Trigonometry

VERSION 2

A Partnership Between Institutions in the Utah System of Higher Education

Salt Lake Community College
University of Utah
Weber State University
Utah Valley University

Acknowledgements

The development of this OER trigonometry textbook was initiated by Salt Lake Community College to provide its students with a low-cost textbook. Early on, the University of Utah joined the project. Weber State University has been assisting with the revision phase since it began in 2019, and Utah Valley University has more recently joined in the development of this textbook. Collectively, a rigorous text has been developed by these institutions. The body of the text conforms to the Trigonometry Learning Outcomes identified by the Utah System of Higher Education.

The content of this textbook has been largely adapted from *College Trigonometry, 3rd Corrected Edition*, authored by Carl Stitz and Jeff Zeager. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. We appreciate the time and effort they invested in creating and sharing their textbook. The reader will note there are many places where we have diverged significantly from their original text. We hope our perspective contributes beneficially to the world of OER mathematics material. Additional content, primarily examples and exercises, are from *Algebra and Trigonometry*, by Jay Abramson, et al. It is licensed under a Creative Commons Attribution 4.0 International License and can be found at https://openstax.org.

The Salt Lake Community College Math Department began this project in 2016 with Ruth Trygstad taking the lead, joined by Shawna Haider and Spencer Bartholomew. University of Utah was represented by Maggie Cummings on the original team. The Pilot Edition of the textbook was completed in 2017, and adopted for use by the mathematics departments of both Salt Lake Community College and University of Utah.

The second version of the textbook is the result of many hours of collaborative revisions, reviews, and meetings beginning in 2019 with completion in Summer 2023. This current revision has been led by Ruth Trygstad and Afshin Ghoreishi (Weber State University). Other team members include Michael van Opstall (University of Utah), Spencer Bartholomew, Sarah Featherstone (Kearns High School), Kyle Costello (SLCC), Piotr Runge (SLCC), and Robert Woodward (SLCC). In Spring 2023, Harish Bhatt (Utah Valley University) joined the team. Jie Gu (SLCC) coordinated the alignment and creation of accompanying online homework exercises. Additionally, Rebecca Noonan-Heale (University of Utah), Robert Woodward, and Shawna Haider developed the online homework that aligns with this textbook. Marilyn Keir (University of Utah) and Kelly MacArthur (University of Utah) created video lectures. Camille Paxman (SLCC) developed the comprehensive textbook index. Many other faculty and staff contributed and their efforts are much appreciated.

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CHAPTER 1
FOUNDATIONS OF TRIGONOMETRY

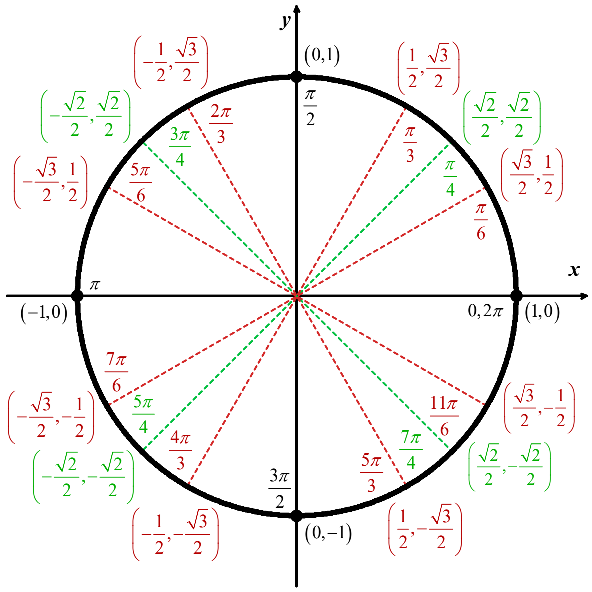


Figure 1.0. 1

Chapter Outline

Introduction

Chances are good that you have some experience with Geometry, which is the study of shapes and surfaces (generally in 1, 2, or 3 dimensions). Trigonometry narrows that focus to the study of triangles – particularly, the properties and relationships of sides and angles – and how these relationships can help define attributes of curves. In this chapter, six basic trigonometric functions are introduced. These functions relate measurements of sides and angles, with an interesting connection to circles. While six new functions may sound like a lot to learn, these functions exhibit numerous interconnecting properties so they will be simple to remember and utilize.

Section 1.1 focuses on angles – what they are, how they are measured, and how they relate to rotation. Some of the terminology here may seem familiar, while some may be new. An important note – in deciding whether you prefer degree or radian measures, keep in mind that both have important uses in the world. Don't completely embrace one while rejecting the other.

