mathbf{v}_n\}$?
2. Suppose that A is an $m \times n$ matrix. Can you guarantee that the equation $A\mathbf{x} = \mathbf{0}$ is consistent?
3. What is $Span\{\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\}$?

? 3

For both parts of this exericse, give a written description of sets of the vectors $\mathbf{b}$ and include a sketch.

1. For which vectors $\mathbf{b}$ in $\mathbb{R}^2$ is the equation

$$\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

   consistent?
2. For which vectors $\mathbf{b}$ in $\mathbb{R}^2$ is the equation

$$\begin{bmatrix} 3 & -6 \\ -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

   consistent?

? 4

Consider the following matrices:

$$A = \begin{bmatrix} 3 & 0 & -1 & 1 \\ 1 & -1 & 3 & 7 \\ 3 & -2 & 1 & 5 \\ -1 & 2 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & -1 & 4 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 1 & 3 \\ -1 & 2 & 2 & 1 \end{bmatrix}.$$

Do the columns of A span $\mathbb{R}^4$? Do the columns of B span $\mathbb{R}^4$?

? 5

Determine whether the following statements are true or false and provide a justification for your response. Throughout, we will assume that the matrix A has columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$; that is,

$$A = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n \,].$$

1. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
2. The equation $A\mathbf{x} = \mathbf{v}_1$ is always consistent.
3. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are vectors in $\mathbb{R}^3$, then their span is $\mathbb{R}^3$.
4. If $\mathbf{b}$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
5. If A is a 8032×427 matrix, then the span of the columns of A is a set of vectors in $\mathbb{R}^{427}$.

? 6

This exercise asks you to construct some matrices whose columns span a given set.

1. Construct a 3×3 matrix whose columns span $\mathbb{R}^3$.
2. Construct a 3×3 matrix whose columns span a plane in $\mathbb{R}^3$.
3. Construct a 3×3 matrix whose columns span a line in $\mathbb{R}^3$.

? 7

Provide a justification for your response to the following questions.

1. Suppose that we have vectors in $\mathbb{R}^8$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}$, whose span is $\mathbb{R}^8$. Can every vector $\mathbf{b}$ in $\mathbb{R}^8$ be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}$?
2. Suppose that we have vectors in $\mathbb{R}^8$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}$, whose span is $\mathbb{R}^8$. Can every vector $\mathbf{b}$ in $\mathbb{R}^8$ be written *uniquely* as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}$?
3. Do the vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   span $\mathbb{R}^3$?
4. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^{438}$. What can you guarantee about the value of n?
5. Can 17 vectors in $\mathbb{R}^{20}$ span $\mathbb{R}^{20}$?

? 8

The following observation will be helpful in this exericse. If we want to find a solution to the equation $AB\mathbf{x} = \mathbf{b}$, we could first find a solution to the equation $A\mathbf{y} = \mathbf{b}$ and then find a solution to the equation $B\mathbf{x} = \mathbf{y}$.

Suppose that A is a 3×4 matrix whose columns span $\mathbb{R}^3$ and B is a 4×5 matrix. In this case, we can form the product AB.

1. What are the dimensions of the product AB?
2. Can you guarantee that the columns of AB span $\mathbb{R}^3$?
3. If you know additionally that the span of the columns of B is $\mathbb{R}^4$, can you guarantee that the columns of AB span $\mathbb{R}^3$?

? 9

Suppose that A is a 12×12 matrix and that, for some vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

1. What can you say about the pivot positions of A?
2. What can you say about the span of the columns of A?
3. If $\mathbf{c}$ is some other vector in $\mathbb{R}^{12}$, what can you conclude about the equation $A\mathbf{x} = \mathbf{c}$?
4. What can you about the solution space to the equation $A\mathbf{x} = \mathbf{0}$?

? 10

Suppose that

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ -2 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -3 \\ -3 \\ -7 \\ 5 \end{pmatrix}.$$

1. Is $\mathbf{v}_3$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? If so, find weights such that $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$.
2. Show that a linear combination

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$$

can be rewritten as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

3. Explain why $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$.

? 11

As defined in this section, the span of a set of vectors is generated by taking all possible linear combinations of those vectors. This exericse will demonstrate the fact that the span can also be realized as the solution space to a linear system.

We will consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

1. Is every vector in $\mathbb{R}^3$ in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? If not, describe the span.
2. To describe $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as the solution space of a linear system, we will write

$$\mathbf{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

If $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then the linear system corresponding to the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b \\ -2 & 0 & 2 & c \end{array}\right]$$

must be consistent. This means that a pivot cannot occur in the rightmost column. Perform row operations to put this augmented matrix into a triangular form. Now identify an equation in a, b, and c that tells us when there is no pivot in the rightmost column. The solution space to this equation describes $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

3. In this example, the matrix formed by the vectors $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_2 \end{bmatrix}$ has two pivot positions. Suppose we were to consider another example in which this matrix had had only one pivot position. How would this have changed the linear system describing $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

2.4: Linear independence

In the previous section, we studied our question concerning the existence of solutions to a linear system by inquiring about the span of a set of vectors. In particular, the span of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the set of vectors $\mathbf{b}$ for which a solution to the linear system $\left[\begin{matrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{matrix}\right]\mathbf{x} = \mathbf{b}$ exists.

In this section, our focus turns to the uniqueness of solutions of a linear system, the second of our two fundamental questions asked in Question 1.4.2. This will lead us to the concept of linear independence.

Linear dependence

In the previous section, we looked at some examples of the span of sets of vectors in $\mathbb{R}^3$. We saw one example in which the span of three vectors formed a plane. In another, the span of three vectors formed $\mathbb{R}^3$. We would like to understand the difference in these two examples.

Preview Activity 2.4.1.

Let's start this activity by studying the span of two different sets of vectors.

1. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

   Describe the span of these vectors, $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

2. We will now consider a set of vectors that looks very much like the first set:

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

   Describe the span of these vectors, $Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

3. Show that the vector $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$ by finding weights such that

$$\mathbf{w}_3 = a\mathbf{w}_1 + b\mathbf{w}_2.$$

4. Explain why any linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$,

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3$$

   can be written as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

5. Explain why

$$Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = Span\{\mathbf{w}_1, \mathbf{w}_2\}.$$

The preview activity presents us with two similar examples that demonstrate quite different behaviors. In the first example, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span $\mathbb{R}^3$, which we recognize because the matrix $\left[\begin{matrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{matrix}\right]$ has a pivot position in every row so that Proposition 2.3.5 applies.

However, the second example is very different. In this case, the matrix $\left[\begin{matrix}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{matrix}\right]$ has only two pivot positions:

$$\left[\begin{matrix}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{matrix}\right] = \begin{bmatrix} 0 & 3 & 3 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let's look at this matrix and change our perspective slightly by considering it to be an augmented matrix.

$$\left[\begin{array}{cc|c} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{array}\right] = \left[\begin{array}{rr|r} 0 & 3 & 3 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

By doing so, we seek to express $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. In fact, the reduced row echelon form shows us that

$$\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_2.$$

Consequently, we can rewrite any linear cominbation of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ so that

$$\begin{aligned} c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 & = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3(\mathbf{w}_1 + \mathbf{w}_2) \\ & = (c_1 + c_3)\mathbf{w}_1 + (c_2 + c_3)\mathbf{w}_2 \end{aligned}.$$

In other words, any linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ may be written as a linear combination using only the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. Since the span of a set of vectors is simply the set of their linear combinations, this shows that

$$Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = Span\{\mathbf{w}_1, \mathbf{w}_2\}text.$$

In other words, adding the vector $\mathbf{w}_3$ to the set of vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ does not change the span.

Before exploring this type of behavior more generally, let's think about this from a geometric point of view. In the first example, we begin with two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and add a third vector $\mathbf{v}_3$.

Because the vector $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, it provides a direction to move that, when creating linear combinations, is independent of $\mathbf{v}_1$ and $\mathbf{v}_2$. The three vectors therefore span $\mathbb{R}^3$.

In the second example, however, the third vector $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$ so it already lies in the plane formed by these two vectors.

Since we can already move in this direction with just the first two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, adding $\mathbf{w}_3$ to the set does not enlarge the span. It remains a plane.

With these examples in mind, we will make the following definition.

Definition 2.4.1

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called *linearly dependent* if one of the vectors is a linear combination of the others. Otherwise, the set of vectors is called *linearly independent*.

