Matrix Algebra for Engineers

Lecture Notes for coursera

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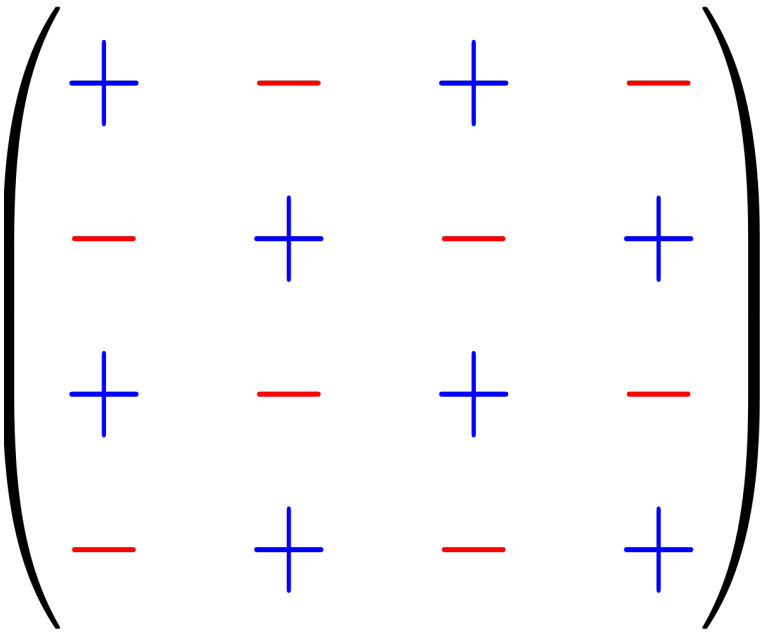

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Preface

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These are my lecture notes for my online Coursera course, Matrix Algebra for Engineers. I have divided these notes into chapters called Lectures, with each Lecture corresponding to a video on Coursera. I have also uploaded all my Coursera videos to YouTube, and links are placed at the top of each Lecture.

There are problems at the end of each lecture chapter and I have tried to choose problems that exemplify the main idea of the lecture. Students taking a formal university course in matrix or linear algebra will usually be assigned many more additional problems, but here I follow the philosophy that less is more. I give enough problems for students to solidify their understanding of the material, but not too many problems that students feel overwhelmed and drop out. I do encourage students to attempt the given problems, but if they get stuck, full solutions can be found in the Appendix.

There are also additional problems at the end of coherent sections that are given as practice quizzes on the Coursera platform. Again, students should attempt these quizzes on the platform, but if a student has trouble obtaining a correct answer, full solutions are also found in the Appendix.

The mathematics in this matrix algebra course is at the level of an advanced high school student, but typically students would take this course after completing a university-level single variable calculus course. There are no derivatives and integrals in this course, but student's are expected to have a certain level of mathematical maturity. Nevertheless, anyone who wants to learn the basics of matrix algebra is welcome to join.

Jeffrey R. Chasnov
Hong Kong
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Contents

I Matrices 1

1 Definition of a matrix 2

2 Addition and multiplication of matrices 4

3 Special matrices 6

Practice quiz: Matrix definitions 8

4 Transpose matrix 9

5 Inner and outer products 11

6 Inverse matrix 13

Practice quiz: Transpose and inverse 15

7 Orthogonal matrices 16

8 Rotation matrices 18

9 Permutation matrices 20

Practice quiz: Orthogonal matrices 22

II Systems of Linear Equations 24

10 Gaussian elimination 25

11 Reduced row echelon form 28

12 Computing inverses 30

Practice quiz: Gaussian elimination 32

13 Elementary matrices 34

14 LU decomposition 36

15 Solving (LU)x = b 38

Practice quiz: LU decomposition 40

III Vector Spaces 42

16 Vector spaces 43

17 Linear independence 45

18 Span, basis and dimension 47

Practice quiz: Vector space definitions 49

19 Gram-Schmidt process 51

20 Gram-Schmidt process (example) 53

Practice quiz: Gram-Schmidt process 55

21 Null space 57

22 Application of the null space 60

23 Column space 62

24 Row space, left null space and rank 64

Practice quiz: Fundamental subspaces 66

25 Orthogonal projections 68

26 The least-squares problem 70

27 Solution of the least-squares problem 72

Practice quiz: Orthogonal projections 75

IV Eigenvalues and Eigenvectors 77

28 Two-by-two and three-by-three determinants 78

29 Laplace expansion 80

30 Leibniz formula 83

31 Properties of a determinant 85

Practice quiz: Determinants 88

32 The eigenvalue problem 89

33 Finding eigenvalues and eigenvectors (Part A) 91

34 Finding eigenvalues and eigenvectors (Part B) 93

Practice quiz: The eigenvalue problem 95

35 Matrix diagonalization 97

36 Matrix diagonalization (example) 99

37 Powers of a matrix 101

38 Powers of a matrix (example) 103

Practice quiz: Matrix diagonalization 105

Problem and practice quiz solutions 106

Module I

Matrices

In this module's lectures, we learn about matrices. Matrices are rectangular arrays of numbers or other mathematical objects and are fundamental to engineering mathematics. We will define matrices and how to add and multiply them, discuss some special matrices such as the identity and zero matrix, learn about transposes and inverses, and define orthogonal and permutation matrices.

Lecture 1 | Definition of a matrix

View this lecture on YouTube

An m-by-n matrix is a rectangular array of numbers (or other mathematical objects) with m rows and n columns. For example, a two-by-two matrix A, with two rows and two columns, looks like

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The first row has elements a and b, the second row has elements c and d. The first column has elements a and c; the second column has elements b and d. As further examples, two-by-three and three-by-two matrices look like

$$\mathrm{B} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

Of special importance are column matrices and row matrices. These matrices are also called vectors. The column vector is in general n-by-one and the row vector is one-by-n. For example, when $n = 3$, we would write a column vector as

$$\mathrm{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and a row vector as

$$\mathrm{y} = \begin{pmatrix} a & b & c \end{pmatrix}.$$

A useful notation for writing a general m-by-n matrix A is

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Here, the matrix element of A in the ith row and the jth column is denoted as a_{ij}.

Problems for Lecture 1

1. The diagonal of a matrix A are the entries a_{ij} where $i = j$.

a) Write down the three-by-three matrix with ones on the diagonal and zeros elsewhere.

b) Write down the three-by-four matrix with ones on the diagonal and zeros elsewhere.

c) Write down the four-by-three matrix with ones on the diagonal and zeros elsewhere.

Solutions to the Problems

Lecture 2 | Addition and multiplication of matrices

View this lecture on YouTube

Matrices can be added only if they have the same dimension. Addition proceeds element by element. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

Matrices can also be multiplied by a scalar. The rule is to just multiply every element of the matrix. For example,

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

Matrices (other than the scalar) can be multiplied only if the number of columns of the left matrix equals the number of rows of the right matrix. In other words, an m-by-n matrix on the left can only be multiplied by an n-by-k matrix on the right. The resulting matrix will be m-by-k. Evidently, matrix multiplication is generally not commutative. We illustrate multiplication using two 2-by-2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}, \qquad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}.$$

First, the first row of the left matrix is multiplied against and summed with the first column of the right matrix to obtain the element in the first row and first column of the product matrix. Second, the first row is multiplied against and summed with the second column. Third, the second row is multiplied against and summed with the first column. And fourth, the second row is multiplied against and summed with the second column.

In general, an element in the resulting product matrix, say in row i and column j, is obtained by multiplying and summing the elements in row i of the left matrix with the elements in column j of the right matrix. We can formally write matrix multiplication in terms of the matrix elements. Let A be an m-by-n matrix with matrix elements a_{ij} and let B be an n-by-p matrix with matrix elements b_{ij}. Then $C = \mathrm{AB}$ is an m-by-p matrix, and its ij matrix element can be written as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Notice that the second index of a and the first index of b are summed over.

Problems for Lecture 2

1. Define the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Compute if defined: $B - 2A$, $3C - E$, AC, CD, CB.

2. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Compute AD and DA.

4. Prove the associative law for matrix multiplication. That is, let A be an m-by-n matrix, B an n-by-p matrix, and C a p-by-q matrix. Then prove that $A(BC) = (AB)C$.

Solutions to the Problems

Lecture 3 | Special matrices

View this lecture on YouTube

The zero matrix, denoted by 0, can be any size and is a matrix consisting of all zero elements. Multiplication by a zero matrix results in a zero matrix. The identity matrix, denoted by I, is a square matrix (number of rows equals number of columns) with ones down the main diagonal. If A and I are the same sized square matrices, then

$$\mathrm{AI} = \mathrm{IA} = \mathrm{A},$$

and multiplication by the identity matrix leaves the matrix unchanged. The zero and identity matrices play the role of the numbers zero and one in matrix multiplication. For example, the two-by-two zero and identity matrices are given by

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A diagonal matrix has its only nonzero elements on the diagonal. For example, a two-by-two diagonal matrix is given by

$$\mathrm{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Usually, diagonal matrices refer to square matrices, but they can also be rectangular.

A band (or banded) matrix has nonzero elements only on diagonal bands. For example, a three-by-three band matrix with nonzero diagonals one above and one below a nonzero main diagonal (called a tridiagonal matrix) is given by

$$\mathrm{B} = \begin{pmatrix} d_1 & a_1 & 0 \\ b_1 & d_2 & a_2 \\ 0 & b_2 & d_3 \end{pmatrix}.$$

An upper or lower triangular matrix is a square matrix that has zero elements below or above the diagonal. For example, three-by-three upper and lower triangular matrices are given by

$$\mathrm{U} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \quad \mathrm{L} = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}.$$

Problems for Lecture 3

1. Let

$$A = \begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix}.$$

Construct a two-by-two matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

2. Verify that

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix}.$$

Prove in general that the product of two diagonal matrices is a diagonal matrix, with elements given by the product of the diagonal elements.

3. Verify that

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 + a_2 b_3 \\ 0 & a_3 b_3 \end{pmatrix}.$$

Prove in general that the product of two upper triangular matrices is an upper triangular matrix, with the diagonal elements of the product given by the product of the diagonal elements.

Solutions to the Problems

Practice Quiz | Matrix definitions

1. Identify the two-by-two matrix with matrix elements $a_{ij} = i - j$.

a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

2. The matrix product $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ is equal to

a) $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

b) $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

c) $\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$

d) $\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$

3. Let A and B be n-by-n matrices with $(\mathrm{AB})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. If A and B are upper triangular matrices, then $a_{ik} = 0$ or $b_{kj} = 0$ when

A. $k < i$ B. $k > i$ C. $k < j$ D. $k > j$

a) A and C only

b) A and D only

c) B and C only

d) B and D only

Solutions to the Practice quiz

Lecture 4 | Transpose matrix

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The transpose of a matrix A, denoted by A^T and spoken as A-transpose, switches the rows and columns of A. That is,

$$\text{if } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \text{then } A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

In other words, we write

$$a_{ij}^T = a_{ji}.$$

Evidently, if A is m-by-n then A^T is n-by-m. As a simple example, view the following transpose pair:

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}^T = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

The following are useful and easy to prove facts:

$$\left(A^T\right)^T = A, \text{ and } (A+B)^T = A^T + B^T.$$

A less obvious fact is that the transpose of the product of matrices is equal to the product of the transposes with the order of multiplication reversed, i.e.,

$$(AB)^T = B^T A^T.$$

If A is a square matrix, and $A^T = A$, then we say that A is *symmetric*. If $A^T = -A$, then we say that A is *skew symmetric*. For example, three-by-three symmetric and skew symmetric matrices look like

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \qquad \begin{pmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{pmatrix}.$$

Notice that the diagonal elements of a skew-symmetric matrix must be zero.

Problems for Lecture 4

1. Prove that $(AB)^T = B^T A^T$.

2. Show using the transpose operator that any square matrix A can be written as the sum of a symmetric and a skew-symmetric matrix.

3. Prove that $A^T A$ is symmetric.

Solutions to the Problems

Lecture 5 | Inner and outer products

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The *inner product* (or dot product or scalar product) between two vectors is obtained from the matrix product of a row vector times a column vector. A row vector can be obtained from a column vector by the transpose operator. With the 3-by-1 column vectors u and v, their inner product is given by

$$\mathrm{u}^{\mathrm{T}}\mathrm{v} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1v_1 + u_2v_2 + u_3v_3.$$

If the inner product between two nonzero vectors is zero, we say that the vectors are *orthogonal*. The *norm* of a vector is defined by

$$||\mathrm{u}|| = \left(\mathrm{u}^{\mathrm{T}}\mathrm{u}\right)^{1/2} = \left(u_1^2 + u_2^2 + u_3^2\right)^{1/2}.$$

If the norm of a vector is equal to one, we say that the vector is *normalized*. If a set of vectors are mutually orthogonal and normalized, we say that these vectors are *orthonormal*.

An *outer product* is also defined, and is used in some applications. The outer product between u and v is given by

$$\mathrm{u}\mathrm{v}^{\mathrm{T}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}.$$

Notice that every column is a multiple of the single vector u, and every row is a multiple of the single vector v^{T}.

Problems for Lecture 5

1. Let A be a rectangular matrix given by $A = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$. Compute $A^T A$ and show that it is a symmetric square matrix. Observe that the sum of the diagonal elements of $A^T A$ is the sum of the squares of all the elements of A.

2. The trace of a square matrix B, denoted as Tr B, is the sum of the diagonal elements of B. Prove that $\mathrm{Tr}(A^T A)$ is the sum of the squares of all the elements of A.

Solutions to the Problems

Lecture 6 | Inverse matrix

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Square matrices may have inverses. When a matrix A has an inverse, we say it is invertible and denote its inverse by A^{-1}. The inverse matrix satisfies

$$AA^{-1} = A^{-1}A = I.$$

If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. Furthermore, if A is invertible then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

It is illuminating to derive the inverse of a general 2-by-2 matrix. Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and try to solve for x_1, y_1, x_2 and y_2 in terms of a, b, c, and d. There are two inhomogeneous and two homogeneous linear equations:

$$\begin{aligned} ax_1 + by_1 &= 1, \qquad & cx_1 + dy_1 &= 0, \\ cx_2 + dy_2 &= 1, \qquad & ax_2 + by_2 &= 0. \end{aligned}$$

To solve, we can eliminate y_1 and y_2 using the two homogeneous equations, and find x_1 and x_2 using the two inhomogeneous equations. The solution for the inverse matrix is found to be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The term $ad - bc$ is just the definition of the determinant of the two-by-two matrix:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

The determinant of a two-by-two matrix is the product of the diagonals minus the product of the off-diagonals. Evidently, a two-by-two matrix A is invertible only if $\det A \neq 0$. Notice that the inverse of a two-by-two matrix, in words, is found by switching the diagonal elements of the matrix, negating the off-diagonal elements, and dividing by the determinant.