Trigonometric ratios are presented in Section 1.2, and that definition is extended in Section 1.3 as we move slowly to defining and regarding these ratios as 'Trigonometric Functions'! After the trigonometric ratios (as they relate to right triangles) are presented in Section 1.2, the sine and cosine functions will be defined for all angles in Section 1.3 using the Unit Circle. Then, Section 1.4 completes the definition of trigonometric functions, using the Unit Circle, by introducing tangent, cosecant, secant, and cotangent functions. Section 1.5 explores connections among these functions and develops the Pythagorean identities. Lastly, Section 1.6 expands the connection of trigonometric functions and the Unit Circle to circles of any radius. This general notion of trigonometric functions will be critical to exploring practical applications of trigonometric functions.

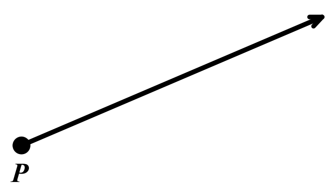
1.1 Degree and Radian Measure of Angles

Learning Objectives

- **Define revolution, degree, and radian measures of angles.**
- **Convert between revolutions, degrees, and radians.**
- **Determine coterminal angle measures.**
- **Determine supplementary and complementary angle measures.**
- **Graph angles in standard position.**

We begin with some basic definitions from Geometry. A **ray**, often described as a half-line, is a subset of a line that contains a point P along with all points lying to one side of P. The point P from which the ray originates is the **initial point** of the ray. The arrowhead indicates that the ray goes on forever.

Figure 1.1. 1



When two rays share a common initial point, they form an **angle** and the common initial point is called the **vertex** of the angle. Following are two angles, the first with vertex *R* and the second with vertex *S*.

Figure 1.1. 2

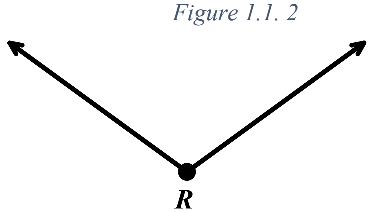
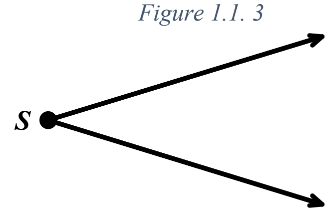


Figure 1.1. 3



The two figures below also depict angles. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**. In the second, the rays are identical, so the angle is indistinguishable from the ray itself.

Figure 1.1. 4

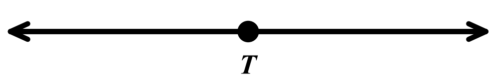
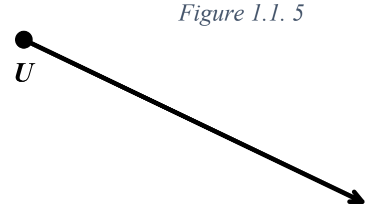


Figure 1.1. 5



The **measure of an angle** is a number that indicates the amount of circular rotation that separates the rays of the angle. There is one immediate problem with this, as the following pictures indicate.

Figure 1.1. 6

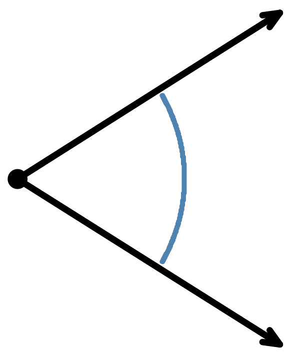
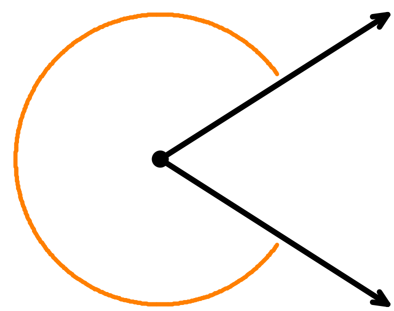
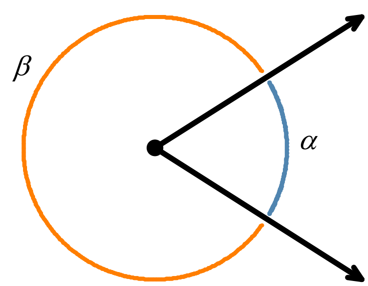


Figure 1.1. 7



Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.[1] Clearly, these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. We often use lower case Greek letters such as α (alpha), β (beta), γ (gamma), and θ (theta) to label angles. For instance, we have

Figure 1.1. 8



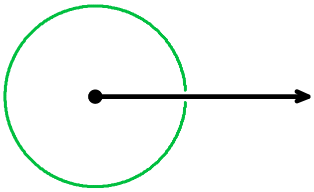
An angle measure refers to a portion of a full revolution, or full rotation. An angle measure may also be stated in units of degrees or radians.