For the sake of completeness, we say that a set of vectors containing only one vector is linearly independent if that vector is not the zero vector, $\mathbf{0}$.

How to recognize linear dependence

Activity 2.4.2.

We would like to develop a means of detecting when a set of vectors is linearly dependent. These questions will point the way.

1. Suppose we have five vectors in $\mathbb{R}^4$ that form the columns of a matrix having reduced row echelon form

$$\left[\begin{array}{ccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{array}\right] \sim \left[\begin{array}{rrrrr} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

   Is it possible to write one of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ as a linear combination of the others? If so, show explicitly how one vector appears as a linear combination of some of the other vectors. Is this set of vectors linearly dependent or independent?

2. Suppose we have another set of three vectors in $\mathbb{R}^4$ that form the columns of a matrix having reduced row echelon form

$$\begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Is it possible to write one of these vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ as a linear combination of the others? If so, show explicitly how one vector appears as a linear combination of some of the other vectors. Is this set of vectors linearly dependent or independent?

3. By looking at the pivot positions, how can you determine whether the columns of a matrix are linearly dependent or independent?
4. If one vector in a set is the zero vector $\mathbf{0}$, can the set of vectors be linearly independent?
5. Suppose a set of vectors in $\mathbb{R}^{10}$ has twelve vectors. Is it possible for this set to be linearly independent?

By now, it shouldn't be too surprising that the pivot positions play an important role in determining whether the columns of a matrix are linearly dependent. Let's discuss the previous activity to make this clear.

- Let's consider the first example from the activity in which we have vectors in $\mathbb{R}^4$ such that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that the third column does not contain a pivot position. Let's focus on the first three columns and consider them as an augmented matrix:

$$\left[\begin{array}{cc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

There is not a pivot in the rightmost column so we know that $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact, we can read the weights from the augmented matrix:

$$\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2.$$

This says that the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ is linearly dependent.

This points to the general observation that a set of vectors is linearly dependent if the matrix they form has a column without a pivot.

In addition, the fifth column of this matrix does not contain a pivot meaning that $\mathbf{v}_5$ can be written as a linear combination:

$$\mathbf{v}_5 = 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_4.$$

This example shows that vectors in columns that do not contain a pivot may be expressed as a linear combination of the vectors in columns that do contain pivots. In this case, we have

$$Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}.$$

- Conversely, the second set of vectors we studied produces a matrix with a pivot in every column.

$$\begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If we interpret this as an augmented matrix again, we see that the linear system is inconsistent since there is a pivot in the rightmost column. This means that $\mathbf{w}_3$ cannot be expressed as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Similarly, $\mathbf{w}_2$ cannot be expressed as a linear combination of $\mathbf{w}_1$. In addition, if $\mathbf{w}_2$ could be expressed as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_3$, we could rearrange that expression to write $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, which we know is impossible.

We can therefore conclude that $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ form a linearly indpendent set of vectors.

This leads to the following proposition.

Proposition 2.4.2.

The columns of a matrix are linearly independent if and only if every column contains a pivot position.

This condition imposes a constraint on how many vectors we can have in a linearly independent set. Here is an example of the reduced row echelon form of a matrix having linearly independent columns. Notice that there are three vectors in $\mathbb{R}^5$ so there are at least as many rows as columns.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

More generally, suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a linearly independent set of vectors in $\mathbb{R}^m$. When these vectors form the columns of a matrix, there must be a pivot position in every column of the matrix. Since every row contains at most one pivot position, the number of columns can be no greater than the number of rows. This means that the number of vectors in a linearly independent set can be no greater than the number of dimensions.

Proposition 2.4.3.

A linearly independent set of vectors in $\mathbb{R}^m$ can contain no more than m vectors.

This says, for instance, that any linearly independent set of vectors in $\mathbb{R}^3$ can contain no more three vectors. Once again, this makes intuitive sense. We usually imagine three independent directions, such as up/down, front/back, left/right, in our three-dimensional world. This proposition tells us that there can be no more independent directions.

The homogeneous equation

Given an $m \times n$ matrix A, we call the equation $A\mathbf{x} = \mathbf{0}$ a *homogenous* equation. The solutions to this equation reflect whether the columns of A are linearly independent or not.

Activity 2.4.3. Linear independence and homogeneous equations.

1. Explain why the homogenous equation $A\mathbf{x} = \mathbf{0}$ is consistent no matter the matrix A.
2. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

whose columns we denote by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly dependent or independent?

3. Give a description of the solution space of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
4. We know that $\mathbf{0}$ is a solution to the homogeneous equation. Find another solution that is different from $\mathbf{0}$. Use your solution to find weights c_1, c_2, and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

5. Use the expression you found in the previous part to write one of the vectors as a linear combination of the others.

For any matrix A, we know that the equation $A\mathbf{x} = \mathbf{0}$ has at least one solution, namely, the vector $\mathbf{x} = \mathbf{0}$. We call this the trivial solution to the homogeneous equation so that any other solution that exists is a $nontrivial$ solution.

If there is no nontrivial solution, then $A\mathbf{x} = \mathbf{0}$ has exactly one solution. There can be no free variables in a description of the solution space so A must have a pivot position in every column. In this case, the columns of A must be linearly independent.

If, however, there is a nontrivial solution, then there are infinitely many solutions so A must have a column without a pivot position. Hence, the columns of A are linearly dependent.

Example 2.4.4

We will make the connection between solutions to the homogeneous equation and the linear independence of the columns more explict by looking at an example. In particular, we will demonstrate how a nontrivial solution to the homogeneous equation shows that one column of A is a linear combination of the others. With the matrix A in the previous activity, the homogeneous equation has the reduced row echelon form

$$\left[\begin{array}{rrr|r} 3 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which implies that

$$\begin{aligned} x_1 \quad + 2x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}.$$

In terms of the free variable x_3, we have

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= 3x_3 \end{aligned}.$$

Any choice for a value of the free variable x_3 produces a solution so let's choose, for convenience, $x_3 = 1$. We then have $(x_1, x_2, x_3) = (-2, 3, 1)$.

Since $(-2, 3, 1)$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, this solution gives weights for a linear combination of the columns of A that create $\mathbf{0}$. That is,

$$-2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

which we rewrite as

$$\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2.$$

As this example demonstrates, there are many ways we can view the question of linear independence. We will record some of these ways in the following proposition.

Proposition 2.4.5.

For a matrix $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n\,]$, the following statements are equivalent:

- The columns of A are linearly dependent.
- One of the vectors in the set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a linear combination of the others.
- The matrix A has a column without a pivot position.
- The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
- There are weights $c_1, c_2, \ldots, c_n$, not all of which are zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}.$$

Summary

In this section, we developed the concept of linear dependence of a set of vectors. At the beginning of the section, we said that this concept addressed the second of our fundamental questions, expressed in Question 1.4.2, concerning the uniqueness of solutions to a linear system. It is worth comparing the results of this section with those of the previous one so that the parallels between them become clear.

As is usual, we will write a matrix as a collection of vectors,

$$A = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n \,].$$

Existence

In the previous section, we asked if we could write a vector $\mathbf{b}$ as a linear combination of the columns of A, which happens precisely when a solution to the equation $A\mathbf{x} = \mathbf{b}$ exists. We saw that every vector $\mathbf{b}$ could be expressed as a linear combination of the columns of A when A has a pivot position in every row. In this case, we said that the span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is $\mathbb{R}^m$. We saw that at least m vectors are needed to span $\mathbb{R}^m$.

Uniqueness

In this section, we studied the uniqueness of solutions to the equation $A\mathbf{x} = \mathbf{0}$, which is always consistent. When a nontrivial solution exists, we saw that one vector of the set $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a linear combination of the others, in which case we said that the set of vectors is linearly dependent. This happens when the matrix A has a column without a pivot position. We saw that there can be no more than m vectors in a set of linearly independent vectors in $\mathbb{R}^m$.

To summarize the specific results of this section, we saw that:

- A set of vectors is linearly dependent if one of the vectors is a linear combination of the others.
- A set of vectors is linearly independent if and only if the vectors form a matrix that has a pivot position in every column.
- A set of linearly independent vectors in $\mathbb{R}^m$ contains no more than m vectors.
- The columns of the matrix A are linearly dependent if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
- A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is linearly dependent if there are weights $c_1, c_2, \dots, c_n$, not all of which are zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}.$$

Exercises 2.4.5Exercises

? 1

Consider the set of vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}.$$

1. Explain why this set of vectors is linearly dependent.
2. Write one of the vectors as a linear combination of the others.
3. Find weights c_1, c_2, c_3, and c_4, not all of which are zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

4. Find a nontrivial solution to the homogenous equation $A\mathbf{x} = \mathbf{0}$ where $A = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \,]$.