Later, we will show that an n-by-n matrix is invertible if and only if its determinant is nonzero. This will require a more general definition of the determinant.

Problems for Lecture 6

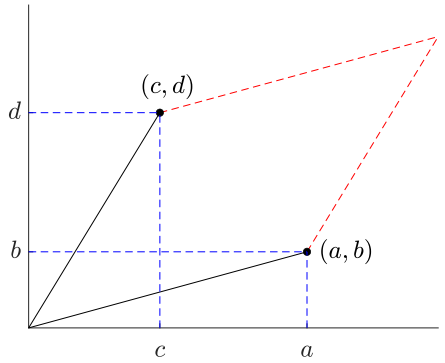
1. Find the inverses of the matrices $\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 4 \\ 3 & 3 \end{pmatrix}$.

2. Prove that if A and B are same-sized invertible matrices , then $(AB)^{-1} = B^{-1}A^{-1}$.

3. Prove that if A is invertible then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

4. Prove that if a matrix is invertible, then its inverse is unique.

5. Consider the parallelogram constructed by the two lines drawn from the origin to the points (a, b) and (c, d), as drawn in the figure.



Show that the area of the parallelogram is given by the absolute value of the determinant

$$\text{Area} = \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|.$$

Solutions to the Problems

Practice Quiz | Transpose and inverse

1. $(ABC)^T$ is equal to

a) $A^TB^TC^T$

b) $A^TC^TB^T$

c) $C^TA^TB^T$

d) $C^TB^TA^T$

2. Suppose A is a square matrix. Which matrix is not symmetric?

a) $A + A^T$

b) AA^T

c) $A - A^T$

d) A^TA

3. Which matrix is the inverse of $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$?

a) $\frac{1}{2}\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

b) $\frac{1}{2}\begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix}$

c) $\frac{1}{2}\begin{pmatrix} 2 & 2 \\ -1 & -2 \end{pmatrix}$

d) $\frac{1}{2}\begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix}$

Solutions to the Practice quiz

Lecture 7 | Orthogonal matrices

View this lecture on YouTube

A square matrix Q with real entries that satisfies

$$Q^{-1} = Q^T$$

is called an *orthogonal matrix*. Another way to write this definition is

$$QQ^T = I \qquad \text{and} \qquad Q^TQ = I.$$

We can more easily understand orthogonal matrices by examining a general two-by-two example. Let Q be the orthogonal matrix given by

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \end{pmatrix},$$

where q_1 and q_2 are the two-by-one column vectors of the matrix Q. Then

$$Q^TQ = \begin{pmatrix} q_1^T \\ q_2^T \end{pmatrix} \begin{pmatrix} q_1 & q_2 \end{pmatrix} = \begin{pmatrix} q_1^Tq_1 & q_1^Tq_2 \\ q_2^Tq_1 & q_2^Tq_2 \end{pmatrix}.$$

If Q is orthogonal, then $Q^TQ = I$ and

$$q_1^Tq_1 = q_2^Tq_2 = 1 \quad \text{and} \quad q_1^Tq_2 = q_2^Tq_1 = 0.$$

That is, the columns of Q form an orthonormal set of vectors. The same argument can also be made for the rows of Q.

Therefore, an equivalent definition of an orthogonal matrix is a square matrix with real entries whose columns (and also rows) form a set of orthonormal vectors.

There is a third equivalent definition of an orthogonal matrix. Let Q be an n-by-n orthogonal matrix, and let x be an n-by-one column vector. Then the length squared of the vector Qx is given by

$$||Qx||^2 = (Qx)^T (Qx) = x^TQ^TQx = x^TIx = x^Tx = ||x||^2.$$

The length of Qx is therefore equal to the length of x, and we say that an orthogonal matrix is a matrix that preserves lengths. In the next lecture, an example of an orthogonal matrix will be the matrix that rotates a two-dimensional vector in the plane.

Problems for Lecture 7

1. Show that the product of two orthogonal matrices is orthogonal.

2. Show that the n-by-n identity matrix is orthogonal.

Solutions to the Problems

Lecture 8 | Rotation matrices

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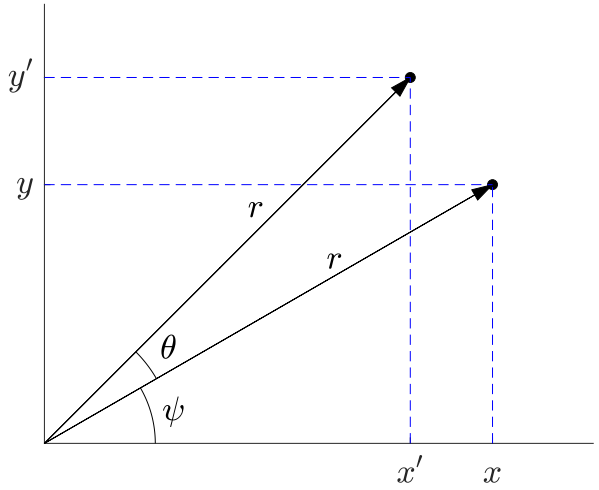

Rotating a vector in the x-y plane.

A matrix that rotates a vector in space doesn't change the vector's length and so should be an orthogonal matrix. Consider the two-by-two rotation matrix that rotates a vector counterclockwise through an angle θ in the x-y plane, shown above. Trigonometry and the addition formula for cosine and sine results in

$$\begin{aligned} x' &= r\cos(\theta+\psi) \\ &= r(\cos\theta\cos\psi - \sin\theta\sin\psi) \\ &= x\cos\theta - y\sin\theta \end{aligned} \qquad \begin{aligned} y' &= r\sin(\theta+\psi) \\ &= r(\sin\theta\cos\psi + \cos\theta\sin\psi) \\ &= x\sin\theta + y\cos\theta. \end{aligned}$$

Writing the equations for x' and y' in matrix form, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The above two-by-two matrix is a rotation matrix and we will denote it by R_θ. Observe that the rows and columns of R_θ are orthonormal and that the inverse of R_θ is just its transpose. The inverse of R_θ rotates a vector by $-\theta$.

Problems for Lecture 8

1. Let $\mathrm{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Show that $\mathrm{R}(-\theta) = \mathrm{R}(\theta)^{-1}$.

2. Find the three-by-three matrix that rotates a three-dimensional vector an angle θ counterclockwise around the z-axis.

Solutions to the Problems

Lecture 9 | Permutation matrices

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Another type of orthogonal matrix is a permutation matrix. A permutation matrix, when multiplying on the left, permutes the rows of a matrix, and when multiplying on the right, permutes the columns. Clearly, permuting the rows of a column vector will not change its length.

For example, let the string $\{1,2\}$ represent the order of the rows of a two-by-two matrix. Then the two possible permutations of the rows are given by $\{1,2\}$ and $\{2,1\}$. The first permutation is no permutation at all, and the corresponding permutation matrix is simply the identity matrix. The second permutation of the rows is achieved by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

The rows of a three-by-three matrix have $3! = 6$ possible permutations, namely $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, $\{3,2,1\}$. For example, the row permutation $\{3,1,2\}$ is achieved by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}.$$

Notice that the permutation matrix is obtained by permuting the corresponding rows of the identity matrix, with the rows of the identity matrix permuted as $\{1,2,3\} \to \{3,1,2\}$. That a permutation matrix is just a row-permuted identity matix is made evident by writing

$$\mathrm{PA} = (\mathrm{PI})\mathrm{A},$$

where P is a permutation matrix and PI is the identity matrix with permuted rows. The identity matrix is orthogonal, and so is the permutation matrix obtained by permuting the rows of the identity matrix.

Problems for Lecture 9

1. Write down the six three-by-three permutation matrices corresponding to the permutations $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, $\{3,2,1\}$.

2. Determine the inverses of all three-by-three permutation matrices, considering that the inverse of a permutation matrix reverses the original permutation. Explain why certain matrices are their own inverses, while others are not.

Solutions to the Problems

Practice Quiz | Orthogonal matrices

1. Which matrix is not orthogonal?

a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

2. Which matrix rotates a three-by-one column vector an angle θ counterclockwise around the x-axis?

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

b) $\begin{pmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{pmatrix}$

c) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. Which matrix, when left multiplying another matrix, moves row one to row two, row two to row three, and row three to row one?

a) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Solutions to the Practice quiz

Module II

Systems of Linear Equations

In this module's lectures, we learn about solving a system of linear equations. A system of linear equations can be written in matrix form, and we can solve using Gaussian elimination. We will learn how to bring a matrix to reduced row echelon form, and how this can be used to compute a matrix inverse. We will also learn how to find the LU decomposition of a matrix, and how to use this decomposition to efficiently solve a system of linear equations.

Lecture 10 | Gaussian elimination

View this lecture on YouTube

Consider the linear system of equations given by

$$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1, \\ 6x_1 - 6x_2 + 7x_3 &= -7, \\ 3x_1 - 4x_2 + 4x_3 &= -6, \end{aligned}$$

which can be written in matrix form as

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix},$$

or symbolically as $\mathrm{Ax} = \mathrm{b}$.

The standard numerical algorithm used to solve a system of linear equations is called *Gaussian elimination*. We first form what is called an *augmented matrix* by combining the matrix A with the column vector b:

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{pmatrix}.$$

Row reduction is then performed on this augmented matrix. Allowed operations are (1) interchange the order of any rows, (2) multiply any row by a constant, (3) add a multiple of one row to another row. These three operations do not change the solution of the original equations. The goal here is to convert the matrix A into upper-triangular form, and then use this form to quickly solve for the unknowns x.

We start with the first row of the matrix and work our way down as follows. First we multiply the first row by 2 and add it to the second row. Then we add the first row to the third row, to obtain

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{pmatrix}.$$

We then go to the second row. We multiply this row by -1 and add it to the third row to obtain

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

The original matrix A has been converted to an upper triangular matrix, and the trans-

formed equations can be determined from the augmented matrix as

$$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1, \\ -2x_2 + 5x_3 &= -9, \\ -2x_3 &= 2. \end{aligned}$$

These equations can be solved by back substitution, starting from the last equation and working backwards. We have

$$\begin{aligned} x_3 &= -1, \\ x_2 &= -\frac{1}{2}(-9 - 5x_3) = 2, \\ x_1 &= -\frac{1}{3}(-1 + x_3 - 2x_2) = 2. \end{aligned}$$

We have thus found the solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

When performing Gaussian elimination, the matrix element that is used during the elimination procedure is called the *pivot*. To obtain the correct multiple, one uses the pivot as the divisor to the matrix elements below the pivot. Gaussian elimination in the way done here will fail if the pivot is zero. If the pivot is zero, a row interchange must first be performed.

Even if no pivots are identically zero, small values can still result in an unstable numerical computation. For very large matrices solved by a computer, the solution vector will be inaccurate unless row interchanges are made. The resulting numerical technique is called Gaussian elimination with partial pivoting, and is usually taught in a standard numerical analysis course.

Problems for Lecture 10

1. Using Gaussian elimination with back substitution, solve the following two systems of equations:

(a)

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= -7, \\ -3x_1 + 5x_2 + x_3 &= 5, \\ 6x_1 - 4x_2 &= 2. \end{aligned}$$

(b)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1, \\ -x_1 + 3x_2 - x_3 &= -1, \\ 2x_1 - 5x_2 + 5x_3 &= 1. \end{aligned}$$

Solutions to the Problems

Lecture 11 | Reduced row echelon form

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A matrix is said to be in reduced row echelon form if the first nonzero entry in every row is a one, all the entries below and above this one are zero, and any zero rows occur at the bottom of the matrix.

The row elimination procedure of Gaussian elimination can be continued to bring a matrix to reduced row echelon form. We notate the reduced row echelon form of a matrix A as $\operatorname{rref}(A)$. For example, consider the three-by-four matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}.$$

Row elimination can proceed as

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

and we therefore have

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We say that the matrix A has two pivot columns, that is, two columns that contain a pivot position with a one in the reduced row echelon form.

Note that rows may need to be exchanged when computing the reduced row echelon form. Also, the reduced row echelon form of a matrix A is unique, and if A is a square invertible matrix, then $\operatorname{rref}(A)$ is the identity matrix.

Problems for Lecture 11

1. Put the following matrices into reduced row echelon form and state which columns are pivot columns:

(a)

$$A = \begin{pmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{pmatrix}$$

Solutions to the Problems

Lecture 12 | Computing inverses

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By bringing an invertible matrix to reduced row echelon form, that is, to the identity matrix, we can compute the matrix inverse. Given a matrix A, consider the equation

$$AA^{-1} = I,$$

for the unknown inverse A^{-1}. Let the columns of A^{-1} be given by the vectors a_1^{-1}, a_2^{-1}, and so on. And let the columns of the identity matrix I be given by e_1, e_2, and so on. The matrix A multiplying the ith column of A^{-1} is the equation

$$Aa_i^{-1} = e_i,$$

where e_i is the ith column of the identity matrix.

This equation for the unknown columns of A^{-1} suggests that we can perform row reduction on an augmented matrix which attaches the n-by-n identity matrix to the n-by-n matrix A. To find A^{-1}, elimination is continued until one obtains $\text{rref}(A) = I$.

We illustrate below:

$$\begin{pmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 6 & -6 & 7 & 0 & 1 & 0 \\ 3 & -4 & 4 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & -2 & 3 & 1 & 0 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} -3 & 0 & 4 & 3 & 1 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} -3 & 0 & 0 & 1 & -1 & 2 \\ 0 & -2 & 0 & -1/2 & -3/2 & 5/2 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 1/4 & 3/4 & -5/4 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{pmatrix};$$

and one can check that

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} -1/3 & 1/3 & -2/3 \\ 1/4 & 3/4 & -5/4 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problems for Lecture 12

1. Compute the inverse of

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}.$$

Solutions to the Problems

Practice Quiz | Gaussian elimination

1. Perform Gaussian elimination without row interchange on the following augmented matrix:
$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -2 \end{pmatrix}$. Which matrix can be the result?

a) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$

2. Which matrix is not in reduced row echelon form?

a) $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. The inverse of $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$ is

a) $\begin{pmatrix} 4/3 & 2/3 & 1/2 \\ 2 & 1 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 2/3 & 1/2 & 4/3 \\ 1 & 1/2 & 2 \\ -3 & -5 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -5 & -3 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}$

Solutions to the Practice quiz

Lecture 13 | Elementary matrices

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The row reduction algorithm of Gaussian elimination can be implemented by multiplying elementary matrices. Here, we show how to construct these elementary matrices, which differ from the identity matrix by a single elementary row operation. Consider the first row reduction step for the following matrix A:

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} = M_1 A, \quad \text{where } M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To construct the elementary matrix M_1, the number two is placed in column-one, row-two. This matrix multiplies the first row by two and adds the result to the second row.