Revolution Measure

The simplest way to describe an angle measure, or the amount of circular rotation, is to describe it as the proportion of one full revolution, 'Rev' for short. The following angles show measures of one revolution, half a revolution, and a quarter of a revolution.

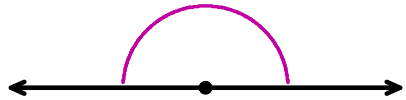
[1] The phrase 'at least' will be justified in short order.

Figure 1.1. 9



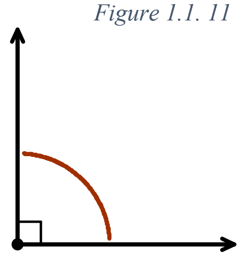
1 Rev

Figure 1.1. 10



$\frac{1}{2}$ Rev

Figure 1.1. 11



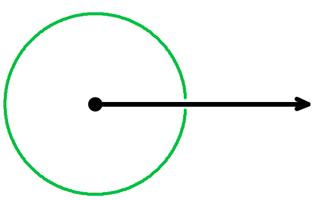
$\frac{1}{4}$ Rev

As seen in the previous figure, we use the small square to denote a right angle, as is commonplace in Geometry. We can determine the revolution measure of any angle if we know the proportion it represents of a full revolution.

Degree Measure

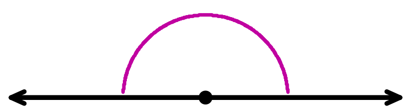
One commonly used system to measure angles is **degree measure**. The measure of an angle in degrees is denoted by a small circle displayed as a superscript. One complete revolution is 360 degrees, 360°, or $1^\circ = \frac{1}{360}$ Rev.[2] Degree measures of angles are determined by their proportion of $1 \text{ Rev} = 360^\circ$. Thus, half of a revolution (a straight angle) measures $\frac{1}{2}\left(360^\circ\right) = 180^\circ$, a quarter of a revolution measures $\frac{1}{4}\left(360^\circ\right) = 90^\circ$, and so on.

Figure 1.1. 12



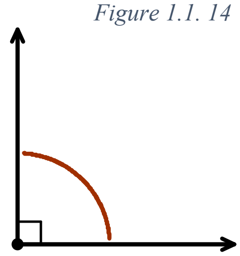
$1 \text{ Rev} = 360^\circ$

Figure 1.1. 13



$\frac{1}{2}\left(360^\circ\right) = 180^\circ$

Figure 1.1. 14



$\frac{1}{4}\left(360^\circ\right) = 90^\circ$

As seen in the previous figure, a right angle measures 90°. If an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between 90° and 180° it is called an **obtuse angle**. An angle with no rotation is measured as 0°. We can determine the degree measure of any angle if we know the proportion it represents of a full revolution or 360°.[3]

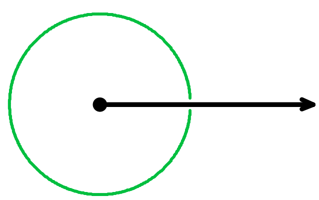
[2] The choice of 360 is most often attributed to the Babylonians.
[3] This is how a protractor is graded.

Radian Measure

Before defining radian measure, we revisit the number $\pi = 3.14159\ldots$. The circumference C of a circle of radius r is $C = 2\pi r$. This tells us that for any circle, the ratio of its circumference to its radius is a constant and that constant is 2π.

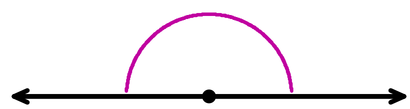
In **radian measure** of angles, one complete revolution is 2π radians, or 1 radian is $\frac{1}{2\pi}$ Rev. Thus, half of a revolution measures $\frac{1}{2}(2\pi) = \pi$, a quarter of a revolution measures $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so on. As explained later, no symbol is used to denote radians. Any number written as an angle measure is assumed to be in radians, but we may include the identifier 'radians' for clarity.