? 2

Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

1. Are these vectors linearly independent or linearly dependent?
2. Describe the $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
3. Suppose that $\mathbf{b}$ is a vector in $\mathbb{R}^3$. Explain why we can guarantee that $\mathbf{b}$ may be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

4. Suppose that $\mathbf{b}$ is a vector in $\mathbb{R}^3$. In how many ways can $\mathbf{b}$ be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?

? 3

Answer the following questions and provide a justification for your responses.

1. If the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly dependent set, must one vector be a scalar multiple of the other?
2. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a linearly independent set of vectors. What can you say about the linear independence or dependence of a subset of these vectors?
3. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a linearly independent set of vectors that form the columns of a matrix A. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, what can you say about the linear independence or dependence of the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{b}$?

? 4

Determine if the following statements are true or false and provide a justification for your response.

1. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then one vector is a scalar multiple of one of the others.
2. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}$ are vectors in $\mathbb{R}^5$, then the set of vectors is linearly dependent.
3. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ are vectors in $\mathbb{R}^{10}$, then the set of vectors is linearly independent.
4. Suppose we have a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and that $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$. Then the set is linearly dependent.
5. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and form the columns of a matrix A. If $A\mathbf{x} = \mathbf{b}$ is consistent, then there is exactly one solution.

? 5

Suppose we have a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbb{R}^5$ that satisfy the relationship:

$$2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

and suppose that A is the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$.

1. Find a nontrivial solution to the equation $A\mathbf{x} = \mathbf{0}$.
2. Explain why the matrix A has a column without a pivot position.
3. Write one of the vectors as a linear combination of the others.
4. Explain why the set of vectors is linearly dependent.

? 6

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a set of vectors in $\mathbb{R}^{27}$ that form the columns of a matrix A.

1. Suppose that the vectors span $\mathbb{R}^{27}$. What can you say about the number of vectors n in this set?
2. Suppose instead that the vectors are linearly independent. What can you say about the number of vectors n in this set?
3. Suppose that the vectors are both linearly independent and span $\mathbb{R}^{27}$. What can you say about the number of vectors in the set?
4. Assume that the vectors are both linearly independent and span $\mathbb{R}^{27}$. Given a vector $\mathbf{b}$ in $\mathbb{R}^{27}$, what can you say about the solution space to the equation $A\mathbf{x} = \mathbf{b}$?

? 7

Given below are some descriptions of sets of vectors that form the columns of a matrix A. For each description, give a possible reduced row echelon form for A or indicate why there is no set of vectors satisfying the description by stating why the required reduced row echelon matrix cannot exist.

1. A set of 4 linearly independent vectors in $\mathbb{R}^5$.
2. A set of 4 linearly independent vectors in $\mathbb{R}^4$.
3. A set of 3 vectors that span $\mathbb{R}^4$.

4. A set of 5 linearly independent vectors in $\mathbb{R}^3$.
5. A set of 5 vectors that span $\mathbb{R}^4$.

? 8

When we explored matrix multiplication in Section 2.2, we saw that some properties that are true for real numbers are not true for matrices. This exercise will investigate that in some more depth.

1. Suppose that A and B are two matrices and that $AB=0$. If $B\neq 0$, what can you say about the linear independence of the columns of A?
2. Suppose that we have matrices A, B and C such that $AB=AC$. We have seen that we cannot generally conclude that $B=C$. If we assume additionally that A is a matrix whose columns are linearly independent, explain why $B=C$. You may wish to begin by rewriting the equation $AB=AC$ as $AB-AC=A(B-C)=0$.

? 9

Suppose that k is an unknown parameter and consider the set of vectors

$$\mathbf{v}_1=\begin{pmatrix}2\\0\\1\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}4\\-2\\-1\end{pmatrix},\mathbf{v}_1=\begin{pmatrix}0\\2\\k\end{pmatrix}.$$

1. For what values of k is the set of vectors linearly dependent?
2. For what values of k does the set of vectors span $\mathbb{R}^3$?

? 10

Given a set of linearly dependent vectors, we can eliminate some of the vectors to create a smaller, linearly independent set of vectors.

1. Suppose that $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Explain why $Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{w}\}=Span\{\mathbf{v}_1,\mathbf{v}_2\}$.
2. Consider the vectors

$$\mathbf{v}_1=\begin{pmatrix}2\\-1\\0\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}1\\2\\1\end{pmatrix},\mathbf{v}_3=\begin{pmatrix}-2\\6\\2\end{pmatrix},\mathbf{v}_4=\begin{pmatrix}7\\-1\\1\end{pmatrix}.$$

   Write one of the vectors as a linear combination of the others. Find a set of three vectors whose span is the same as $Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.
3. Are the three vectors you are left with linearly independent? If not, express one of the vectors as a linear combination of the others and find a set of two vectors whose span is the same as $Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.
4. Give a geometric description of $Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ in $\mathbb{R}^3$ as we did in Section 2.3.

2.5: Matrix transformations

The past few sections introduced us to vectors and linear combinations as a means of thinking geometrically about the solutions to a linear system. Using matrix-vector multiplication, we rewrote a linear system as a matrix equation $A\mathbf{x}=\mathbf{b}$ and used the concepts of span and linear independence to understand when solutions exist and when they are unique.

In this section, we will explore how matrix-vector multiplication defines certain types of functions, which we call *matrix transformations*, similar to those encountered in previous algebra courses. In particular, we will develop some algebraic tools for thinking about matrix transformations and look at some motivating examples. In the next section, we will see how matrix transformations describe important geometric operations and how they are used in computer animation.

Preview Activity 2.5.1.

We will begin by considering a more familiar situation; namely, the function $f(x)=x^2$, which takes a real number x as an input and produces its square x^2 as its output.

1. What is the value of $f(3)$?
2. Can we solve the equation $f(x)=4$? If so, is the solution unique?
3. Can we solve the equation $f(x)=-10$? If so, is the solution unique?
4. Sketch a graph of the function $f(x)=x^2$ in Figure 2.5.1

Figure 2.5.1. Graph the function $f(x)=x^2$ above.

- Remember that the range of a function is the set of all possible outputs. What is the range of the function f?
- We will now consider functions having the form $g(x)=mx$. Draw a graph of the function $g(x)=2x$ on the left in Figure 2.5.2.

Figure 2.5.2. Graphs of the function $g(x)=2x$ and $h(x)=-\frac{1}{3}x$.

- Draw a graph of the funcion $h(x)=-\frac{1}{3}x$ on the right of Figure 2.5.2.
- Remember that composing two functions means we use the output from one function as the input into the other. That is, $g\circ h(x)=g(h(x))$. What function results from composing $g\circ h(x)$? How is the composite function related to the two functions g and h?

Matrix transformations

In the preview activity, we considered simple linear functions, such as $g(x)=\frac{1}{2}x$ whose graph is the line shown in Figure 2.5.3. We construct a function like this by choosing a number m; when given an input x, the output $g(x)=mx$ is formed by multiplying x by m.

Figure 2.5.3. The graph of the function $g(x)=\frac{1}{2}x$.

In this section, we will consider functions defined through matrix-vector multiplication. That is, we will choose a matrix A; when given an input $\mathbf{x}$, the function $T(\mathbf{x})=A\mathbf{x}$ forms the product $A\mathbf{x}$ as its output. Such a function is called a *matrix transformation*.

Activity 2.5.2.

In this activity, we will look at some examples of matrix transformations.

1. To begin, suppose that A is the matrix

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}.$$

We define the matrix transformation $T(\mathbf{x})=A\mathbf{x}$ so that

$$T\left[\begin{pmatrix}-2\\ 3\end{pmatrix}\right]=A\begin{pmatrix}-2\\ 3\end{pmatrix}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{pmatrix}-2\\ 3\end{pmatrix}=\begin{pmatrix}-1\\ 4\end{pmatrix}.$$

The function T takes the vector $\begin{pmatrix}-2\\3\end{pmatrix}$ as an input and gives us $\begin{pmatrix}-1\\4\end{pmatrix}$ as the output.