The next step in row elimination is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_2 M_1 A, \quad \text{where } M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Here, to construct M_2 the number one is placed in column-one, row-three, and the matrix multiplies the first row by one and adds the result to the third row.

The last step in row elimination is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = M_3 M_2 M_1 A, \quad \text{where } M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Here, to construct M_3 the number negative-one is placed in column-two, row-three, and this matrix multiplies the second row by negative-one and adds the result to the third row.

We have thus found that

$$M_3 M_2 M_1 A = U,$$

where U is an upper triangular matrix. This discussion will be continued in the next lecture.

Problems for Lecture 13

1. Construct the elementary matrix that multiplies the second row of a four-by-four matrix by two and adds the result to the fourth row.

Solutions to the Problems

Lecture 14 | LU decomposition

View this lecture on YouTube

In the last lecture, we have found that row reduction of a matrix A can be written as

$$M_3 M_2 M_1 A = U,$$

where U is upper triangular. Upon inverting the elementary matrices, we have

$$A = M_1^{-1} M_2^{-1} M_3^{-1} U.$$

Now, the matrix M_1 multiples the first row by two and adds it to the second row. To invert this operation, we simply need to multiply the first row by negative-two and add it to the second row, so that

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}; \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$L = M_1^{-1} M_2^{-1} M_3^{-1}$$

is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix},$$

which is lower triangular. Also, the non-diagonal elements of the elementary inverse matrices are simply combined to form L. Our LU decomposition of A is therefore

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}.$$

Problems for Lecture 14

1. Find the LU decomposition of

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}.$$

Solutions to the Problems

Lecture 15 | Solving (LU)x = b

View this lecture on YouTube

The LU decomposition is useful when one needs to solve $Ax = b$ for many right-hand sides. With the LU decomposition in hand, one writes

$$(LU)x = L(Ux) = b,$$

and lets $y = Ux$. Then we solve $Ly = b$ for y by forward substitution, starting from the first equation and working forward to complete the solution, and $Ux = y$ for x by backward substitution. It is possible to show that for large matrices, solving $(LU)x = b$ is substantially faster than solving $Ax = b$ directly.

We now illustrate the solution of $LUx = b$, with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}.$$

With $y = Ux$, we first solve $Ly = b$, that is,

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}.$$

Using forward substitution,

$$\begin{aligned} y_1 &= -1, \\ y_2 &= -7 + 2y_1 = -9, \\ y_3 &= -6 + y_1 - y_2 = 2. \end{aligned}$$

We then solve $Ux = y$, that is,

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix}.$$

Using back substitution,

$$\begin{aligned} x_3 &= -1, \\ x_2 &= -\frac{1}{2}(-9 - 5x_3) = 2, \\ x_1 &= -\frac{1}{3}(-1 - 2x_2 + x_3) = 2, \end{aligned}$$

and we have found $\begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} = \begin{pmatrix} 2, & 2, & -1 \end{pmatrix}$.

Problems for Lecture 15

1. Using

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU,$$

compute the solution to $Ax = b$ with

$$\text{(a) } b = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}, \qquad \text{(b) } b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Solutions to the Problems

Practice Quiz | LU decomposition

1. Which of the following is the elementary matrix that multiplies the second row of a four-by-four matrix by 2 and adds the result to the third row?

a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

2. Which of the following is the LU decomposition of $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$?

a) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1/2 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 2 & -10 & 6 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ -6 & 14 & 3 \end{pmatrix}$

3. Suppose $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Solve $LUx = b$ by letting $y = Ux$. The solutions for y and x are

a) $y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} 1/6 \\ 1/2 \\ -1 \end{pmatrix}$

b) $y = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $x = \begin{pmatrix} -1/6 \\ -1/2 \\ 1 \end{pmatrix}$

c) $y = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $x = \begin{pmatrix} 1/6 \\ -1/2 \\ 1 \end{pmatrix}$

d) $y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} -1/6 \\ 1/2 \\ 1 \end{pmatrix}$

Solutions to the Practice quiz

Module III

Vector Spaces

In this module's lectures, we learn about vector spaces. A vector space consists of a set of vectors and a set of scalars that is closed under vector addition and scalar multiplication and that satisfies the usual rules of arithmetic. We will learn some of the vocabulary and phrases of linear algebra, such as linear independence, span, basis and dimension. We will learn about the four fundamental subspaces of a matrix, the Gram-Schmidt process, orthogonal projection, and the matrix formulation of the least-squares problem of drawing a straight line to fit noisy data.

Lecture 16 | Vector spaces

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A *vector space* consists of a set of vectors and a set of scalars. Although vectors can be quite general, for the purpose of this course we will only consider vectors that are real column matrices, and scalars that are real numbers.

For the set of vectors and scalars to form a vector space, the set of vectors must be closed under vector addition and scalar multiplication. That is, when you multiply any two vectors in the set by real numbers and add them, the resulting vector must still be in the set.

As an example, consider the set of vectors consisting of all three-by-one matrices, and let u and v be two of these vectors. Let $\mathrm{w} = a\mathrm{u} + b\mathrm{v}$ be the sum of these two vectors multiplied by the real numbers a and b. If w is still a three-by-one matrix, then this set of vectors is closed under scalar multiplication and vector addition, and is indeed a vector space. The proof is rather simple. If we let

$$\mathrm{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathrm{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

then

$$\mathrm{w} = a\mathrm{u} + b\mathrm{v} = \begin{pmatrix} au_1 + bv_1 \\ au_2 + bv_2 \\ au_3 + bv_3 \end{pmatrix}$$

is evidently a three-by-one matrix. This vector space is called $\mathbb{R}^3$.

Our main interest in vector spaces is to determine the vector spaces associated with matrices. There are four fundamental vector spaces of an m-by-n matrix A. They are called the *null space*, the *column space*, the *row space*, and the *left null space*. We will meet these vector spaces in later lectures.

Problems for Lecture 16

1. Explain why, in a vector space with real numbers as the set of scalars, the zero vector must be a member of every vector space.

2. Explain why the following sets of three-by-one matrices (with real number scalars) are vector spaces:

(a) The set of three-by-one matrices with zero in the first row;

(b) The set of three-by-one matrices with first row equal to the second row;

(c) The set of three-by-one matrices with first row a constant multiple of the third row.

Solutions to the Problems

Lecture 17 | Linear independence

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The vectors $\{u_1, u_2, \ldots, u_n\}$ are *linearly independent* if for any scalars c_1, c_2, $\ldots$, c_n, the equation

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0$$

has only the solution $c_1 = c_2 = \cdots = c_n = 0$. What this means is that one is unable to write any of the vectors $u_1, u_2, \ldots, u_n$ as a linear combination of any of the other vectors. For instance, if there was a solution to the above equation with $c_1 \neq 0$, then we could solve that equation for u_1 in terms of the other vectors with nonzero coefficients.

As an example consider whether the following three three-by-one column vectors are linearly independent:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

Indeed, they are not linearly independent, that is, they are *linearly dependent*, because w can be written in terms of u and v. In fact, $w = 2u + 3v$.

Now consider the three three-by-one column vectors given by

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These three vectors are linearly independent because you cannot write any one of these vectors as a linear combination of the other two. If we go back to our definition of linear independence, we can see that the equation

$$au + bv + cw = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has as its only solution $a = b = c = 0$.

For simple examples, visual inspection can often decide if a set of vectors are linearly independent. For a more algorithmic procedure, place the vectors as the rows of a matrix and compute the reduced row echelon form. If the last row becomes all zeros, then the vectors are linearly dependent, and if not all zeros, then they are linearly independent.

Problems for Lecture 17

1. Which of the following sets of vectors are linearly independent?

(a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Solutions to the Problems

Lecture 18 | Span, basis and dimension

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Given a set of vectors, one can generate a vector space by forming all linear combinations of that set of vectors. The *span* of the set of vectors $\{v_1, v_2, \ldots, v_n\}$ is the vector space consisting of all linear combinations of $v_1, v_2, \ldots, v_n$. We say that a set of vectors spans a vector space.

For example, the set of vectors given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$$

spans the vector space of all three-by-one matrices with zero in the third row. This vector space is a *vector subspace* of all three-by-one matrices.

One doesn't need all three of these vectors to span this vector subspace because any one of these vectors is linearly dependent on the other two. The smallest set of vectors needed to span a vector space forms a *basis* for that vector space. Here, given the set of vectors above, we can construct a basis for the vector subspace of all three-by-one matrices with zero in the third row by simply choosing two out of three vectors from the above spanning set. Three possible basis vectors are given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

Although all three combinations form a basis for the vector subspace, the first combination is usually preferred because this is an orthonormal basis. The vectors in this basis are mutually orthogonal and of unit norm.

The number of vectors in a basis gives the dimension of the vector space. Here, the dimension of the vector space of all three-by-one matrices with zero in the third row is two.

Problems for Lecture 18

1. Find an orthonormal basis for the vector space of all three-by-one matrices with first row equal to second row. What is the dimension of this vector space?

Solutions to the Problems

Practice Quiz | Vector space definitions

1. Which set of three-by-one matrices (with real number scalars) is not a vector space?

a) The set of three-by-one matrices with zero in the second row.

b) The set of three-by-one matrices with the sum of all rows equal to one.

c) The set of three-by-one matrices with the first row equal to the third row.

d) The set of three-by-one matrices with the first row equal to the sum of the second and third rows.

2. Which one of the following sets of vectors is linearly independent?

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} \right\}$

c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

3. Which one of the following is an orthonormal basis for the vector space of all three-by-one matrices with the sum of all rows equal to zero?

a) $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

b) $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$

c) $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

d) $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$

Solutions to the Practice quiz

Lecture 19 | Gram-Schmidt process

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Given any basis for a vector space, we can use an algorithm called the Gram-Schmidt process to construct an orthonormal basis for that space. Let the vectors $v_1, v_2, \ldots, v_n$ be a basis for some n-dimensional vector space. We will assume here that these vectors are column matrices, but this process also applies more generally.

We will construct an orthogonal basis $u_1, u_2, \ldots, u_n$, and then normalize each vector to obtain an orthonormal basis. First, define $u_1 = v_1$. To find the next orthogonal basis vector, define

$$u_2 = v_2 - \frac{(u_1^T v_2)u_1}{u_1^T u_1}.$$

Observe that u_2 is equal to v_2 minus the component of v_2 that is parallel to u_1. By multiplying both sides of this equation with u_1^T, it is easy to see that $u_1^T u_2 = 0$ so that these two vectors are orthogonal.

The next orthogonal vector in the new basis can be found from

$$u_3 = v_3 - \frac{(u_1^T v_3)u_1}{u_1^T u_1} - \frac{(u_2^T v_3)u_2}{u_2^T u_2}.$$

Here, u_3 is equal to v_3 minus the components of v_3 that are parallel to u_1 and u_2. We can continue in this fashion to construct n orthogonal basis vectors. These vectors can then be normalized via

$$\widehat{u}_1 = \frac{u_1}{(u_1^T u_1)^{1/2}}, \quad \text{etc.}$$

Since u_k is a linear combination of $v_1, v_2, \ldots, v_k$, the vector subspace spanned by the first k basis vectors of the original vector space is the same as the subspace spanned by the first k orthonormal vectors generated through the Gram-Schmidt process. We can write this result as

$$\text{span}\{u_1, u_2, \ldots, u_k\} = \text{span}\{v_1, v_2, \ldots, v_k\}.$$

Problems for Lecture 19

1. Suppose the four basis vectors $\{v_1, v_2, v_3, v_4\}$ are given, and one performs the Gram-Schmidt process on these vectors in order. Write down the equation to find the fourth orthogonal vector u_4. Do not normalize.

Solutions to the Problems

Lecture 20 | Gram-Schmidt process (example)

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As an example of the Gram-Schmidt process, consider a subspace of three-by-one column matrices with the basis

$$\{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\},$$

and construct an orthonormal basis for this subspace. Let $u_1 = v_1$. Then u_2 is found from

$$\begin{aligned} u_2 &= v_2 - \frac{(u_1^T v_2) u_1}{u_1^T u_1} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Normalizing the two vectors, we obtain the orthonormal basis

$$\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Notice that the initial two vectors v_1 and v_2 span the vector subspace of three-by-one column matrices for which the second and third rows are equal. Clearly, the orthonormal basis vectors constructed from the Gram-Schmidt process span the same subspace.

Problems for Lecture 20

1. Consider the vector subspace of three-by-one column vectors with the third row equal to the negative of the second row, and with the following given basis:

$$W = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Use the Gram-Schmidt process to construct an orthonormal basis for this subspace.

2. Consider a subspace of all four-by-one column vectors with the following basis:

$$W = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Use the Gram-Schmidt process to construct an orthonormal basis for this subspace.

Solutions to the Problems

Practice Quiz | Gram-Schmidt process

1. In the fourth step of the Gram-Schmidt process, the vector

$$u_4 = v_4 - \frac{(u_1^T v_4)u_1}{u_1^T u_1} - \frac{(u_2^T v_4)u_2}{u_2^T u_2} - \frac{(u_3^T v_4)u_3}{u_3^T u_3}$$

is always perpendicular to

a) v_1

b) v_2

c) v_3

d) v_4

2. The Gram-Schmidt process applied to $\{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ results in

a) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

b) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

c) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

d) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

3. The Gram-Schmidt process applied to $\{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ results in

a) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

b) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\}$

c) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

d) $\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solutions to the Practice quiz

Lecture 21 | Null space

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The null space of a matrix A, which we denote as Null(A), is the vector space spanned by all column vectors x that satisfy the matrix equation

$$\mathrm{A}\mathbf{x} = 0.$$

Clearly, if x and y are in the null space of A, then so is $a\mathbf{x} + b\mathbf{y}$ so that the null space is closed under vector addition and scalar multiplication. If the matrix A is m-by-n, then Null(A) is a vector subspace of all n-by-one column matrices. If A is a square invertible matrix, then Null(A) consists of just the zero vector.