Figure 1.1. 15



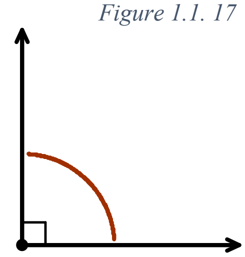
$1 \text{ Rev} = 360^\circ = 2\pi$

Figure 1.1. 16



$\frac{1}{2}(2\pi) = \pi$ radians

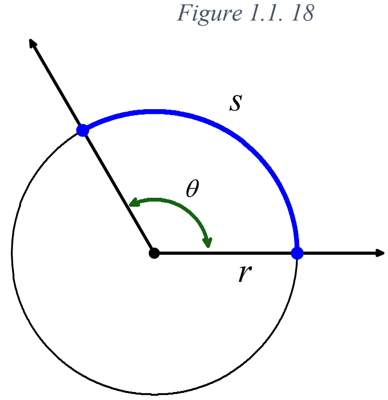
Figure 1.1. 17



$\frac{1}{4}(2\pi) = \frac{\pi}{2}$

Now we develop a way to visualize an angle with any radian measure. First, an angle with its vertex at the center of a circle is called a **central angle**. For a central angle, there is a corresponding arc on the circle, and we say that the central angle is subtended by that arc. Consider a central angle measuring θ radians, or $\frac{\theta}{2\pi}$ Rev, in a circle of radius r where the central angle is subtended by an arc of length s.

Figure 1.1. 18



We can find s as follows. Since our angle is $\frac{\theta}{2\pi}$ of one revolution, then s is also $\frac{\theta}{2\pi}$ of the arc length for one revolution, which is the circumference of the circle.

$$\text{circumference} = \text{arc length for central angle of 1 Rev} = 2\pi r$$

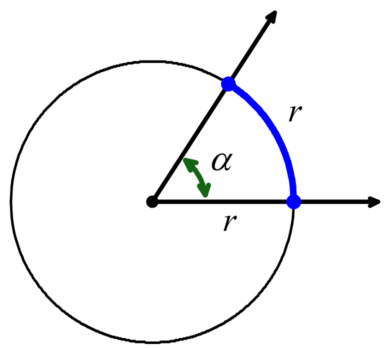
$$s = \text{arc length for central angle of } \frac{\theta}{2\pi} \text{ Rev} = \frac{\theta}{2\pi}(2\pi r) = \theta r$$

So, $s = \theta r$ where θ is in radians. We have the following.

Definition 1.1. The **radian measure** θ of a central angle is the ratio of the length s of the arc subtending that angle to the radius r of the circle: $\theta = \frac{s}{r}$.

For a central angle α of measure 1 radian, $s = 1r = r$. That is, an angle of 1 radian is the angle subtended by an arc with length equal to the radius of the circle.

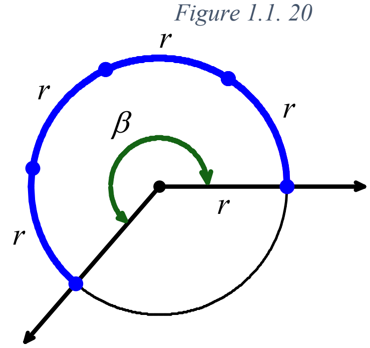
Figure 1.1. 19



α has radian measure 1

According to the relationship $s = \theta r$, the radian measure of an angle θ tells us how many 'radius lengths' we need to sweep out along the circle to subtend the angle θ. For example, an angle β of radian measure 4 is subtended by an arc of length 4 times the radius.

Figure 1.1. 20



β has radian measure 4

The reason no symbol is used to denote radian measure is that $\theta = \frac{s}{r}$, being the ratio of two lengths, is just a number. As with degree measure, the distinction between the angle itself and its measure is often blurred in practice so that when we write $\theta = \frac{\pi}{2}$, we mean 'θ is an angle that measures $\frac{\pi}{2}$ radians'.

Supplementary and Complementary Angles

Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°, or π radians. Two acute angles are called **complementary angles** if their measures add to 90°, or $\frac{\pi}{2}$ if the angles are measured in radians. In the diagrams below, the angles α and β are supplementary angles while the pair γ and φ are complementary angles.

Figure 1.1. 21

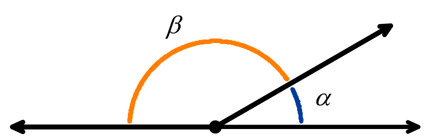
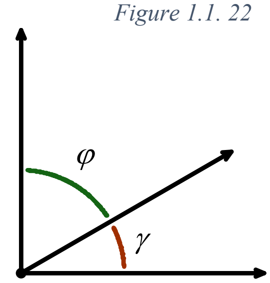


Figure 1.1. 22



The Greek letter phi, φ, is pronounced 'fee', or 'fie' to avoid possible confusion.[4]

Example 1.1.1.