1. What is $T\left[\begin{pmatrix}1\\-2\end{pmatrix}\right]$?
2. What is $T\left[\begin{pmatrix}1\\0\end{pmatrix}\right]$?
3. What is $T\left[\begin{pmatrix}0\\1\end{pmatrix}\right]$?
4. Is there a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \begin{pmatrix}3\\0\end{pmatrix}$?

2. Suppose that $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 3 & 3 & -2 & 1 \\ 0 & 2 & 1 & -3 \\ -2 & 1 & 4 & -4 \end{bmatrix}.$$

1. What is the dimension of the vectors $\mathbf{x}$ that are inputs for T?
2. What is the dimension of the vectors $T(\mathbf{x}) = A\mathbf{x}$ that are outputs?
3. Describe the vectors $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{0}$.

3. If A is the matrix $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]$, what is $T\left[\begin{pmatrix}0\\1\end{pmatrix}\right]$ in terms of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$?
4. Suppose that A is a 3×2 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If

$$T\left[\begin{pmatrix}1\\0\end{pmatrix}\right] = \begin{pmatrix}3\\-1\\1\end{pmatrix}, T\left[\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}2\\2\\-1\end{pmatrix},$$

what is the matrix A?

Let's discuss a few of the issues that appear in this activity. First, if A is an $m\times n$ matrix, we can form the matrix product $A\mathbf{x}$ when $\mathbf{x}$ is an n-dimensional vector in $\mathbb{R}^n$. The resulting product $A\mathbf{x}$ is an m-dimensional vector in $\mathbb{R}^m$. If $T(\mathbf{x}) = A\mathbf{x}$, we therefore write $T:\mathbb{R}^n \to \mathbb{R}^m$ meaning T takes vectors in $\mathbb{R}^n$ as inputs and produces vectors in $\mathbb{R}^m$ as outputs. For instance, if

$$A = \begin{bmatrix} 4 & 0 & -3 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix},$$

then $T:\mathbb{R}^4 \to \mathbb{R}^2$.

If we know the matrix A, then we can form the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. However, if we only know the values of the matrix transformation T, we can reconstruct the matrix A. The key is to remember that matrix-vector multiplication constructs a linear combination. For instance, if A is a $m\times 2$ matrix $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]$, then

$$T\left[\begin{pmatrix}1\\0\end{pmatrix}\right] = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]\begin{pmatrix}1\\0\end{pmatrix} = 1\mathbf{v}_1 + 0\mathbf{v}_2 = \mathbf{v}_1.$$

That is, we can find the first column of A by evaluating $T\left[\begin{pmatrix}1\\0\end{pmatrix}\right]$. Similarly, the second column of A is found by evaluating $T\left[\begin{pmatrix}0\\1\end{pmatrix}\right]$.

More generally, we will write

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

so that

$$T(\mathbf{e}_j) = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,]\,\mathbf{e}_j = \mathbf{v}_j.$$

This means that the j^{th} column of A is found by evaluating $T(\mathbf{e}_j)$. We record this fact in the following proposition.

Proposition 2.5.4.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation given by $T(\mathbf{x}) = A\mathbf{x},$ then the matrix A has columns $T(\mathbf{e}_j)$; that is,

$$A = [\,T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)\,].$$

We will look at some examples of matrix transformations in the following activity.

Activity 2.5.3.

Suppose that we work for a company that produces baked goods, including cakes, donuts, and eclairs. Our company operates two plants, Plant 1 and Plant 2. In one hour of operation,

- Plant 1 produces 10 cakes, 50 donuts, and 30 eclairs.
- Plant 2 produces 20 cakes, 30 donuts, and 30 eclairs.

1. If plant 1 operates for x_1 hours and Plant 2 for x_2 hours, how many cakes C does the company produce? How many donuts D? How many eclairs E?
2. We define a matrix transformation $T(\mathbf{x}) = \begin{pmatrix} C \\ D \\ E \end{pmatrix}$ where $\begin{pmatrix} C \\ D \\ E \end{pmatrix}$ represents the number of baked goods produced when the plants are operated for times $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. If $T(\mathbf{x}) = A\mathbf{x},$ what are the dimensions of the matrix A?
3. Find the vector $T\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ and the vector $T\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$ and use your results to write the matrix A.
4. If we operate Plant 1 for 40 hours and Plant 2 for 50 hours, how many baked goods have we produced?
5. Suppose the marketing department says we need to produce 1500 cakes, 4700 donuts, and 3300 eclairs. Is it possible to meet this order? If so, how long should the two plants operate?
6. Let's now consider the needed ingredients:
   - Each cake requires 4 units of flour and and 2 units of sugar.
   - Each donut requires 1 unit of flour and 1 unit of sugar.
   - Each eclair requires 1 units of flour and 2 units of sugar.

   Suppose we make C cakes, D donuts, and E eclairs. How many units of flour F are required? How many units of sugar S?
7. Write a matrix B that defines the matrix transformation $R\left[\begin{pmatrix} C \\ D \\ E \end{pmatrix}\right] = \begin{pmatrix} F \\ S \end{pmatrix}$.
8. If Plant 1 operates for 30 hours and Plant 2 operates for 20 hours, how many units of flour and sugar are required?
9. We can consider the matrix transformation $P(\mathbf{x}) = \begin{pmatrix} F \\ S \end{pmatrix}$ that tells us how many units of flour and sugar are required when we operate the plants for x_1 and x_2 hours. Find the matrix that defines the transformation P.

In this activity, we considered two matrix transformations and constructed a third using composition. We began with the matrix transformation T that tells us the number of baked goods produced when the plants are operated for a certain amount of time. If we write the times as $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents the situation where Plant 1 operates for one hour and Plant 2 is not operated. We are told that, in this one hour, Plant 1 produces 10 cakes, 50 donuts, and 30 eclairs. We therefore have

$$T\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 10 \\ 50 \\ 30 \end{pmatrix}.$$

Similarly,

$$T\left[\begin{pmatrix}0\\1\end{pmatrix}\right]=\begin{pmatrix}20\\30\\30\end{pmatrix},$$

which tells us that the matrix A that defines T is

$$A=\begin{bmatrix}10 & 20\\50 & 30\\30 & 30\end{bmatrix}.$$

In the same way, we use the matrix transformation $R\left[\begin{pmatrix}C\\D\\E\end{pmatrix}\right]=\begin{pmatrix}F\\S\end{pmatrix}$ to describe the ingredients required to make a certain number of cakes, donuts, and eclairs. We see that

$$R\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right]=\begin{pmatrix}4\\2\end{pmatrix},\qquad R\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\right]=\begin{pmatrix}1\\1\end{pmatrix},\qquad R\left[\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\begin{pmatrix}1\\2\end{pmatrix},$$

which means that the matrix defining R is

$$B=\begin{bmatrix}4 & 1 & 1\\2 & 1 & 2\end{bmatrix}.$$

Finally, we wish to compose these two matrix transformations. For instance, if we operate the plants for times given by the vector $\mathbf{x}$, we would like to know the required amounts of the ingredients. To determine this, notice that $T(\mathbf{x})=A\mathbf{x}$ tells us how many cakes, donuts, and eclairs we produce. The ingredients required are then given by

$$R(T(\mathbf{x}))=R(A\mathbf{x})=B(A\mathbf{x})=BA\mathbf{x}.$$

Notice that the matrix that defines the composition is given by the product of the two matrices defining the matrix transformations.

In this case, we have

$$BA=\begin{bmatrix}4 & 1 & 1\\2 & 1 & 2\end{bmatrix}\begin{bmatrix}10 & 20\\50 & 30\\30 & 30\end{bmatrix}=\begin{bmatrix}120 & 140\\130 & 130\end{bmatrix}.$$

This means that the matrix transformation that tells us the required amount of ingredients given the amount of time that the plants are operated is described by

$$P(\mathbf{x})=R\circ T(\mathbf{x})=\begin{bmatrix}120 & 140\\130 & 130\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}F\\S\end{pmatrix}.$$

For instance, if Plant 1 operates for 30 hours and Plant 2 for 20 hours, we have

$$P\left[\begin{pmatrix}30\\20\end{pmatrix}\right]=\begin{bmatrix}120 & 140\\130 & 130\end{bmatrix}\begin{pmatrix}30\\20\end{pmatrix}=\begin{pmatrix}6400\\6500\end{pmatrix}.$$

In other words, we need 6400 units of flour and 6500 units of sugar.

This activity shows that the composition of matrix transformations corresponds to the product of matrices, an important observation that we summarize in the following proposition.