To find a basis for the null space of a noninvertible matrix, we bring A to reduced row echelon form. We demonstrate by example. Consider the three-by-five matrix given by

$$\mathrm{A} = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

By judiciously permuting rows to simplify the arithmetic, one pathway to construct rref(A) is

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We call the variables associated with the pivot columns, x_1 and x_3, *basic variables*, and the variables associated with the non-pivot columns, x_2, x_4 and x_5, *free variables*. Writing the basic variables on the left-hand side of the $\mathrm{A}\mathbf{x} = 0$ equations, we have from the first and second rows

$$x_1 = 2x_2 + x_4 - 3x_5,$$
$$x_3 = -2x_4 + 2x_5.$$

Eliminating x_1 and x_3, we can write the general solution for vectors in $\text{Null}(\text{A})$ as

$$\begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix},$$

where the free variables x_2, x_4, and x_5 can take any values. By writing the null space in this form, a basis for $\text{Null}(\text{A})$ is made evident, and is given by

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The null space of A is seen to be a three-dimensional subspace of all five-by-one column matrices. In general, the dimension of $\text{Null}(\text{A})$ is equal to the number of non-pivot columns of $\text{rref}(\text{A})$.

Problems for Lecture 21

1. Determine a basis for the null space of

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Solutions to the Problems

Lecture 22 | Application of the null space

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An under-determined system of linear equations $\mathrm{Ax} = \mathrm{b}$ with more unknowns than equations may not have a unique solution. If u is the general form of a vector in the null space of A, and v is any vector that satisfies $\mathrm{Av} = \mathrm{b}$, then $x = \mathrm{u} + \mathrm{v}$ satisfies $\mathrm{Ax} = \mathrm{A}(\mathrm{u} + \mathrm{v}) = \mathrm{Au} + \mathrm{Av} = 0 + b = b$. The general solution of $\mathrm{Ax} = \mathrm{b}$ can therefore be written as the sum of a general vector in $\mathrm{Null}(\mathrm{A})$ and a particular vector that satisfies the under-determined system.

As an example, suppose we want to find the general solution to the linear system of two equations and three unknowns given by

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 0, \\ 2x_1 - 2x_2 - x_3 &= 1, \end{aligned}$$

which in matrix form is given by

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We first bring the augmented matrix to reduced row echelon form:

$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & -2 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix}.$$

The null space satisfying $\mathrm{Au} = 0$ is determined from $u_1 = 0$ and $u_2 = -u_3/2$, and we can write

$$\mathrm{Null}(\mathrm{A}) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

A particular solution for the inhomogeneous system satisfying $\mathrm{Av} = b$ is found by solving $v_1 = 1/4$ and $v_2 + v_3/2 = -1/4$. Here, we simply take the free variable v_3 to be zero, and we find $v_1 = 1/4$ and $v_2 = -1/4$. The general solution to the original underdetermined linear system is the sum of the null space and the particular solution and is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Problems for Lecture 22

1. Find the general solution to the system of equations given by

$$\begin{aligned} -3x_1 + 6x_2 - x_3 + x_4 &= -7, \\ x_1 - 2x_2 + 2x_3 + 3x_4 &= -1, \\ 2x_1 - 4x_2 + 5x_3 + 8x_4 &= -4. \end{aligned}$$

Solutions to the Problems

Lecture 23 | Column space

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The column space of a matrix is the vector space spanned by the columns of the matrix. When a matrix is multiplied by a column vector, the resulting vector is in the column space of the matrix, as can be seen from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix}.$$

In general, Ax is a linear combination of the columns of A. Given an m-by-n matrix A, what is the dimension of the column space of A, and how do we find a basis? Note that since A has m rows, the column space of A is a subspace of all m-by-one column matrices.

Fortunately, a basis for the column space of A can be found from rref(A). Consider the example

$$\mathrm{A} = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}, \qquad \mathrm{rref}(\mathrm{A}) = \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix equation $\mathrm{Ax} = 0$ expresses the linear dependence of the columns of A, and row operations on A do not change the dependence relations. For example, the second column of A above is -2 times the first column, and after several row operations, the second column of $\mathrm{rref}(\mathrm{A})$ is still -2 times the first column.

It should be self-evident that only the pivot columns of $\mathrm{rref}(\mathrm{A})$ are linearly independent, and the dimension of the column space of A is therefore equal to its number of pivot columns; here it is two. A basis for the column space is given by the first and third columns of A, (not $\mathrm{rref}(\mathrm{A})$), and is

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}.$$

Recall that the dimension of the null space is the number of non-pivot columns—equal to the number of free variables—so that the sum of the dimensions of the null space and the column space is equal to the total number of columns. A statement of this theorem is as follows. Let A be an m-by-n matrix. Then

$$\dim(\mathrm{Col}(\mathrm{A})) + \dim(\mathrm{Null}(\mathrm{A})) = n.$$

Problems for Lecture 23

1. Determine the dimension and find a basis for the column space of

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Solutions to the Problems

Lecture 24 | Row space, left null space and rank

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In addition to the column space and the null space, a matrix A has two more vector spaces associated with it, namely the column space and null space of A^T, which are called the row space and the left null space.

If A is an m-by-n matrix, then the row space and the null space are subspaces of all n-by-one column matrices, and the column space and the left null space are subspaces of all m-by-one column matrices.

The null space consists of all vectors x such that $Ax = 0$, that is, the null space is the set of all vectors that are orthogonal to the row space of A. We say that these two vector spaces are orthogonal.

A basis for the row space of a matrix can be found from computing $\text{rref}(A)$, and is found to be rows of $\text{rref}(A)$ (written as column vectors) with pivot columns. The dimension of the row space of A is therefore equal to the number of pivot columns, while the dimension of the null space of A is equal to the number of nonpivot columns. The union of these two subspaces make up the vector space of all n-by-one matrices and we say that these subspaces are *orthogonal complements* of each other.

Furthermore, the dimension of the column space of A is also equal to the number of pivot columns, so that the dimensions of the column space and the row space of a matrix are equal. We have

$$\dim(\text{Col}(A)) = \dim(\text{Row}(A)).$$

We call this dimension the rank of the matrix A. This is an amazing result since the column space and row space are subspaces of two different vector spaces. In general, we must have $\text{rank}(A) \leq \min(m, n)$. When the equality holds, we say that the matrix is of full rank. And when A is a square matrix and of full rank, then the dimension of the null space is zero and A is invertible.

Problems for Lecture 24

1. Find a basis for the column space, row space, null space and left null space of the four-by-five matrix A, where

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 & 2 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 1 & -2 & 3 & -1 & -3 \end{pmatrix}$$

Check to see that null space is the orthogonal complement of the row space, and the left null space is the orthogonal complement of the column space. Find $\text{rank}(A)$. Is this matrix of full rank?

Solutions to the Problems

Practice Quiz | Fundamental subspaces

1. Which of the following sets of vectors form a basis for the null space of $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix}$?

a) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

c) $\left\{ \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

2. The general solution to the system of equations given by

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 1, \\ 2x_1 + 4x_2 + x_3 + x_4 &= 1, \\ 3x_1 + 6x_2 + x_3 + x_4 &= 1, \end{aligned}$$

is

a) $a\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

b) $a\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

c) $a\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix}$

d) $a\begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3. What is the rank of the matrix $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix}$?

a) 1

b) 2

c) 3

d) 4

Solutions to the Practice quiz

Lecture 25 | Orthogonal projections

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Suppose that V is the n-dimensional vector space of all n-by-one matrices and W is a p-dimensional subspace of V. Let $\{s_1, s_2, \ldots, s_p\}$ be an orthonormal basis for W. Extending the basis for W, let $\{s_1, s_2, \ldots, s_p, t_1, t_2, \ldots, t_{n-p}\}$ be an orthonormal basis for V.

Any vector v in V can be expanded using the basis for V as

$$v = a_1 s_1 + a_2 s_2 + \cdots + a_p s_p + b_1 t_1 + b_2 t_2 + b_{n-p} t_{n-p},$$

where the a's and b's are scalar coefficients. The orthogonal projection of v onto W is then defined as

$$v_{\text{proj}_W} = a_1 s_1 + a_2 s_2 + \cdots + a_p s_p,$$

that is, the part of v that lies in W.

If you only know the vector v and the orthonormal basis for W, then the orthogonal projection of v onto W can be computed from

$$v_{\text{proj}_W} = (v^T s_1) s_1 + (v^T s_2) s_2 + \cdots + (v^T s_p) s_p,$$

that is, $a_1 = v^T s_1$, $a_2 = v^T s_2$, etc.

We can prove that the vector v_{proj_W} is the vector in W that is closest to v. Let w be any vector in W different than v_{proj_W}, and expand w in terms of the basis vectors for W:

$$w = c_1 s_1 + c_2 s_2 + \cdots + c_p s_p.$$

The distance between v and w is given by the norm $||v - w||$, and we have

$$\begin{aligned} ||v - w||^2 &= (a_1 - c_1)^2 + (a_2 - c_2)^2 + \cdots + (a_p - c_p)^2 + b_1^2 + b_2^2 + \cdots + b_{n-p}^2 \\ &\geq b_1^2 + b_2^2 + \cdots + b_{n-p}^2 = ||v - v_{\text{proj}_W}||^2, \end{aligned}$$

or $||v - v_{\text{proj}_W}|| \leq ||v - w||$, a result that will be used later in the problem of least squares.

Problems for Lecture 25

1. Find the general orthogonal projection of $\mathbf{v}$ onto W, where $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. What are the projections when $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and when $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$?

Solutions to the Problems

Lecture 26 | The least-squares problem

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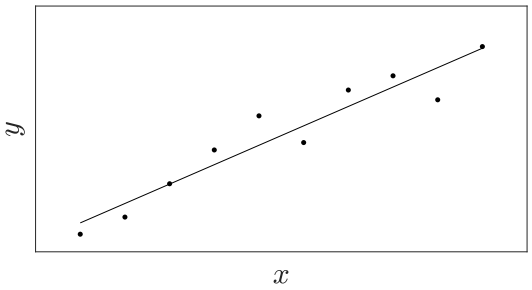
Suppose there is some experimental data that you want to fit by a straight line. This is called a linear regression problem and an illustrative example is shown below.



In general, let the data consist of a set of n points given by (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n). Here, we assume that the x values are exact, and the y values are noisy. We further assume that the best fit line to the data takes the form $y = \beta_0 + \beta_1 x$. Although we know that the line will not go through all of the data points, we can still write down the equations as if it does. We have

$$y_1 = \beta_0 + \beta_1 x_1, \quad y_2 = \beta_0 + \beta_1 x_2, \quad \ldots, \quad y_n = \beta_0 + \beta_1 x_n.$$

These equations constitute a system of n equations in the two unknowns β_0 and β_1. The corresponding matrix equation is given by

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

This is an overdetermined system of equations with no solution. The problem of least squares is to find the best solution.

We can generalize this problem as follows. Suppose we are given a matrix equation, $\mathrm{Ax} = \mathrm{b}$, that has no solution because b is not in the column space of A. So instead we solve $\mathrm{Ax} = \mathrm{b}_{\mathrm{proj}_{\mathrm{Col(A)}}}$, where $\mathrm{b}_{\mathrm{proj}_{\mathrm{Col(A)}}}$ is the projection of b onto the column space of A. The solution is then called the least-squares solution for x.

Problems for Lecture 26

1. Suppose we have data points given by $(x_i, y_i) = (0,1)$, $(1,3)$, $(2,3)$, and $(3,4)$. If the data is to be fit by the line $y = \beta_0 + \beta_1 x$, write down the overdetermined matrix expression for the set of equations $y_i = \beta_0 + \beta_1 x_i$.

Solutions to the Problems

Lecture 27 | Solution of the least-squares problem

View this lecture on YouTube

We want to find the least-squares solution to an overdetermined matrix equation $Ax = b$. We write $b = b_{\text{proj}_{\text{Col}(A)}} + (b - b_{\text{proj}_{\text{Col}(A)}})$, where $b_{\text{proj}_{\text{Col}(A)}}$ is the projection of b onto the column space of A. Since $(b - b_{\text{proj}_{\text{Col}(A)}})$ is orthogonal to the column space of A, it is in the nullspace of A^T. Multiplication of the overdetermined matrix equation by A^T then results in a solvable set of equations, called the *normal equations* for $Ax = b$, given by

$$A^T Ax = A^T b.$$

A unique solution to this matrix equation exists when the columns of A are linearly independent.

An interesting formula exists for the matrix which projects b onto the column space of A. Multiplying the normal equations on the left by $A(A^TA)^{-1}$, we obtain

$$Ax = A(A^TA)^{-1}A^Tb = b_{\text{proj}_{\text{Col}(A)}}.$$

Notice that the projection matrix $P = A(A^TA)^{-1}A^T$ satisfies $P^2 = P$, that is, two projections is the same as one. If A itself is a square invertible matrix, then $P = I$ and b is already in the column space of A.

As an example of the application of the normal equations, consider the toy least-squares problem of fitting a line through the three data points $(1,1)$, $(2,3)$ and $(3,2)$. With the line given by $y = \beta_0 + \beta_1 x$, the overdetermined system of equations is given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

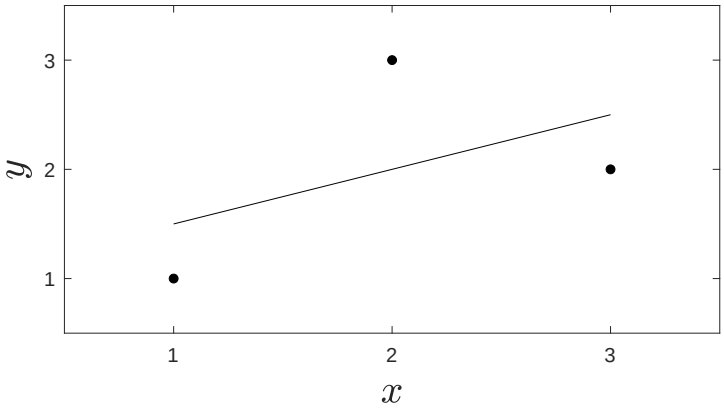
The least-squares solution is determined by solving

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},$$

or

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \end{pmatrix}.$$

We can use Gaussian elimination to determine $\beta_0 = 1$ and $\beta_1 = 1/2$, and the least-squares line is given by $y = 1 + x/2$. The graph of the data and the line is shown on the next page.