Proposition 2.5.5.

If we have a matrix transformation T defined by the matrix A and a matrix transformation S defined by the matrix B, then the composition of the matrix transformations is a new matrix transformation $S\circ T$ defined by the matrix BA.

Discrete Dynamical Systems

In Section 4.4, we will give considerable attention to a specific type of matrix transformation, which is illustrated in the next activity.

Activity 2.5.4.

Suppose we run a company that has two warehouses, which we will call P and Q, and a fleet of 1000 delivery trucks. Every day, a delivery truck goes out from one of the warehouses and returns every evening to one of the warehouses. Every evening,

- 70% of the trucks that leave P return to P. The other 30% return to Q.
- 50% of the trucks that leave Q return to Q and 50% return to P.

We will use the vector $\mathbf{x} = \begin{pmatrix} P \\ Q \end{pmatrix}$ to represent the number of trucks at location P and Q in the morning. We consider the matrix transformation $T(\mathbf{x}) = \begin{pmatrix} P' \\ Q' \end{pmatrix}$ that describes the number of trucks at location P and Q in the evening.

1. Suppose that all 1000 trucks begin the day at location P and none at Q. How many trucks are at each location at the end of the day? Therefore, what is the vector $T\left[\begin{pmatrix} 1000 \\ 0 \end{pmatrix}\right]$?

   Using this result, what is $T\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$?
2. In the same way, suppose that all 1000 trucks begin the day at location Q and none at P. How many trucks are at each location at the end of the day? What is the result $T\left[\begin{pmatrix} 0 \\ 1000 \end{pmatrix}\right]$?
3. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.
4. Suppose that there are 100 trucks at P and 900 at Q at the beginning of the day. How many are there at the two locations at the end of the day?
5. Suppose that there are 550 trucks at P and 450 at Q at the end of the day. How many trucks were there at the two locations at the beginning of the day?
6. Suppose that all of the trucks are at location Q on Monday morning?
   1. How many trucks are at each location Monday evening?
   2. How many trucks are at each location Tuesday evening?
   3. How many trucks are at each location Wednesday evening?
7. Suppose that S is the matrix transformation that transforms the distribution of trucks $\mathbf{x}$ one morning into the distribution of trucks two mornings later. What is the matrix that defines the transformation S?
8. Suppose that R is the matrix transformation that transforms the distribution of trucks $\mathbf{x}$ one morning into the distribution of trucks one week later. What is the matrix that defines the transformation R?
9. What happens to the distribution of trucks after a very long time?

This is type of situation occurs frequently. We have a vector $\mathbf{x}$ that describes the state of some system; in this case, $\mathbf{x}$ describes the distribution of trucks between the two locations at a particular time. Then we have a matrix A that defines a matrix transformation with $T(\mathbf{x}) = A\mathbf{x}$ describing the state at some later time. We call $\mathbf{x}$ the *state* vector and T the *transition* function, as it describes the transition of the state vector from one time to the next.

We begin in an initial state $\mathbf{x}_0 = \left[\begin{pmatrix} 0 \\ 1000 \end{pmatrix}\right]$. The state one day later will be the vector $\mathbf{x}_1 = T(\mathbf{x}_0) = A\mathbf{x}_0$. In the example from our activity, we have

$$A = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}\left(\begin{pmatrix} 0 \\ 1000 \end{pmatrix}\right] = \left(\begin{pmatrix} 500 \\ 500 \end{pmatrix}\right].$$

We can, of course, repeat this process. The vector $\mathbf{x}_1$ describes the state after one day. After a second day, we have the state vector

$$\mathbf{x}_2 = T(\mathbf{x}_1) = A\mathbf{x}_1 = A^2\mathbf{x}_0 = \left[\begin{pmatrix} 600 \\ 400 \end{pmatrix}\right].$$

We can continue this process finding $\mathbf{x}_k$, the state after k days using $\mathbf{x}_k = A\mathbf{x}_{k-1} = A^k\mathbf{x}_0$. In this way, we see that the long-term behavior of the state vector is determined by the powers of the matrix A.

Using Sage, we can compute A^k for some very large powers of A. For instance,

$$A^{100} \approx \begin{bmatrix} 0.625 & 0.625 \\ 0.375 & 0.375 \end{bmatrix}.$$

In fact, all large powers of A look very close to this matrix. Therefore, after a very long time, the state vector is very close to

$$\begin{bmatrix} 0.625 & 0.625 \\ 0.375 & 0.375 \end{bmatrix}\left[\begin{pmatrix} 0 \\ 1000 \end{pmatrix}\right] = \left[\begin{pmatrix} 625 \\ 375 \end{pmatrix}\right].$$

This means that, eventually, 625 cars are at location P every day and 375 are at Q.

We call this sitution in which the state of a system evolves from one time to the next according to the rule $\mathbf{x}_{k+1} = A\mathbf{x}_k$ a *discrete dynamical system*. In Chapter 4, we will develop a theory that enables us to easily make long-term predictions without needing to compute large powers of the matrix.

Summary

This section introduced matrix transformations, functions that are defined by matrix-vector multiplication, such as $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A.

- If A is an $m \times n$ matrix, then $T : \mathbb{R}^n \to \mathbb{R}^m$.
- The columns of the matrix A are given by evaluating the transformation T on the vectors $\mathbf{e}_j$; that is,

$$A = [\, T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n) \,].$$

- The composition of matrix transformations corresponds to matrix multiplication.
- A discrete dynamical system consists of a state vector $\mathbf{x}$ along with a transition function $T(\mathbf{x}) = A\mathbf{x}$ that describes how the state vector evolves from one time to the next. Powers of the matrix A determine the long-term behavior of the state vector.

Exercises 2.5.4

? 1

Suppose that T is the matrix transformation defined by the matrix A and S is the matrix transformation defined by B where

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & 2 \\ -1 & 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \end{bmatrix}.$$

1. If $T : \mathbb{R}^n \to \mathbb{R}^m$, what are the values of m and n? What values of m and n are appropriate for the transformation S?
2. Evaluate the matrix transformation $T\left[\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right]$.
3. Evaluate the matrix transformation $S\left[\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right]$.
4. Evaluate the matrix transformation $S \circ T\left[\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right]$.
5. Find the matrix C that defines the matrix transformation $S \circ T$.

? 2

Determine whether the following statements are true or false and provide a justification for your response.

1. A matrix transformation $T : \mathbb{R}^4 \to \mathbb{R}^5$ is defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is a 4×5 matrix.
2. If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a matrix transformation, then there are infinitely many vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.
3. If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a matrix transformation, then it is possible that every equation $T(\mathbf{x}) = \mathbf{b}$ has a solution for every vector $\mathbf{b}$.
4. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, then the equation $T(\mathbf{x}) = \mathbf{0}$ always has a solution.
5. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation and $\mathbf{v}$ and $\mathbf{w}$ two vectors in $\mathbb{R}^n$, then the vectors $T(\mathbf{v} + t\mathbf{w})$ form a line in $\mathbb{R}^m$.

? 3

This problem concerns the identification of matrix transformations.

1. Check that the following function $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a matrix transformation by finding a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.
$$T\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right] = \begin{bmatrix} 3x_1 - x_2 + 4x_3 \\ 5x_2 - x_3 \end{bmatrix}.$$
2. Explain why
$$T\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right] = \begin{bmatrix} 3x_1^4 - x_2 + 4x_3 \\ 5x_2 - x_3 \end{bmatrix}$$
is not a matrix transformation.

? 4

Suppose that the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 5 \\ 0 & 2 & 2 \end{bmatrix}$$

defines the matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

1. Describe the vectors $\mathbf{x}$ that satisfy $T(\mathbf{x}) = \mathbf{0}$.
2. Describe the vectors $\mathbf{x}$ that satisfy $T(\mathbf{x}) = \begin{pmatrix} -8 \\ 9 \\ 2 \end{pmatrix}$.
3. Describe the vectors $\mathbf{x}$ that satisfy $T(\mathbf{x}) = \begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix}$.

? 5

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a matrix transformation with $T(\mathbf{e}_j) = \mathbf{v}_j$ where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are as shown in Figure 2.5.6.

Figure 2.5.6. The vectors $T(\mathbf{e}_j) = \mathbf{v}_j$.

1. Sketch the vector $T\left[\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right]$.

2. What is the vector $T\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\right]$?
3. Find all the vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

? 6

Suppose that a company has three plants, called Plants 1, 2, and 3, that produce milk M and yogurt Y. For every hour of operation,

- Plant 1 produces 20 units of milk and 15 units of yogurt.
- Plant 2 produces 30 units of milk and 5 units of yogurt.
- Plant 3 produces 0 units of milk and 40 units of yogurt.

1. Suppose that x_1, x_2, and x_3 record the amounts of time that the three plants are operated. Find expressions for the number of units of milk M and yogurt Y produced.
2. If we write $\mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ and $\mathbf{y}) = \begin{pmatrix}M\\Y\end{pmatrix}$, find the matrix A that defines the matrix transformation $T(\mathbf{x}) = \mathbf{y})$.
3. Furthermore, suppose that producing each unit of
   - milk requires 5 units of electricity and 8 units of labor.
   - yogurt requires 6 units of electricity and 10 units of labor.

   Write expressions for the required amounts of electricity E and labor L in terms of M and Y.
4. If we write the vector $\mathbf{z}) = \begin{pmatrix}E\\L\end{pmatrix}$ to record the required amounts of electricity and labor, find the matrix B that defines the matrix transformation $S(\mathbf{y})) = \mathbf{z})$.
5. If $\mathbf{x} = \begin{pmatrix}30\\20\\10\end{pmatrix}$ describes the amounts of time that the three plants are operated, how much milk and yogurt is produced? How much electricity and labor are required?
6. Find the matrix C that describes the matrix transformation $R(\mathbf{x}) = \mathbf{z})$ that gives the required amounts of electricity and labor when the plants are operated times given by vector $\mathbf{x}$.

? 7

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a matrix transformation and that

$$T\left[\begin{pmatrix}1\\1\end{pmatrix}\right] = \begin{pmatrix}3\\-2\end{pmatrix}, \qquad T\left[\begin{pmatrix}-1\\1\end{pmatrix}\right] = \begin{pmatrix}1\\2\end{pmatrix}.$$

1. Find the vector $T\left[\begin{pmatrix}1\\0\end{pmatrix}\right]$.
2. Find the matrix A that defines T.
3. Find the vector $T\left[\begin{pmatrix}4\\-5\end{pmatrix}\right]$.

? 8

Suppose that two species P and Q interact with one another and that we measure their populations every month. We record their populations in a state vector $\mathbf{x} = \begin{pmatrix}p\\q\end{pmatrix}$, where p and q are the populations of P and Q, respectively. We observe that there is a matrix

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.7 & 1.2 \end{bmatrix}$$

such that the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is the transition function describing how the state vector evolves from month to month. We also observe that, at the beginning of July, the populations are described by the state vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1. What will the populations be at the beginning of August?
2. What were the populations at the beginning of June?
3. What will the populations be at the beginning of December?
4. What will the populations be at the beginning of July in the following year?

? 9

Students in a school are sometimes absent due to an illness. Suppose that

- 95% of the students who attend school will attend school the next day.
- 50% of the students are absent one day will be absent the next day.

We will record the number of present students p and the number of absent students a in a state vector $\mathbf{x} = \begin{pmatrix} p \\ a \end{pmatrix}$. On Tuesday, the state vector is $\mathbf{x} = \left[\begin{pmatrix} 1700 \\ 100 \end{pmatrix}\right]$. The state vector evolves from one day to the next according to the transition function $T : \mathbb{R}^2 \to \mathbb{R}^2$.

1. Suppose we initially have 1000 students who are present and none absent. Find T\left[\left(\twovec{1000}{0}\right]\text{.}
2. Suppose we initially have 1000 students who are absent and none present. Find T\left[\left(\twovec{0}{1000}\right]\text{.}
3. Use the results of parts a and b to find the matrix A that defines the matrix transformation T.
4. If $\mathbf{x} = \left(\begin{pmatrix} 1700 \\ 100 \end{pmatrix}\right]$ on Tuesday, how are the students distributed on Wednesday?
5. How many students were present on Monday?
6. How many students are present on the following Tuesday?
7. What happens to the number of students who are present after a very long time?

2.6: The geometry of matrix transformations

Matrix transformations, which we explored in the last section, allow us to describe certain functions $T : \mathbb{R}^n \to \mathbb{R}^m$. In this section, we will demonstrate how matrix transformations provide a convenient way to describe geometric operations, such as rotations, reflections, and scalings. We will then explore how matrix transformations are used in computer animation.

Preview Activity 2.6.1.

Suppose that we wish to describe the geometric operation that reflects 2-dimensional vectors in the horiztonal axis. For instance, Figure 2.6.1 illustrates how a vector $\mathbf{x}$ is reflected into the vector $T(\mathbf{x})$.

Figure 2.6.1. A vector $\mathbf{x}$ and its reflection $T(\mathbf{x})$ in the horizontal axis.

1. If $\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, what is the vector $T(\mathbf{x})$? Sketch the vectors $\mathbf{x}$ and $T(\mathbf{x})$.
2. More generally, if $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, what is $T(\mathbf{x})$?
3. Find the vectors $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ and $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$.
4. Use your results to write the matrix A so that $T(\mathbf{x}) = A\mathbf{x}$. Then verify that $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$ agrees with what you found in part b.
5. Describe the transformation that results from composing T with itself; that is, what is the transformation $T \circ T$? Explain how matrix multiplication can be used to justify your response.

The geometry of $\mathbf{2 \times 2}$ matrix transformations

The preview activity demonstrates how the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ defines a matrix transformation that has the effect of reflecting 2-dimensional vectors in the horizontal axis. The following activity shows, more generally, that matrix transformations can perform a variety of important geometric operations.

Activity 2.6.2.

The diagram below demonstrates the effect of a matrix transformation on the plane. You may modify the matrix A defining the matrix transformation T through the sliders at the top. You may also move the red vector $\mathbf{x}$ on the left, by clicking in the head of the vector, and observe $T(\mathbf{x})$ on the right.

For the following matrices A given below, use the diagram to study the effect of the corresponding matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. For each transformation, describe the geometric effect of the transformation on the plane.

1. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.
2. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
3. The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
4. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
5. The matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
6. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
7. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
8. The matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$.

The previous activity presented some examples in which matrix transformations perform interesting geometric actions, such as rotations, scalings, and reflections. Let's turn this question around: Suppose we have a specific geometric action that we would like to perform. Can we find a matrix A that represents this action through the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$?

The linearity of matrix-vector multiplication Proposition 2.2.3 provides the key to answering this question. Remember that if A is a matrix, $\mathbf{v}$ and $\mathbf{w}$ vectors, and c a scalar, then

$$\begin{aligned} A(c\mathbf{v}) &= cA\mathbf{v} \\ A(\mathbf{v}+\mathbf{w}) &= A\mathbf{v} + A\mathbf{w}. \end{aligned}$$

This means that a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ satisfies the corresponding linearity property:

$$\begin{aligned} T(c\mathbf{v}) &= cT(\mathbf{v}) \\ T(\mathbf{v}+\mathbf{w}) &= T(\mathbf{v})+T(\mathbf{w}). \end{aligned}$$

It turns out that, if $T:\mathbb{R}^n\to\mathbb{R}^m$ satisfies these two linearity properties, then we can find a matrix A such that $T(\mathbf{x})=A\mathbf{x}$. In fact, Proposition 2.5.4 tells us how to form A; we simply write

$$A=\left[\, T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots T(\mathbf{e}_n) \,\right].$$

We will now check that $T(\mathbf{x})=A\mathbf{x}$ using the linearity of T:

$$\begin{aligned} T(\mathbf{x}) = T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) &= T(x_1\mathbf{e}_1+x_2\mathbf{e}_2+\ldots+x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1)+x_2T(\mathbf{e}_2)+\ldots+x_nT(\mathbf{e}_n) \\ &= x_1A\mathbf{e}_1+x_2A\mathbf{e}_2+\ldots+x_nA\mathbf{e}_n \\ &= A(x_1\mathbf{e}_1+x_2\mathbf{e}_2+\ldots+x_n\mathbf{e}_n) \\ &= A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= A\mathbf{x} \end{aligned}.$$

The result is the following proposition.

Proposition 2.6.2.