Solution of a toy least-squares problem.

Problems for Lecture 27

1. Suppose we have data points given by $(x_n, y_n) = (0,1)$, $(1,3)$, $(2,3)$, and $(3,4)$. By solving the normal equations, fit the data by the line $y = \beta_0 + \beta_1 x$.

Solutions to the Problems

Practice Quiz | Orthogonal projections

1. Which vector is the orthogonal projection of $\mathrm{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ onto $W = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$?

a) $\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

b) $\frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

c) $\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

d) $\frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

2. Suppose we have data points given by $(x_n, y_n) = (1,1)$, $(2,1)$ and $(3,3)$. If the data is to be fit by the line $y = \beta_0 + \beta_1 x$, which is the overdetermined equation for β_0 and β_1?

a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

3. Suppose we have data points given by $(x_n, y_n) = (1,1)$, $(2,1)$ and $(3,3)$. Which is the best fit line to the data?

a) $y = \frac{1}{3} + x$

b) $y = -\frac{1}{3} + x$

c) $y = 1 + \frac{1}{3}x$

d) $y = 1 - \frac{1}{3}x$

Solutions to the Practice quiz

Module IV

Eigenvalues and Eigenvectors

In this module's lectures, we will learn about determinants and the eigenvalue problem. We will learn how to compute determinants using a Laplace expansion, the Leibniz formula, or by row or column elimination. We will formulate the eigenvalue problem and learn how to find the eigenvalues and eigenvectors of a matrix. We will learn how to diagonalize a matrix using its eigenvalues and eigenvectors, and how this leads to an easy calculation of a matrix raised to a power.

Lecture 28 | Two-by-two and three-by-three determinants

View this lecture on YouTube

We already showed that a two-by-two matrix A is invertible when its determinant is nonzero, where

$$\det \mathrm{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If A is invertible, then the equation $\mathrm{A}x = b$ has the unique solution $x = \mathrm{A}^{-1}b$. But if A is not invertible, then $\mathrm{A}x = b$ may have no solution or an infinite number of solutions. When $\det \mathrm{A} = 0$, we say that the matrix A is singular.

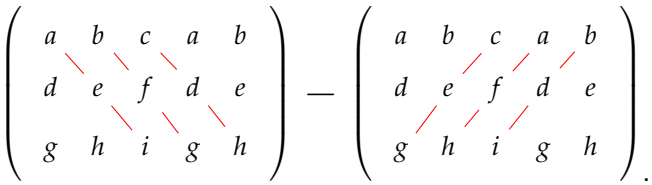
It is also straightforward to define the determinant for a three-by-three matrix. We consider the system of equations $\mathrm{A}\mathbf{x} = 0$ and determine the condition for which $\mathbf{x} = 0$ is the only solution. With

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

one can do the messy algebra of elimination to solve for x_1, x_2, and x_3. One finds that $x_1 = x_2 = x_3 = 0$ is the only solution when $\det \mathrm{A} \neq 0$, where the definition, apart from a constant, is given by

$$\det \mathrm{A} = aei + bfg + cdh - ceg - bdi - afh.$$

An easy way to remember this result is to mentally draw the following picture:



The matrix A is periodically extended two columns to the right, drawn explicitly here but usually only imagined. Then the six terms comprising the determinant are made evident, with the lines slanting down towards the right getting the plus signs and the lines slanting down towards the left getting the minus signs. Unfortunately, this mnemonic only works for three-by-three matrices.

Problems for Lecture 28

1. Find the determinant of the three-by-three identity matrix.

2. Show that the three-by-three determinant changes sign when the first two rows are interchanged.

3. Let A and B be two-by-two matrices. Prove by direct computation that $\det AB = \det A \det B$.

Solutions to the Problems

Lecture 29 | Laplace expansion

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There is a way to write the three-by-three determinant that generalizes. It is called a Laplace expansion (also called a cofactor expansion or expansion by minors). For the three-by-three determinant, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg),$$

which can be written suggestively as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Evidently, the three-by-three determinant can be computed from lower-order two-by-two determinants, called minors. The rule here for a general n-by-n matrix is that one goes across the first row of the matrix, multiplying each element in the row by the determinant of the matrix obtained by crossing out that element's row and column, and adding the results with alternating signs.

In fact, this expansion in minors can be done across any row or down any column. When the minor is obtained by deleting the ith-row and j-th column, then the sign of the term is given by $(-1)^{i+j}$. An easy way to remember the signs is to form a checkerboard pattern, exhibited here for the three-by-three and four-by-four matrices:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}, \quad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}.$$

Example: Compute the determinant of

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 2 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{pmatrix}.$$

We first expand in minors down the second column. The only nonzero contribution comes from the two in the third row, and we cross out the second column and third row (and multiply by a minus sign) to obtain a three-by-three determinant:

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 2 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{vmatrix}.$$

We then again expand in minors down the second column. The only nonzero contribution comes from the five in the third row, and we cross out the second column and third row (and mutiply by a minus sign) to obtain a two-by-two determinant, which we then compute:

$$-2 \begin{vmatrix} 1 & 0 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{vmatrix} = 10 \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = 80.$$

The trick here is to expand by minors across the row or column containing the most zeros.

Problems for Lecture 29

1. Compute the determinant of

$$A = \begin{pmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & 4 & 1 & 0 \\ 8 & -5 & 6 & 7 & -2 \\ -2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 3 & 2 & 0 \end{pmatrix}.$$

Solutions to the Problems

Lecture 30 | Leibniz formula

View this lecture on YouTube

Another way to generalize the three-by-three determinant is called the Leibniz formula, or more descriptively, the big formula. The three-by-three determinant can be written as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh + bfg - bdi + cdh - ceg,$$

where each term in the formula contains a single element from each row and from each column. For example, to obtain the third term bfg, b comes from the first row and second column, f comes from the second row and third column, and g comes from the third row and first column. As we can choose one of three elements from the first row, then one of two elements from the second row, and only one element from the third row, there are $3! = 6$ terms in the formula, and the general n-by-n matrix without any zero entries will have $n!$ terms.

The sign of each term depends on whether the choice of columns as we go down the rows is an even or odd permutation of the columns ordered as $\{1, 2, 3, \ldots, n\}$. An even permutation is when columns are interchanged an even number of times, and an odd permutation is when they are interchanged an odd number of times. Even permutations get a plus sign and odd permutations get a minus sign.

For the determinant of the three-by-three matrix, the plus terms aei, bfg, and cdh correspond to the column orderings $\{1, 2, 3\}$, $\{2, 3, 1\}$ and $\{3, 1, 2\}$, which are even permutations of $\{1, 2, 3\}$, and the minus terms afh, bdi, and ceg correspond to the column orderings $\{1, 3, 2\}$, $\{2, 1, 3\}$ and $\{3, 2, 1\}$, which are odd permutations.

Problems for Lecture 30

1. Using the Leibniz formula, compute the determinant of the following four-by-four matrix:

$$\mathrm{A} = \begin{pmatrix} a & b & c & d \\ e & f & 0 & 0 \\ 0 & g & h & 0 \\ 0 & 0 & i & j \end{pmatrix}.$$

Solutions to the Problems

Lecture 31 | Properties of a determinant

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The determinant is a function that maps a square matrix to a scalar. It is uniquely defined by the following three properties:

Property 1: The determinant of the identity matrix is one;

Property 2: The determinant changes sign under row interchange;

Property 3: The determinant is a linear function of the first row, holding all other rows fixed.

Using two-by-two matrices, the first two properties are illustrated by

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix};$$

and the third property is illustrated by

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Both the Laplace expansion and Leibniz formula for the determinant can be proved from these three properties. Other useful properties of the determinant can also be proved:

- The determinant is a linear function of any row, holding all other rows fixed;
- If a matrix has two equal rows, then the determinant is zero;
- If we add k times row-i to row-j, the determinant doesn't change;
- The determinant of a matrix with a row of zeros is zero;
- A matrix with a zero determinant is not invertible;
- The determinant of a diagonal matrix is the product of the diagonal elements;
- The determinant of an upper or lower triangular matrix is the product of the diagonal elements;
- The determinant of the product of two matrices is equal to the product of the determinants;
- The determinant of the inverse matrix is equal to the reciprical of the determinant;
- The determinant of the transpose of a matrix is equal to the determinant of the matrix.

Notably, these properties imply that Gaussian elimination, done on rows or columns or both, can be used to simplify the computation of a determinant. Row interchanges and multiplication of a row by a constant change the determinant and must be treated correctly.

Problems for Lecture 31

1. Using the defining properties of a determinant, prove that if a matrix has two equal rows, then the determinant is zero.

2. Using the defining properties of a determinant, prove that the determinant is a linear function of any row, holding all other rows fixed.

3. Using the results of the above problems, prove that if we add k times row-i to row-j, the determinant doesn't change.

4. Use Gaussian elimination to find the determinant of the following matrix:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Solutions to the Problems

Practice Quiz | Determinants

1. The determinant of $\begin{pmatrix} -3 & 0 & -2 & 0 & 0 \\ 2 & -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 3 & 0 & -3 & 2 & -3 \\ -3 & 3 & 3 & 0 & -2 \end{pmatrix}$ is equal to

a) 48

b) 42

c) -42

d) -48

2. The determinant of $\begin{pmatrix} a & e & 0 & 0 \\ b & f & g & 0 \\ c & 0 & h & i \\ d & 0 & 0 & j \end{pmatrix}$ is equal to

a) $afhj + behj - cegj - degi$

b) $afhj - behj + cegj - degi$

c) $agij - beij + cefj - defh$

d) $agij + beij - cefj - defh$

3. Assume A and B are invertible n-by-n matrices. Which of the following identities is false?

a) $\det \mathrm{A}^{-1} = 1/\det \mathrm{A}$

b) $\det \mathrm{A}^{\mathrm{T}} = \det \mathrm{A}$

c) $\det(\mathrm{A} + \mathrm{B}) = \det \mathrm{A} + \det \mathrm{B}$

d) $\det(\mathrm{AB}) = \det \mathrm{A} \det \mathrm{B}$

Solutions to the Practice quiz

Lecture 32 | The eigenvalue problem

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Let A be a square matrix, x a column vector, and λ a scalar. The eigenvalue problem for A solves

$$\mathrm{A}\mathrm{x} = \lambda \mathrm{x}$$

for eigenvalues λ_i with corresponding eigenvectors x_i. Making use of the identity matrix I, the eigenvalue problem can be rewritten as the homogeneous equation

$$(\mathrm{A} - \lambda \mathrm{I})\mathrm{x} = 0,$$

where the matrix $(\mathrm{A} - \lambda \mathrm{I})$ is just the matrix A with λ subtracted from its diagonal. For there to be nonzero eigenvectors, the matrix $(\mathrm{A} - \lambda \mathrm{I})$ must be singular, that is,

$$\det(\mathrm{A} - \lambda \mathrm{I}) = 0.$$

This equation is called the *characteristic equation* of the matrix A. From the Leibniz formula, the characteristic equation of an n-by-n matrix is an n-th order polynomial equation in λ. For each found λ_i, a corresponding eigenvector $\mathrm{x_i}$ can be determined directly by solving $(\mathrm{A} - \lambda_i \mathrm{I})\mathrm{x} = 0$ for x.

For illustration, we compute the eigenvalues of a general two-by-two matrix. We have

$$0 = \det(\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc);$$

and this characteristic equation can be rewritten as

$$\lambda^2 - \mathrm{Tr}\,\mathrm{A}\ \lambda + \det \mathrm{A} = 0,$$

where Tr A is the trace, or sum of the diagonal elements, of the matrix A.

Since the characteristic equation of a two-by-two matrix is a quadratic equation, it can have either (i) two distinct real roots; (ii) two distinct complex conjugate roots; or (iii) one degenerate real root. More generally, eigenvalues can be real or complex, and an n-by-n matrix may have less than n distinct eigenvalues.

Problems for Lecture 32

1. Using the formula for a three-by-three determinant, determine the characteristic equation for a general three-by-three matrix A. This equation should be written as a cubic equation in λ.

Solutions to the Problems

Lecture 33 | Finding eigenvalues and eigenvectors (Part A)

View this lecture on YouTube

We compute here the two real eigenvalues and eigenvectors of a two-by-two matrix.

Example: Find the eigenvalues and eigenvectors of $\mathrm{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The characteristic equation of A is given by

$$\lambda^2 - 1 = 0,$$

with solutions $\lambda_1 = 1$ and $\lambda_2 = -1$. The first eigenvector is found by solving $(\mathrm{A} - \lambda_1 \mathrm{I})\mathrm{x} = 0$, or

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

The equation from the second row is just a constant multiple of the equation from the first row and this will always be the case for two-by-two matrices. From the first row, say, we find $x_2 = x_1$. The second eigenvector is found by solving $(\mathrm{A} - \lambda_2 \mathrm{I})\mathrm{x} = 0$, or

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

so that $x_2 = -x_1$. The eigenvalues and eigenvectors are therefore given by

$$\lambda_1 = 1, \quad \mathrm{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -1, \quad \mathrm{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvectors can be multiplied by an arbitrary nonzero constant. Notice that $\lambda_1 + \lambda_2 = \mathrm{Tr}\,\mathrm{A}$ and that $\lambda_1 \lambda_2 = \det \mathrm{A}$, and analogous relations are true for any n-by-n matrix. In particular, comparing the sum over all the eigenvalues and the matrix trace provides a simple algebra check.

Problems for Lecture 33

1. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$.

2. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Solutions to the Problems

Lecture 34 | Finding eigenvalues and eigenvectors (Part B)

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We compute some more eigenvalues and eigenvectors.

Example: Find the eigenvalues and eigenvectors of $\mathrm{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The characteristic equation of B is given by

$$\lambda^2 = 0,$$

so that there is a degenerate eigenvalue of zero. The eigenvector associated with the zero eigenvalue is found from $\mathrm{Bx} = 0$ and has zero second component. This matrix therefore has only one eigenvalue and eigenvector, given by

$$\lambda = 0, \quad \mathrm{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Example: Find the eigenvalues of $\mathrm{C} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The characteristic equation of C is given by

$$\lambda^2 + 1 = 0,$$

which has the imaginary solutions $\lambda = \pm i$. Matrices with complex eigenvalues play an important role in the theory of linear differential equations.

Problems for Lecture 34

1. Find the eigenvalues of $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Solutions to the Problems

Practice Quiz | The eigenvalue problem

1. Which of the following are the eigenvalues of $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$?

a) $\frac{3}{2} \pm \frac{\sqrt{3}}{2}$

b) $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$

c) $\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

d) $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

2. Which of the following are the eigenvalues of $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$?

a) $1 \pm 3i$

b) $1 \pm \sqrt{3}$

c) $3\sqrt{3} \pm 1$

d) $3 \pm i$

3. Which of the following is an eigenvector of $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$?

a) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

d) $\begin{pmatrix} \sqrt{2} \\ 1 \\ \sqrt{2} \end{pmatrix}$

Solutions to the Practice quiz

Lecture 35 | Matrix diagonalization

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For concreteness, consider a two-by-two matrix A with eigenvalues and eigenvectors given by

$$\lambda_1, \quad x_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}; \quad \lambda_2, \quad x_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.$$

And consider the matrix product and factorization given by

$$A \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} \\ \lambda_1 x_{21} & \lambda_2 x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Generalizing, we define S to be the matrix whose columns are the eigenvectors of A, and Λ to be the diagonal matrix with eigenvalues down the diagonal. Then for any n-by-n matrix with n linearly independent eigenvectors, we have

$$AS = S\Lambda,$$

where S is an invertible matrix. Multiplying both sides on the right or the left by S^{-1}, we derive the relations

$$A = S\Lambda S^{-1} \quad \text{and} \quad \Lambda = S^{-1}AS.$$

To remember the order of the S and S^{-1} matrices in these formulas, just remember that A should be multiplied on the right by the eigenvectors placed in the columns of S.

Problems for Lecture 35

1. Prove that two eigenvectors corresponding to distinct eigenvalues are linearly independent.

2. Prove that if the columns of an n-by-n matrix are linearly independent, then the matrix is invertible. (An n-by-n matrix whose columns are eigenvectors corresponding to distinct eigenvalues is therefore invertible.)

Solutions to the Problems

Lecture 36 | Matrix diagonalization (example)

View this lecture on YouTube

Example: Diagonalize the matrix $\mathrm{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

The eigenvalues of A are determined from

$$\det(\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = 0.$$

Solving for λ, the two eigenvalues are given by $\lambda_1 = a + b$ and $\lambda_2 = a - b$. The corresponding eigenvector for λ_1 is found from $(\mathrm{A} - \lambda_1 \mathrm{I})\mathrm{x}_1 = 0$, or

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

and the corresponding eigenvector for λ_2 is found from $(\mathrm{A} - \lambda_2 \mathrm{I})\mathrm{x}_2 = 0$, or

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solving for the eigenvectors and normalizing them, the eigenvalues and eigenvectors are given by

$$\lambda_1 = a + b, \quad \mathrm{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad \lambda_2 = a - b, \quad \mathrm{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The matrix S of eigenvectors can be seen to be orthogonal so that $\mathrm{S}^{-1} = \mathrm{S}^{\mathrm{T}}$. We then have

$$\mathrm{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathrm{S}^{-1} = \mathrm{S}^{\mathrm{T}} = \mathrm{S};$$

and the diagonalization result is given by

$$\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problems for Lecture 36

1. Diagonalize the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Solutions to the Problems

Lecture 37 | Powers of a matrix

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Diagonalizing a matrix facilitates finding powers of that matrix. Suppose that A is diagonalizable, and consider

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1},$$

where in the two-by-two example, Λ^2 is simply

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}.$$

In general, Λ^p has the eigenvalues raised to the power of p down the diagonal, and

$$A^p = S\Lambda^p S^{-1}.$$

Problems for Lecture 37

1. From calculus, the exponential function is sometimes defined from the power series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots.$$

In analogy, the matrix exponential of an n-by-n matrix A can be defined by

$$e^{\mathrm{A}} = \mathrm{I} + \mathrm{A} + \frac{1}{2!}\mathrm{A}^2 + \frac{1}{3!}\mathrm{A}^3 + \dots.$$

If A is diagonalizable, show that

$$e^{\mathrm{A}} = \mathrm{S}e^{\Lambda}\mathrm{S}^{-1},$$

where

$$e^{\Lambda} = \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix}.$$

Solutions to the Problems

Lecture 38 | Powers of a matrix (example)

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Example: Determine a general formula for $\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n$*, where* n *is a positive integer.*

We have previously determined that the matrix can be written as

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Raising the matrix to the nth power, we obtain

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} (a+b)^n & 0 \\ 0 & (a-b)^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

And multiplying the matrices, we obtain

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix}.$$

Problems for Lecture 38

1. Determine $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^n$, where n is a positive integer.

Solutions to the Problems

Practice Quiz | Matrix diagonalization

1. Let λ_1 and λ_2 be distinct eigenvalues of a two-by-two matrix A. Which of the following cannot be the associated eigenvectors?

a) $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

d) $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2. Which matrix is equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{100}$?

a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. Which matrix is equal to e^{I}, where I is the two-by-two identity matrix?

a) $\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Solutions to the Practice quiz

Problem and practice quiz solutions

Solutions to the Problems for Lecture 1

1.

$$a)\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad b)\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \quad c)\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

Solutions to the Problems for Lecture 2

1. $\mathrm{B}-2\mathrm{A}=\begin{pmatrix}0&-4&3\\0&-2&-4\end{pmatrix}$, $3\mathrm{C}-E:$ not defined, $\mathrm{AC}:$ not defined,

$\mathrm{CD}=\begin{pmatrix}11&10\\10&11\end{pmatrix}$, $\mathrm{CB}=\begin{pmatrix}8&-10&-3\\10&-8&0\end{pmatrix}$.

2. $\mathrm{AB}=\mathrm{AC}=\begin{pmatrix}4&7\\8&14\end{pmatrix}$.

3. $\mathrm{AD}=\begin{pmatrix}2&3&4\\2&6&12\\2&9&16\end{pmatrix}$, $\mathrm{DA}=\begin{pmatrix}2&2&2\\3&6&9\\4&12&16\end{pmatrix}$.

4. $[\mathrm{A}(\mathrm{BC})]_{ij}=\sum_{k=1}^{n}a_{ik}[\mathrm{BC}]_{kj}=\sum_{k=1}^{n}\sum_{l=1}^{p}a_{ik}b_{kl}c_{lj}=\sum_{l=1}^{p}\sum_{k=1}^{n}a_{ik}b_{kl}c_{lj}=\sum_{l=1}^{p}[\mathrm{AB}]_{il}c_{lj}=[(\mathrm{AB})\mathrm{C}]_{ij}$.

Solutions to the Problems for Lecture 3

1. In general, matrix multiplication can be written as

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}ax_1+bx_2\\cx_1+dx_2\end{pmatrix}=x_1\begin{pmatrix}a\\c\end{pmatrix}+x_2\begin{pmatrix}b\\d\end{pmatrix}.$$

Let $\mathrm{A}=\begin{pmatrix}-1&2\\4&-8\end{pmatrix}$. Notice that the second column of A is -2 times the first column. Then AB will be zero if the first row of B is 2 times the second row. For example, if we take our second row of B to be $(2\ \ 3)$, then we have

$$\begin{pmatrix}-1&2\\4&-8\end{pmatrix}\begin{pmatrix}4&6\\2&3\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

2. Let A be an m-by-p diagonal matrix, B a p-by-n diagonal matrix, and let C = AB. The ij element of C is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Since A is a diagonal matrix, the only nonzero term in the sum is $k = i$ and we have $c_{ij} = a_{ii} b_{ij}$. And since B is a diagonal matrix, the only nonzero elements of C are the diagonal elements $c_{ii} = a_{ii} b_{ii}$.

3. Let A and B be n-by-n upper triangular matrices, and let C = AB. The ij element of C is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Since A and B are upper triangular, we have $a_{ik} = 0$ when $k < i$ and $b_{kj} = 0$ when $k > j$. Excluding the zero terms from the summation, we have

$$c_{ij} = \sum_{k=i}^{j} a_{ik} b_{kj},$$

which is equal to zero when $i > j$ proving that C is upper triangular. Furthermore, $c_{ii} = a_{ii} b_{ii}$.

Solutions to the Practice quiz: Matrix definitions

1. d. With $a_{ij} = i - j$, we have $a_{11} = a_{22} = 0$, $a_{12} = -1$, and $a_{21} = 1$. Therefore $\mathrm{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2. a. $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

3. b. For upper triangular matrices A and B, $a_{ik} = 0$ when $k < i$ and $b_{kj} = 0$ when $k > j$.

Solutions to the Problems for Lecture 4

1. Let A be an m-by-p matrix, B a p-by-n matrix. We have

$$\left[(\mathrm{AB})^{\mathrm{T}}\right]_{ij} = [\mathrm{AB}]_{ji} = \sum_{k=1}^{p} a_{jk} b_{ki} = \sum_{k=1}^{p} b_{ik}^{\mathrm{T}} a_{kj}^{\mathrm{T}} = \left[\mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\right]_{ij}.$$

Therefore, $(\mathrm{AB})^{\mathrm{T}} = \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$.

2. The square matrix $A + A^T$ is symmetric, and the square matrix $A - A^T$ is skew symmetric. Using these two matrices, we can write

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right).$$

3. Let A be a m-by-n matrix. Then using $(AB)^T = B^T A^T$ and $(A^T)^T = A$, we have

$$(A^T A)^T = A^T A.$$

Solutions to the Problems for Lecture 5

1.

$$A^T A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 & ad + be + cf \\ ad + be + cf & d^2 + e^2 + f^2 \end{pmatrix}.$$

2. Let A be an m-by-n matrix. Then

$$\mathrm{Tr}(A^T A) = \sum_{j=1}^{n} (A^T A)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ji}^T a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2,$$

which is the sum of the squares of all the elements of A.

Solutions to the Problems for Lecture 6

1. $\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -6 \\ -4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 4 \\ 3 & 3 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -4 \\ -3 & 6 \end{pmatrix}$.

2. From the definition of an inverse,

$$(AB)^{-1}(AB) = I, \quad (AB)(AB)^{-1} = I.$$

Either multiply on the right by B^{-1}, and then by A^{-1}, or multiply on the left by A^{-1}, and then by B^{-1}, to obtain

$$(AB)^{-1} = B^{-1} A^{-1}.$$

3. We assume that A is invertible so that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

Taking the transpose of both sides of these two equations, using both $I^T = I$ and $(AB)^T = B^T A^T$, we obtain

$$(AA^{-1})^T = (A^{-1})^T A^T = I^T = I \quad \text{and} \quad (A^{-1}A)^T = A^T (A^{-1})^T = I^T = I.$$

Therefore, since we have found that

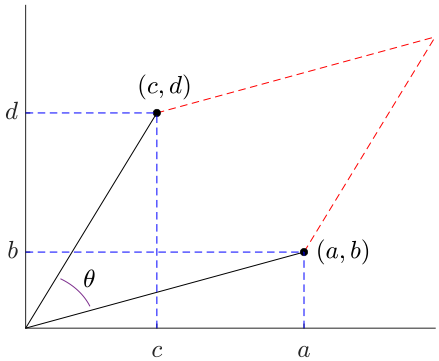
$$(A^{-1})^T A^T = I \quad \text{and} \quad A^T (A^{-1})^T = I,$$

we conclude that A^T is invertible and that $(A^T)^{-1} = (A^{-1})^T$.

4. Let A be an invertible matrix, and suppose B and C are its inverse. To prove that $B = C$, we write

$$B = BI = B(AC) = (BA)C = IC = C.$$

5. Consider the parallelogram drawn below.



The area of the parallelogram is given by

$$\text{Area} = (\text{base}) \times (\text{height}) = \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}\sin\theta.$$

We can use the Pythagorean theorem and the Law of Cosines to determine $\sin\theta$. We have

$$\sin\theta = \sqrt{1 - \cos^2\theta},$$

and

$$(a - c)^2 + (b - d)^2 = (a^2 + b^2) + (c^2 + d^2) - 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}\cos\theta.$$

Simplifying and solving for $\cos\theta$, we find

$$\cos\theta = \frac{(ac + bd)}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}},$$

so that

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{|ad - bc|}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}}.$$

The area of the parallelogram is therefore

$$\text{Area} = \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sin\theta = |ad-bc| = \left|\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right|.$$

Solutions to the Practice quiz: Transpose and inverse

1. d. $(ABC)^T = ((AB)C)^T = C^T(AB)^T = C^TB^TA^T$.

2. c. A symmetric matrix C satisfies $C^T = C$. We can test all four matrices.
$(A + A^T)^T = A^T + A = A + A^T$;
$(AA^T)^T = AA^T$;
$(A - A^T)^T = A^T - A = -(A - A^T)$;
$(A^TA)^T = A^TA$.
Only the third matrix is not symmetric. It is a skew-symmetric matrix, where $C^T = -C$.

3. a. Exchange the diagonal elements, negate the off-diagonal elements, and divide by the determinant. We have $\begin{pmatrix} 2 & 2 \\ 1 & 2\end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -1 & 2\end{pmatrix}$.

Solutions to the Problems for Lecture 7

1. Let Q_1 and Q_2 be orthogonal matrices. Then

$$(Q_1Q_2)^{-1} = Q_2^{-1}Q_1^{-1} = Q_2^TQ_1^T = (Q_1Q_2)^T.$$

2. Since $I\,I = I$, we have $I^{-1} = I$. And since $I^T = I$, we have $I^{-1} = I^T$ and I is an orthogonal matrix.

Solutions to the Problems for Lecture 8

1. $R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} = R(\theta)^{-1}$.