The function $T:\mathbb{R}^n\to\mathbb{R}^m$ is a matrix transformation where $T(\mathbf{x})=A\mathbf{x}$ for some $m\times n$ matrix A if and only if

$$\begin{aligned} T(c\mathbf{v}) &= cT(\mathbf{v}) \\ T(\mathbf{v}+\mathbf{w}) &= T(\mathbf{v})+T(\mathbf{w}). \end{aligned}.$$

In this case, A is the matrix whose columns are $T(\mathbf{e}_j)$; that is,

$$A=\left[\, T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n) \,\right].$$

We will put this proposition to use in the following example by finding the matrix whose matrix transformation performs a specific geometric operation.

Example 2.6.3

In this example, we will find the matrix defining a matrix transformation that performs a 45° counterclockwise rotation.

We first need to know that this geometric operation can be represented by a matrix transformation. To begin, we will define the function $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T(\mathbf{x})$ is obtained by rotating $\mathbf{x}$ counterclockwise by 45°, as shown in Figure 2.6.4.

Figure 2.6.4. The function T rotates a vector counterclockwise by 45°.

We need to check that T is a matrix transformation; by Proposition 2.6.2, this means that we should make sure that

$$\begin{aligned} T(c\mathbf{v}) &= cT(\mathbf{v}) \\ T(\mathbf{v}+\mathbf{w}) &= T(\mathbf{v})+T(\mathbf{w}). \end{aligned}$$

The next two figures illustrate these properties. For instance, Figure 2.6.5 shows that relationship between $T(\mathbf{v})$ and $T(c\mathbf{v})$ when c is a scalar. We easily see that $T(c\mathbf{v})$ is a scalar multiple of $T(\mathbf{v})$ and hence that $T(c\mathbf{v}) = cT(\mathbf{v})$.

Figure 2.6.5. We see that the vector $T(c\mathbf{v})$ is a scalar multiple to $T(\mathbf{v})$ so that $T(c\mathbf{v}) = cT(\mathbf{v})$.

Similarly, Figure 2.6.6 shows the relationship between $T(\mathbf{v}+\mathbf{w})$, $T(\mathbf{v})$, and $T(\mathbf{w})$. In this way, we see that $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v})+T(\mathbf{w})$.

Figure 2.6.6. We see that the vector $T(\mathbf{v}+\mathbf{w})$ is the sum of $T(\mathbf{v})$ and $T(\mathbf{w})$ so that $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v})+T(\mathbf{w})$.

This shows that the function T, which rotates vectors by 45° is a matrix transformation. We may therefore write it as $T(\mathbf{x}) = A\mathbf{x}$ where A is the 2×2 matrix $A = [\,T(\mathbf{e}_1) \quad T(\mathbf{e}_2)\,]$. The columns of this matrix, $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, are shown in Figure 2.6.7.

Figure 2.6.7. The effect of T on $\mathbf{e}_1$ and $\mathbf{e}_2$.

To find the components of these vectors, notice that they form an isosceles right triangle, as shown in Figure 2.6.8. Since the length of $\mathbf{e}_1$ is 1, the length of $T(\mathbf{e}_1)$, the hypotenuse of the triangle, is 1.

Figure 2.6.8. The vector $T(\mathbf{e}_1)$ forms a right isosceles triangle whose hypotenuse has length 1.

This leads to

$$T(\mathbf{e}_1) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, T(\mathbf{e}_2) = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Hence, the matrix A is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

You may wish to check this using the interactive diagram in the previous activity using the approximation $1/\sqrt{2} \approx 0.7$.

In this example, we found that the desired geometric operation, a rotation in the plane, was in fact a matrix transformation T by checking that

$$\begin{aligned} T(c\mathbf{v}) &= cT(\mathbf{v}) \\ T(\mathbf{v}+\mathbf{w}) &= T(\mathbf{v})+T(\mathbf{w}). \end{aligned}.$$

In general, the same kind of thinking applies to show that rotations, reflections, and scalings are matrix transformations so we will not bother with that step in the future.

Activity 2.6.3.

In this activity, we seek to describe various matrix transformations by finding the matrix that gives the desired transformation. All of the transformations that we study here have the form $T : \mathbb{R}^2 \to \mathbb{R}^2$.

1. Find the matrix of the transformation that has no effect on vectors; that is, $T(\mathbf{x}) = \mathbf{x}$. We call this matrix the *identity* and denote it by I.
2. Find the matrix of the transformation that reflects vectors in $\mathbb{R}^2$ over the line $y = x$.
3. What is the result of composing the reflection you found in the previous part with itself; that is, what is the effect of reflecting in the line $y = x$ and then reflecting in this line again. Provide a geometric explanation for your result as well as an algebraic one obtained by multiplying matrices.
4. Find the matrix that rotates vectors counterclockwise in the plane by 90°.
5. Compare the result of rotating by 90° and then reflecting in the line $y = x$ to the result of first reflecting in $y = x$ and then rotating 90°.

6. Find the matrix that results from composing a 90° rotation with itself. Explain the geometric meaning of this operation.
7. Find the matrix that results from composing a 90° rotation with itself four times; that is, if T is the matrix transformation that rotates vectors by 90°, find the matrix for $T\circ T\circ T\circ T$. Explain why your result makes sense geometrically.
8. Explain why the matrix that rotates vectors counterclockwise by an angle θ is

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

In the first part of this activity, we encountered the *identity* matrix, which, as an $n\times n$ matrix, has the form

$$I=\begin{bmatrix}1 & 0 & \ldots & 0\\ 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & 0\\ 0 & 0 & \ldots & 1\end{bmatrix}=\left[\begin{array}{cccc}\mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n\end{array}\right].$$

The matrix transformation $T(\mathbf{x})=I\mathbf{x}$ leaves vectors unchanged; that is, $T(\mathbf{x})=\mathbf{x}$ so that $I\mathbf{x}=\mathbf{x}.$ Notice that the columns of I are simply the vectors $\mathbf{e}_j$.

Matrix transformations and computer animation

Linear algebra plays a significant role in computer animation. We will now illustrate how matrix transformations and some of the ideas we have developed in this section are used by computer animators to create the illusion of motion in their characters.

Figure 2.6.9 shows a test character used by Pixar animators. On the left is the original definition of the character; on the right, we see that the character has been moved into a different pose. To make it appear that the character is moving, animators create a sequence of frames in which the character's pose is modified slightly from one frame to the next. Matrix transformations play an important role in doing this.

Figure 2.6.9. Computer animators define a character and create motion by drawing it in a sequence of poses. © Disney/Pixar

For instance, Figure 2.6.10 shows the character Remy from Pixar's *Ratatouille*. Clearly, a lot goes into transforming the model on the left into the engaging character on the right, such as the addition of fur and eyes. We will focus only on the motion of the character.

Figure 2.6.10. Remy from the Pixar movie *Ratatouille*. © Disney/Pixar.

Of course, realistic characters will be drawn in three-dimensions. To keep things a little more simple, however, we will look at this two-dimensional character and devise matrix transformations that move them into different poses.

Of course, the first thing we may wish to do is simply move them to a different position in the plane, such as that shown in Figure 2.6.11. Motions like this are called *translations*.

Figure 2.6.11. Translating our character to a new position in the plane.

This presents a problem because a matrix transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ has the property that $T(\mathbf{0})=\mathbf{0}.$ This means that a matrix transformation cannot move the origin of the coordinate plane. To address this restriction, animators use *homogeneous coordinates*, which are formed by placing the two-dimensional coordinate plane inside $\mathbb{R}^3$ as the plane $z=1.$ This is shown in Figure 2.6.12.

Figure 2.6.12. Include the plane in $\mathbb{R}^3$ as the plane $z=1$ so that we can translate the character.

Therefore, rather than describing points in the plane as vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, we describe them as three-dimensional vectors $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$. As we see in the next activity, this allows us to translate our character in the plane.

Activity 2.6.4.

In this activity, we will use homogeneous coordinates and matrix transformations to move our character into a variety of poses.

1. Since we regard our character as living in $\mathbb{R}^3$, we will consider matrix transformations defined by matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}.$$

Verify that such a matrix transformation transforms points in the plane $z = 1$ into other points in this plane; that is, verify that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}.$$

Express the coordinates of the resulting point x' and y' in terms of the coordinates of the original point x and y.

The diagram below allows you to create matrix transformations of this form to move our character into different poses. You may use it to help address the following questions.

Compose Reset

2. Find the matrix transformation that translates our character to a new position in the plane, as shown in Figure 2.6.13

Figure 2.6.13. Translating to a new position.

3. As originally drawn, our character is waving with one of their hands. In one of the movie's scenes, we would like her to wave with their other hand, as shown in Figure 2.6.14. Find the matrix transformation that moves them into this pose.