2. The z-coordinate stays fixed, and the vector rotates an angle θ in the x-y plane. Therefore,

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}.$$

Solutions to the Problems for Lecture 9

1.

$$\mathrm{P}_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{P}_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{P}_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{P}_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{P}_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{P}_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.

$$\mathrm{P}_{123}^{-1} = \mathrm{P}_{123}, \quad \mathrm{P}_{132}^{-1} = \mathrm{P}_{132}, \quad \mathrm{P}_{213}^{-1} = \mathrm{P}_{213}, \quad \mathrm{P}_{321}^{-1} = \mathrm{P}_{321},$$
$$\mathrm{P}_{231}^{-1} = \mathrm{P}_{312}, \quad \mathrm{P}_{312}^{-1} = \mathrm{P}_{231}.$$

The matrices that are their own inverses correspond to either no permutation or a single permutation of rows, e.g., $\{1,3,2\}$, which permutes rows two and three. The matrices that are not their own inverses correspond to two permutations, e.g., $\{2,3,1\}$, which permutes rows one and two, and then two and three. For example, commuting rows by left multiplication, we have

$$P_{231} = P_{132}P_{213},$$

so that the inverse matrix is given by

$$P_{231}^{-1} = P_{213}^{-1}P_{132}^{-1} = P_{213}P_{132}.$$

Because matrices in general do not commute, $P_{231}^{-1} \neq P_{231}$. Note also that the permutation matrices are orthogonal, so that the inverse matrices are equal to the transpose matrices. Therefore, only the symmetric permutation matrices can be their own inverses.

Solutions to the Practice quiz: Orthogonal matrices

1. d. An orthogonal matrix has orthonormal rows and columns. The rows and columns of the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ are not orthonormal and therefore this matrix is not an orthogonal matrix.

2. a. The rotation matrix representing a counterclockwise rotation around the x-axis in the y-z plane can be obtained from the rotation matrix representing a counterclockwise rotation around the z-axis in the x-y plane by shifting the elements to the right one column and down one row, assuming a periodic extension of the matrix. The result is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.

3. b. Interchange the rows of the identity matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Solutions to the Problems for Lecture 10

1.

(a) Row reduction of the augmented matrix proceeds as follows:

$$\begin{pmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{pmatrix}.$$

Solution by back substitution is given by

$$\begin{aligned} x_3 &= -6, \\ x_2 &= -\frac{1}{2}(x_3 - 2) = 4, \\ x_1 &= \frac{1}{3}(7x_2 + 2x_3 - 7) = 3. \end{aligned}$$

The solution is therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}.$$

(b) Row reduction of the augmented matrix proceeds as follows:

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ -1 & 3 & -1 & -1 \\ 2 & -5 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Solution by back substitution is given by

$$\begin{aligned} x_3 &= -1, \\ x_2 &= -2x_3 = 2, \\ x_1 &= 2x_2 - 3x_3 + 1 = 8. \end{aligned}$$

The solution is therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 11

1.

(a) Row reduction proceeds as follows:

$$A = \begin{pmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 3/2 & 0 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{pmatrix}$$
$$\to \begin{pmatrix} 3 & 0 & 0 & 9 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & -1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{pmatrix}.$$

Here, columns one, two, and three are pivot columns.

(b) Row reduction proceeds as follows:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, columns one and three are pivot columns.

Solutions to the Problems for Lecture 12

1.

$$\left(\begin{array}{ccc|ccc} 3 & -7 & -2 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 & 1 & 0 \\ 6 & -4 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 3 & -7 & -2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 10 & 4 & -2 & 0 & 1 \end{array}\right) \to$$
$$\left(\begin{array}{ccc|ccc} 3 & 0 & 3/2 & -5/2 & -7/2 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 & 5 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 2 & 4 & 3/2 \\ 0 & -2 & 0 & -2 & -4 & -1 \\ 0 & 0 & -1 & 3 & 5 & 1 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 4/3 & 1/2 \\ 0 & 1 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{array}\right).$$

Therefore,

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}.$$

Solutions to the Practice quiz: Gaussian elimination

1. a. $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -3 & -2 \end{pmatrix}$
$\to \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 \end{pmatrix}$.

2. c. A matrix in reduced row echelon form has all its pivots equal to one, and all the entries above and below the pivots eliminated. The only matrix that is not in reduced row echelon form is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. The pivot in the third row, third column has a one above it in the first row, third column.

3. d. There are many ways to do this computation by hand, and here is one way:
$\begin{pmatrix} 3 & -7 & -2 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 & 1 & 0 \\ 6 & -4 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 10 & 4 & -2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 & 5 & 1 \end{pmatrix}$
$\to \begin{pmatrix} 3 & -7 & 0 & -5 & -10 & -2 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & 0 & -5 & -10 & -2 \\ 0 & -2 & 0 & -2 & -4 & -1 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix}$
$\to \begin{pmatrix} 3 & 0 & 0 & 2 & 4 & 3/2 \\ 0 & 1 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2/3 & 4/3 & 1/2 \\ 0 & 1 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix}$.
Therefore, $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}$.

Solutions to the Problems for Lecture 13

1.
$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 14

1.
$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 15

1. We know

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU.$$

To solve $LUx = b$, we let $y = Ux$, solve $Ly = b$ for y, and then solve $Ux = y$ for x.

(a)

$$b = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$$

The equations $Ly = b$ are given by

$$\begin{aligned} y_1 &= -3, \\ -y_1 + y_2 &= 3, \\ 2y_1 - 5y_2 + y_3 &= 2, \end{aligned}$$

with solution $y_1 = -3$, $y_2 = 0$, and $y_3 = 8$. The equations $Ux = y$ are given by

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= -3 \\ -2x_2 - x_3 &= 0 \\ -x_3 &= 8, \end{aligned}$$

with solution $x_3 = -8$, $x_2 = 4$, and $x_1 = 3$.

(b)

$$b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

The equations Ly = b are given by

$$\begin{aligned} y_1 &= 1, \\ -y_1 + y_2 &= -1, \\ 2y_1 - 5y_2 + y_3 &= 1, \end{aligned}$$

with solution $y_1 = 1$, $y_2 = 0$, and $y_3 = -1$. The equations Ux = y are given by

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= 1 \\ -2x_2 - x_3 &= 0 \\ -x_3 &= -1, \end{aligned}$$

with solution $x_3 = 1$, $x_2 = -1/2$, and $x_1 = -1/6$.

Solutions to the Practice quiz: LU decomposition

1. c. Start with the identity matrix. In the third row (changed row) and second column (the row which is multiplied by 2) , place a 2. The elementary matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

2. b.

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} = M_1 A, \quad \text{where } M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} = M_2 M_1 A, \quad \text{where } M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = M_3 M_2 M_1 A, \quad \text{where } M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}.$$

Therefore,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

3. b. To solve LUx = b, let y = Ux. Then solve Ly = b for y and Ux = y for x. The

equations given by Ly = b are

$$\begin{aligned} y_1 &= 1, \\ -y_1 + y_2 &= -1, \\ 2y_1 - 5y_2 + y_3 &= 1. \end{aligned}$$

Solution by forward substitution gives $\mathrm{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The equations given by Ux = y are

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= 1, \\ -2x_2 - x_3 &= 0, \\ -x_3 &= -1. \end{aligned}$$

Solution by backward substitution gives $\mathrm{x} = \begin{pmatrix} -1/6 \\ -1/2 \\ 1 \end{pmatrix}$.

Solutions to the Problems for Lecture 16

1. Let v be a vector in the vector space. Then both 0v and $\mathrm{v} + (-1)\mathrm{v}$ must be vectors in the vector space and both of them are the zero vector.

2. In all of the examples, the vector spaces are closed under scalar multiplication and vector addition.

Solutions to the Problems for Lecture 17

1.

(a) We place the vectors as the rows of a matrix and compute the reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Because the reduced row echelon form is the identity matrix, the vectors are linearly independent.

(b) We place the vectors as the rows of a matrix and compute the reduced row echelon form:

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Because the reduced row echelon form is the identity matrix, the vectors are linearly independent.

(c) By inspection,

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

The vectors are therefore linearly dependent.

Solutions to the Problems for Lecture 18

1. One possible orthonormal basis is

$$\left\{ \frac{1}{2}\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\-\sqrt{2}\end{pmatrix} \right\}.$$

The dimension of this vector space is two.

Solutions to the Practice quiz: Vector space definitions

1. b. A vector space must be closed under vector addition and scalar multiplication. The set of three-by-one matrices with the sum of all rows equal to one is not a closed set. For example, k times a vector whose sum of all rows is equal to one results in a vector whose sum of all rows is equal to k.

2. d. One can find the relations

$$\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad 8\begin{pmatrix}1\\-1\\2\end{pmatrix} + 3\begin{pmatrix}4\\6\\-2\end{pmatrix} = 10\begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix},$$

so that these sets of three matrices are linearly dependent. The remaining set of vectors can be put in the rows of a matrix and the reduced row echelon form can be computed:

$$\begin{pmatrix}3&2&1\\3&1&2\\2&1&0\end{pmatrix} \to \begin{pmatrix}3&2&1\\0&-1&1\\0&-1/3&-2/3\end{pmatrix} \to \begin{pmatrix}3&0&3\\0&1&-1\\0&0&-1\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Because the reduced row echelon form is the identity matrix, the vectors are linearly independent.

3. b. Since a three-by-one matrix has three degrees of freedom, and the constraint that the sum of all rows equals zero eliminates one degree of freedom, the basis should consist of

two vectors. We can arbitrarily take the first unnormalized vector to be $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. The vector orthogonal to this first vector with sum of all rows equal to zero is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. Normalizing both of these vectors, we get $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$.

Solutions to the Problems for Lecture 19

1.

$$u_4 = v_4 - \frac{(u_1^T v_4)u_1}{u_1^T u_1} - \frac{(u_2^T v_4)u_2}{u_2^T u_2} - \frac{(u_3^T v_4)u_3}{u_3^T u_3}.$$

Solutions to the Problems for Lecture 20

1. Define

$$\{v_1, v_2\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Let $u_1 = v_1$. Then u_2 is found from

$$\begin{aligned} u_2 &= v_2 - \frac{(u_1^T v_2)u_1}{u_1^T u_1} \\ &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Normalizing, we obtain the orthonormal basis

$$\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

2. Define

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Let $u_1 = v_1$. Then u_2 is found from

$$u_2 = v_2 - \frac{(u_1^T v_2)u_1}{u_1^T u_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix};$$

and u_3 is found from

$$u_3 = v_3 - \frac{(u_1^T v_3)u_1}{u_1^T u_1} - \frac{(u_2^T v_3)u_2}{u_2^T u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

Normalizing the three vectors, we obtain the orthonormal basis

$$\{\hat{u}_1, \hat{u}_2, \hat{u}_3\} = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Solutions to the Practice quiz: Gram-Schmidt process

1. a. The vector u_4 is orthogonal to u_1, u_2, and u_3. Since $u_1 = v_1$, then u_4 is orthogonal to v_1.

2. a. Since the vectors are already orthogonal, we need only normalize them to find

$$\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

3. b. Let $u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Then,

$$u_2 = v_2 - \frac{(u_1^T v_2)u_1}{u_1^T u_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

Normalizing the vectors, we have

$$\{\hat{u}_1, \hat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Solutions to the Problems for Lecture 21

1. We bring A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The equation $Ax = 0$ with the pivot variables on the left-hand sides is given by

$$x_1 = -2x_4, \quad x_2 = x_4, \quad x_3 = x_4,$$

and a general vector in the nullspace can be written as $\begin{pmatrix} -2x_4 \\ x_4 \\ x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. A basis for the null space is therefore given by the single vector $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Solutions to the Problems for Lecture 22

1. The system in matrix form is given by

$$\begin{pmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix}.$$

We form the augmented matrix and bring the first four columns to reduced row echelon form:

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The null space is found from the first four columns solving $\mathrm{Au} = 0$, and writing the basic variables on the left-hand side, we have the system

$$u_1 = 2u_2 + u_4, \quad u_3 = -2u_4;$$

from which we can write the general form of the null space as

$$\begin{pmatrix} 2u_2 + u_4 \\ u_2 \\ -2u_4 \\ u_4 \end{pmatrix} = u_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

A particular solution is found by solving $\mathrm{Av} = \mathrm{b}$, and we have

$$v_1 - 2v_2 - v_4 = 3, \quad v_3 + 2v_4 = -2.$$

The free variables v_2 and v_4 can be set to zero, and the particular solution is determined to be $v_1 = 3$ and $v_3 = -2$. The general solution to the underdetermined system of equations is therefore given by

$$\mathrm{x} = a \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}.$$

Solutions to the Problems for Lecture 23

1. We find

$$\mathrm{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad \mathrm{rref}(\mathrm{A}) = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

and $\dim(\mathrm{Col}(\mathrm{A})) = 3$, with a basis for the column space given by the first three columns of A.

Solutions to the Problems for Lecture 24

1. We find the reduced row echelon from of A and A^{T}:

$$\mathrm{A} = \begin{pmatrix} 2 & 3 & -1 & 1 & 2 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 1 & -2 & 3 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A^T = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 3 & -1 & 2 & -2 \\ -1 & 0 & -1 & 3 \\ 1 & -1 & 1 & -1 \\ 2 & 1 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Columns one, two, and four are pivot columns of A and columns one, two, and three are pivot columns of A^T. Therefore, the column space of A is given by

$$\text{Col}(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\};$$

and the row space of A (the column space of A^T) is given by

$$\text{Row}(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The null space of A are found from the equations

$$x_1 = -x_3 + x_5, \quad x_2 = x_3 - 2x_5, \quad x_4 = 2x_5,$$

and a vector in the null space has the general form

$$\begin{pmatrix} -x_3 + x_5 \\ x_3 - 2x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, the null space of A is given by

$$\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

The null space of A^T are found from the equations

$$x_1 = -2x_4, \quad x_2 = 2x_4, \quad x_3 = 5x_4,$$

and a vector in the null space has the general form

$$\begin{pmatrix} -2x_4 \\ 2x_4 \\ 5x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -2 \\ 2 \\ 5 \\ 1 \end{pmatrix}.$$

Therefore, the left null space of A is given by

$$\text{LeftNull}(\text{A}) = \text{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ 5 \\ 1 \end{pmatrix} \right\}.$$

It can be checked that the null space is the orthogonal complement of the row space and the left null space is the orthogonal complement of the column space. The $\text{rank}(\text{A}) = 3$, and A is not of full rank.

Solutions to the Practice quiz: Fundamental subspaces

1. d. To find the null space of a matrix, bring it to reduced row echelon form. We have

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With $x_1 = -2x_2$, $x_3 = 0$, and $x_4 = 0$, a basis for the null space is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

2. b. This system of linear equations is underdetermined, and the solution will be a general vector in the null space of the multiplying matrix plus a particular vector that satisfies the underdetermined system of equations. The linear system in matrix form is given by

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We bring the augmented matrix to reduced row echelon form:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 & 1 \\ 3 & 6 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

A basis for the null space is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, and a particular solution can be found by setting the free variable $x_2 = 0$. Therefore, $x_1 = x_3 = 0$ and $x_4 = 1$, and the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where a is a free constant.

3. c. The matrix in reduced row echelon form is $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The number of pivot columns is three, and this is the rank.

Solutions to the Problems for Lecture 25

1. We first use the Gram-Schmidt process to find an orthonormal basis for W. We assign $u_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Then,

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

An orthonormal basis for W is therefore

$$s_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

The projection of v onto W is then given by

$$v_{\text{proj}_W} = (v^T s_1)s_1 + (v^T s_2)s_2 = \frac{1}{3}(a+b+c) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6}(-2a+b+c) \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

When $a = 1$, $b = c = 0$, we have

$$v_{\text{proj}_W} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix};$$

and when $b = 1$, $a = c = 0$, we have

$$v_{\text{proj}_W} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-2\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Solutions to the Problems for Lecture 26

1.

$$\begin{pmatrix}1&0\\1&1\\1&2\\1&3\end{pmatrix}\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix} = \begin{pmatrix}1\\3\\3\\4\end{pmatrix}$$

Solutions to the Problems for Lecture 27

1. The normal equations are given by

$$\begin{pmatrix}1&1&1&1\\0&1&2&3\end{pmatrix}\begin{pmatrix}1&0\\1&1\\1&2\\1&3\end{pmatrix}\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix} = \begin{pmatrix}1&1&1&1\\0&1&2&3\end{pmatrix}\begin{pmatrix}1\\3\\3\\4\end{pmatrix},$$

or

$$\begin{pmatrix}4&6\\6&14\end{pmatrix}\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix} = \begin{pmatrix}11\\21\end{pmatrix}.$$

The solution is $\beta_0 = 7/5$ and $\beta_1 = 9/10$, and the least-squares line is given by $y = 7/5 + 9x/10$.

Solutions to the Practice quiz: Orthogonal projections

1. b. We first normalize the vectors in W to obtain the orthonormal basis

$$w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \qquad w_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

Then the orthogonal projection of $\mathrm{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ onto W is given by

$$\mathrm{v}_{\mathrm{proj}_W} = (\mathrm{v}^{\mathrm{T}}\mathrm{w}_1)\mathrm{w}_1 + (\mathrm{v}^{\mathrm{T}}\mathrm{w}_2)\mathrm{w}_2 = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

2. d. The overdetermined system of equations is given by

$$\begin{aligned} \beta_0 + \beta_1 x_1 &= y_1, \\ \beta_0 + \beta_1 x_2 &= y_2, \\ \beta_0 + \beta_1 x_3 &= y_3. \end{aligned}$$

Substituting in the values for x and y, and writing in matrix form, we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

3. b. The normal equations are given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Multiplying out, we have

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}.$$

Inverting the two-by-two matrix, we have

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -2 \\ 6 \end{pmatrix}.$$

The best fit line is therefore $y = -\frac{1}{3} + x$.

Solutions to the Problems for Lecture 28

1.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times 1 \times 1 = 1.$$

2.

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = dbi + ecg + fah - fbg - eai - dch$$

$$= -(aei + bfg + cdh - ceg - bdi - afh) = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

3. Let

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\mathrm{AB} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

and

$$\begin{aligned} \det \mathrm{AB} &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= (acef + adeh + bcfg + bdgh) - (acef + adfg + bceh + bdgh) \\ &= (adeh + bcfg) - (adfg + bceh) \\ &= ad(eh - fg) - bc(eh - fg) \\ &= (ad - bc)(eh - fg) \\ &= \det \mathrm{A} \det \mathrm{B}. \end{aligned}$$

Solutions to the Problems for Lecture 29

1. We first expand in minors across the fourth row:

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & 4 & 1 & 0 \\ 8 & -5 & 6 & 7 & -2 \\ -2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 3 & 2 & 0 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ -5 & 6 & 7 & -2 \\ 0 & 3 & 2 & 0 \end{vmatrix}.$$

We then expand in minors down the fourth column:

$$2 \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ -5 & 6 & 7 & -2 \\ 0 & 3 & 2 & 0 \end{vmatrix} = 4 \begin{vmatrix} 3 & 2 & 4 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \end{vmatrix}.$$

Finally, we expand in minors down the first column:

$$4\begin{vmatrix} 3 & 2 & 4 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \end{vmatrix} = 12\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 60.$$

Solutions to the Problems for Lecture 30

1. For each element chosen from the first row, there is only a single way to choose nonzero elements from all subsequent rows. Considering whether the columns chosen are even or odd permutations of the ordered set $\{1,2,3,4\}$, we obtain

$$\begin{vmatrix} a & b & c & d \\ e & f & 0 & 0 \\ 0 & g & h & 0 \\ 0 & 0 & i & j \end{vmatrix} = afhj - behj + cegj - degi.$$

Solutions to the Problems for Lecture 31

1. Suppose the square matrix A has two equal rows. If we interchange these two rows, the determinant of A changes sign according to Property 2, even though A doesn't change. Therefore, $\det A = -\det A$, or $\det A = 0$.

2. To prove that the determinant is a linear function of row i, interchange rows 1 and row i using Property 2. Use Property 3, then interchange rows 1 and row i again.

3. Consider a general n-by-n matrix. Using the linear property of the jth row, and that a matrix with two equal rows has zero determinant, we have

$$\begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1}+ka_{i1} & \dots & a_{jn}+ka_{in} \\ \vdots & \ddots & \vdots \end{vmatrix} = \begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \end{vmatrix} + k\begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \end{vmatrix} = \begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \end{vmatrix}.$$

Therefore, the determinant doesn't change by adding k times row-i to row-j.

4.

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & 5/2 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & 5/2 \\ 0 & 0 & 7/2 \end{vmatrix} = 2 \times 1 \times 7/2 = 7.$$

Solutions to the Practice quiz: Determinants

1. a. To find the determinant of a matrix with many zero elements, perform a Laplace expansion across the row or down the column with the most zeros. Choose the correct sign. We have for one expansion choice,

$$\begin{vmatrix} -3 & 0 & -2 & 0 & 0 \\ 2 & -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 3 & 0 & -3 & 2 & -3 \\ -3 & 3 & 3 & 0 & -2 \end{vmatrix} = 2 \begin{vmatrix} -3 & 0 & -2 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ -3 & 3 & 3 & -2 \end{vmatrix} = -4 \begin{vmatrix} -3 & 0 & -2 \\ 2 & -2 & -2 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} -3 & 0 \\ 2 & -2 \end{vmatrix} = 48.$$

2. b. We can apply the Leibniz formula by going down the first column. For each element in the first column there is only one possible choice of elements from the other three columns. We have

$$\begin{pmatrix} a & e & 0 & 0 \\ b & f & g & 0 \\ c & 0 & h & i \\ d & 0 & 0 & j \end{pmatrix} = afhj - behj + cegj - degi.$$

The signs are obtained by considering whether the following permutations of the rows $\{1,2,3,4\}$ are even or odd: $afhj = \{1,2,3,4\}$ (even); $behj = \{2,1,3,4\}$ (odd); $cegj = \{3,1,2,4\}$ (even); $degi = \{4,1,2,3\}$ (odd).

3. c. The only identity which is false is $\det(A+B) = \det A + \det B$.

Solutions to the Problems for Lecture 32

1. The characteristic equation is given by

$$0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix}$$

$$= (a-\lambda)(e-\lambda)(i-\lambda) + bfg + cdh - c(e-\lambda)g - bd(i-\lambda) - (a-\lambda)fh$$

$$= -\lambda^3 + (a+e+i)\lambda^2 - (ae+ai+ei-bd-cg-fh)\lambda + aei + bfg + cdh - ceg - bdi - afh.$$

The result can be made more memorable if we recall that

$$\mathrm{Tr}\{A\} = a+e+i, \qquad \det A = aei + bfg + cdh - ceg - bdi - afh.$$

The coefficient of the λ term can be rewritten as

$$ae + ai + ei - bd - cg - fh = (ei - fh) + (ai - cg) + (ae - bd) = \begin{vmatrix} e & f \\ h & i \end{vmatrix} + \begin{vmatrix} a & c \\ g & i \end{vmatrix} + \begin{vmatrix} a & b \\ d & e \end{vmatrix},$$

which are the sum of the minors obtained by crossing out the rows and columns of the diagonal elements. The cubic equation in more memorable form is therefore

$$\lambda^3 - \mathrm{Tr}\{\mathrm{A}\}\lambda^2 + \sum(\text{minors of the diagonal elements of A})\lambda - \det \mathrm{A} = 0.$$

Solutions to the Problems for Lecture 33

1. Let $\mathrm{A} = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$. The eigenvalues of A are found from

$$0 = \det(\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 49.$$

Therefore, $2 - \lambda = \pm 7$, and the eigenvalues are $\lambda_1 = -5$, $\lambda_2 = 9$. The eigenvector for $\lambda_1 = -5$ is found from

$$\begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

or $x_1 + x_2 = 0$. The eigenvector for $\lambda_2 = 9$ is found from

$$\begin{pmatrix} -7 & 7 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

or $x_1 - x_2 = 0$. The eigenvalues and corresponding eigenvectors are therefore given by

$$\lambda_1 = -5, \quad \mathrm{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \lambda_2 = 9, \quad \mathrm{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

2. Let $\mathrm{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. The eigenvalues of A are found from

$$0 = \det(\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)^2 - 2\right).$$

Therefore, $\lambda_1 = 2$, $\lambda_2 = 2 - \sqrt{2}$, and $\lambda_3 = 2 + \sqrt{2}$. The eigenvector for $\lambda_1 = 2$ are found

from

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

or $x_2 = 0$ and $x_1 + x_3 = 0$, or $x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The eigenvector for $\lambda_2 = 2 - \sqrt{2}$ is found from

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Gaussian elimination gives us

$$\text{rref} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $x_1 = x_3$ and $x_2 = -\sqrt{2}x_3$ and an eigenvector is $x_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$. Similarly, the third eigenvector is $x_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

Solutions to the Problems for Lecture 34

34. Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The eigenvalues of A are found from

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1.$$

Therefore, $1 - \lambda = \pm i$, and the eigenvalues are $\lambda_1 = 1 - i$, $\lambda_2 = 1 + i$.

Solutions to the Practice quiz: The eigenvalue problem

1. b. The characteristic equation $\det(A - \lambda I) = 0$ for a two-by-two matrix A results in the quadratic equation $\lambda^2 - \text{Tr}A\,\lambda + \det A = 0$, which for the given matrix yields $\lambda^2 - 3\lambda + 1 = 0$. Application of the quadratic formula results in $\lambda_\pm = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$.

2. d. The characteristic equation is $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 1 = 0$. With $i = \sqrt{-1}$, the solution is $\lambda_\pm = 3 \pm i$.

3. b. One can either compute the eigenvalues and eigenvectors of the matrix, or test the given possible answers. If we test the answers, then only one is an eigenvector, and we have

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ 2+2\sqrt{2} \\ \sqrt{2}+2 \end{pmatrix} = (2+\sqrt{2}) \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 35

1. Let λ_1 and λ_2 be distinct eigenvalues of A, with corresponding eigenvectors x_1 and x_2. Write

$$c_1x_1 + c_2x_2 = 0.$$

To prove that x_1 and x_2 are linearly independent, we need to show that $c_1 = c_2 = 0$. Multiply the above equation on the left by A and use $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$ to obtain

$$c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0.$$

From the first equation, we write $c_2x_2 = -c_1x_1$, and eliminating c_2x_2 from the second equation we obtain

$$(\lambda_1 - \lambda_2)c_1x_1 = 0.$$

From the first equation, we can also write $c_1x_1 = -c_2x_2$, and eliminating c_1x_1 from the second equation we obtain

$$(\lambda_2 - \lambda_1)c_2x_2 = 0.$$

Therefore, if $\lambda_1 \neq \lambda_2$, then $c_1 = c_2 = 0$ and x_1 and x_2 are linearly independent.

2. Let A be an n-by-n matrix. We have

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Null}(A)) = n.$$

Since the columns of A are linearly independent, we have

$$\dim(\mathrm{Col}(A)) = n \quad \text{and} \quad \dim(\mathrm{Null}(A)) = 0.$$

If the only solution to $Ax = 0$ is the zero vector, then $\det A \neq 0$ and A is invertible.

Solutions to the Problems for Lecture 36

1. The eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ are

$$\lambda_1 = 2,\ x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_2 = 2-\sqrt{2},\ x_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}; \quad \lambda_3 = 2+\sqrt{2},\ x_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Notice that the three eigenvectors are mutually orthogonal. This will happen when the matrix is symmetric. If we normalize the eigenvectors, the matrix with eigenvectors as columns will be an orthogonal matrix. Normalizing the orthogonal eigenvectors (so that $S^{-1} = S^T$) , we have

$$S = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}.$$

We therefore find

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}$$
$$= \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}.$$

Solutions to the Problems for Lecture 37

1.

$$\begin{aligned} e^A &= e^{S\Lambda S^{-1}} \\ &= I + S\Lambda S^{-1} + \frac{S\Lambda^2 S^{-1}}{2!} + \frac{S\Lambda^3 S^{-1}}{3!} + \dots \\ &= S\left(I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \dots\right) S^{-1} \\ &= S e^{\Lambda} S^{-1}. \end{aligned}$$

Because Λ is a diagonal matrix, the powers of Λ are also diagonal matrices with the diagonal elements raised to the specified power. Each diagonal element of e^{Λ} contains a power series of the form

$$1 + \lambda_i + \frac{\lambda_i^2}{2!} + \frac{\lambda_i^3}{3!} + \dots,$$

which is the power series for e^{λ_i}.

Solutions to the Problems for Lecture 38

1. We use the result

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n = \frac{1}{2} \begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix}$$

to find

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}.$$

Solutions to the Practice quiz: Matrix diagonalization

1. c. Eigenvectors with distinct eigenvalues must be linearly independent. All the listed pairs of eigenvectors are linearly independent except $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where $x_2 = -x_1$.

2. d. A simple calculation shows that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Therefore,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{100} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \right)^{50} = I^{50} = I.$$

A more complicated calculation diagonalizes this symmetric matrix. The eigenvalues and orthonormal eigenvectors are found to be

$$\lambda_1 = 1,\ v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -1,\ v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The diagonalization then takes the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{100} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{100} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = I.$$

3. a. We have

$$e^I = I + I + \frac{I^2}{2!} + \frac{I^3}{3!} + \cdots = I\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) = Ie^1 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}.$$