Figure 2.6.14. Waving with the other hand.

4. Later, our chracter performs a cartwheel by moving through the sequence of poses shown in Figure 2.6.15. Find the matrix transformations that create these poses.

Figure 2.6.15. Performing a cartwheel.

5. Next, we would like to find the transformations that zoom in on our character's face, as shown in Figure 2.6.16. To do this, you should think about composing matrix transformations. This can be accomplished in the diagram by using the *Compose* button, which makes the current pose, displayed on the right, the new beginning pose, displayed on the left. What is the matrix transformation that moves the character from the original pose, shown in the upper left, to the final pose, shown in the lower right?

Figure 2.6.16. Zooming in on our characters' face.

6. We would also like to create our character's shadow, shown in the sequence of poses in Figure 2.6.17. Find the sequence of matrix transformations that achieves this. In particular, find the matrix transformation that take our character from their original pose to their shadow in the lower right.

Figure 2.6.17. Casting a shadow.

7. Write a final scene to the movie and describe how to construct a sequence of matrix transformations that create your scene.

Summary

This section explored how geometric operations, such as rotations, reflections, and scalings, are performed by matrix transformations.

- A matrix of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ represents a horizontal scaling by a factor a and a vertical scaling by b.
- A matrix of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ defines a rotation by an angle θ.
- Composing geometric operations corresponds to matrix multiplication.
- Computer animators use matrix transformations to create the illusion of motion. Homogeneous coordinates are used so that translations can be realized as matrix transformations.

Exercises 2.6.4Exercises

? 1

For each of the following geometric operations in the plane, find a 2×2 matrix that defines the matrix transformation performing the operation.

1. Rotates vectors by 180°.
2. Reflects vectors in the vertical axis.

3. Reflects vectors in the line $y = -x$.
4. Rotates vectors counterclockwise by 60°.
5. First rotates vectors counterclockwise by 60° and then reflects in the line $y = x$.

? 2

This exercise investigates the composition of reflections in the plane.

1. Find the result of first reflecting in the line $y = 0$ and then $y = x$. What familiar operation is the cumulative effect of this composition?
2. What happens if you compose the operations in the opposite order; that is, what happens if you first reflect in $y = x$ and then $y = 0$? What familiar operation results?
3. What familiar geometric operation results if you first reflect in the line $y = x$ and then $y = -x$?
4. What familiar geometric operation results if you first rotate by 90° and then reflect in the line $y = x$?

It is a general fact that the composition of two reflections results in a rotation through twice the angle from the first line of reflection to the second. We will investigate this more generally in Exercise 2.6.4.8

? 3

Shown below in Figure 2.6.18 are the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in $\mathbb{R}^3$.

Figure 2.6.18. The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in $\mathbb{R}^3$.

1. Imagine that the thumb of your right hand points in the direction of $\mathbf{e}_1$. A positive rotation about the x axis corresponds to a rotation in the direction in which your fingers point. Find the matrix defining the matrix transformation T that rotates vectors by 90° around the x-axis.
2. In the same way, find the matrix that rotates vectors by 90° around the y-axis.
3. Find the matrix that rotates vectors by 90° around the z-axis.
4. What is the cumulative effect of rotating by 90° about the x-axis, followed by a 90° rotation about the y-axis, followed by a -90° rotation about the x-axis.

? 4

We have seen how a matrix transformation can perform a geometric operation; now we would like to find a matrix transformation that undoes that operation.

1. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the matrix transformation that rotates vectors by 90°. Find a matrix transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ that undoes the rotation; that is, S takes $T(\mathbf{x})$ back into $\mathbf{x}$ so that $S \circ T(\mathbf{x}) = \mathbf{x}$. Think geometrically about what the transformation S should be and then verify it algebraically.

   We say that S is the *inverse* of T and we will write it as T^{-1}.
2. Verify algebraically that the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ across the line $y = x$ is its own inverse; that is, $R^{-1} = R$.
3. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

   is called a *shear*. Find the inverse of T.
4. Describe the geometric effect of the matrix transformation defined by

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$

   and then find its inverse.

? 5

We have seen that the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

performs a rotation through an angle θ about the origin. Suppose instead that we would like to rotate by 90° about the point $(1,2)$. Using homogeneous coordinates, we will develop a matrix that performs this operation.

Our strategy is to

- begin with a vector whose tail is at the point $(1,2)$,
- translate the vector so that its tail is at the origin,
- rotate by 90°, and
- translate the vector so that its tail is back at $(1,2)$.

This is shown in Figure 2.6.19.

Figure 2.6.19. A sequence of matrix transformations that, when read right to left and top to bottom, rotate a vector about the point $(1,2)$.

Remember that, when working with homogeneous coordinates, we consider matrices of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}.$$

1. The first operation is a translation by $(-1,-2)$. Find the matrix that performs this translation.
2. The second operation is a 90° rotation about the origin. Find the matrix that performs this rotation.
3. The third operation is a translation by $(1,2)$. Find the matrix that performs this translation.
4. Use these matrices to find the matrix that performs a 90° rotation about $(1,2)$.
5. Use your matrix to determine where the point $(-10,5)$ ends up if rotated by 90° about the $(1,2)$.

? 6

This exercise concerns matrix transformations called *projections*.

1. Consider the matrix transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ that assigns to a vector $\mathbf{x}$ the closest vector on horizontal axis as illustrated in Figure 2.6.20. This transformation is called the projection onto the horizontal axis. You may imagine $T(\mathbf{x})$ as the shadow cast by $\mathbf{x}$ from a flashlight far up on the positive y-axis.

   Figure 2.6.20. Projection onto the x-axis.

   Find the matrix that defines this matrix transformation T.
2. Find the matrix that defines projection on the vertical axis.
3. What is the result of composing the projection onto the horizontal axis with the projection onto the vertical axis?
4. Find the matrix that defines projection onto the line $y=x$.

? 7

This exericse concerns the matrix transformations defined by matrices of the form

$$A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Let's begin by looking at two special types of these matrices.

1. First, consider the matrix where $a=2$ and $b=0$ so that

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Describe the geometric effect of this matrix. More generally, suppose we have

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix},$$

where r is a positive number. What is the geometric effort of A on vectors in the plane?

2. Suppose now that $a = 0$ and $b = 1$ so that

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

What is the geometric effect of A on vectors in the plane? More generally, suppose we have

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

What is the geometric effect of A on vectors in the plane?

3. In general, the composition of matrix transformation depends on the order in which we compose them. For these transformations, however, it is not the case. Check this by verifying that

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

4. Let's now look at the general case where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

We will draw the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in the plane and express it using polar coordinates r and θ as shown in Figure 2.6.21.

Figure 2.6.21. A vector may be expressed in polar coordinates.

We then have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

Show that the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

5. Using this description, describe the geometric effect on vectors in the plane of the matrix transformation defined by

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

6. Suppose we have a matrix transformation T defined by a matrix A and another transformation S defined by B where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}.$$

Describe the geometric effect of the composition $S \circ T$ in terms of the a, b, c, and d.

The matrices of this form give a model for the complex numbers and will play an important role in Section 4.4.

? 8

We saw earlier that the rotation in the plane through an angle θ is given by the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We would like to find a similar expression for the matrix that represents the reflection in L_θ, the line passing through the origin and making an angle of θ with the positive x-axis, as shown in Figure 2.6.22.

Figure 2.6.22. The reflection in the line L_θ.

1. To do this, notice that this reflection can be obtained by composing three separate transformations as shown in Figure 2.6.23. Beginning with the vector $\mathbf{x}$, we apply the transformation R to rotate by $-\theta$ and obtain $R(\mathbf{x})$. Next, we apply S, a reflection in the horizontal axis, followed by T, a rotation by θ. We see that $T(S(R(\mathbf{x})))$ is the same as the reflection of $\mathbf{x}$ in the original line L_θ.

   Figure 2.6.23. Reflection in the line L_θ as a composition of three transformations.

   Using this decomposition, show that the reflection in the line L_θ is described by the matrix

   $$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

   You will need to remember the trigonometric identities:

   $$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}.$$

2. Now that we have a matrix that describes the reflection in the line L_θ, show that the composition of the reflection in the horizontal axis followed by the reflection in L_θ is a counterclockwise rotation by an angle 2θ. We saw some examples of this earlier in Exercise 2.6.4.2.