1. Find a supplementary angle for $\alpha = 111^\circ$.
2. Find a complementary angle for $\beta = \frac{\pi}{5}$.

Solution.

1. To find a supplementary angle for $\alpha = 111^\circ$, we seek an angle θ so that $\alpha + \theta = 180^\circ$. Then

$$\begin{aligned}\theta &= 180^\circ - \alpha \\ &= 180^\circ - 111^\circ \\ &= 69^\circ\end{aligned}$$

2. To find a complementary angle for $\beta = \frac{\pi}{5}$, we seek an angle γ so that $\beta + \gamma = \frac{\pi}{2}$. We get

[4] The symbol φ, or ϕ, represents the small Greek letter phi. We will occasionally use the symbol Φ to represent the uppercase Greek letter phi.

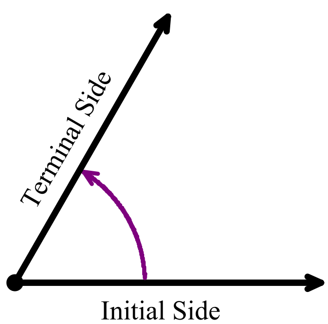
$$\begin{aligned}\gamma &= \frac{\pi}{2} - \beta \\ &= \frac{\pi}{2} - \frac{\pi}{5} \\ &= \frac{3\pi}{10}\end{aligned}$$

□

Oriented Angles

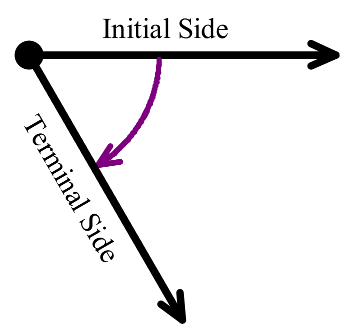
Up to this point, we have discussed only angles that measure between 0 Rev and 1 Rev, between 0° and 360°, or between 0 radians and 2π radians, inclusive. Ultimately, we want to extend their applicability to other real-world phenomena. A first step in this direction is to introduce the concept of an **oriented angle**. As its name suggests, for an oriented angle, the direction of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown below. When the rotation is counter-clockwise from initial side to terminal side, we say that the angle measure is **positive**; when the rotation is clockwise, we say the angle measure is **negative**.

Figure 1.1. 23



A positive angle, 60°

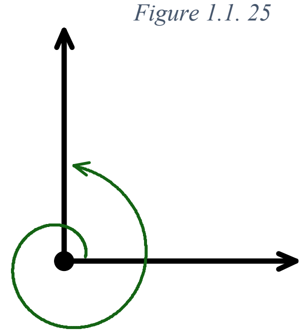
Figure 1.1. 24



A negative angle, −60°

We also extend our allowable rotations to include angles that encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the first 360°), then continue with an additional 90° counter-clockwise rotation, as follows.

Figure 1.1. 25



450°

Converting Between Revolutions, Degrees and Radians

For converting between angle measures, we use the fact that one revolution is 360°, or 2π radians.

Converting Between Angle Measures

- To convert degree measure to revolution measure, divide by 360. To convert revolution measure to degree measure, multiply by 360.
- To convert radian measure to revolution measure, divide by 2π. To convert revolution measure to radian measure, multiply by 2π.
- To convert degree measure to radian measure, multiply by $\frac{\pi}{180}$.
- To convert radian measure to degree measure, multiply by $\frac{180}{\pi}$.

Example 1.1.2. Convert the following measures.

1. 30° to revolutions
2. $\frac{1}{6}$ revolution to radians
3. 60° to radians
4. $-\frac{5\pi}{4}$ radians to degrees
5. 1 radian to degrees

Solution.

1. To convert 30° to revolutions, we simply divide by the number of degrees in one revolution.

$$30^\circ = \frac{30}{360}\text{ Rev} = \frac{1}{12}\text{ Rev}$$

2. To convert $\frac{1}{6}$ Rev to radians, we multiply by the number of radians in one revolution.

$$\frac{1}{6}\text{ Rev} = \frac{1}{6}(2\pi)\text{ radians} = \frac{\pi}{3}\text{ radians}$$

3. To convert 60° to radians, we multiply by $\frac{\pi}{180}$.

$$60^\circ = 60\left(\frac{\pi}{180}\right)\text{ radians} = \frac{\pi}{3}\text{ radians}$$

4. To convert $-\frac{5\pi}{4}$ radians to degrees, we multiply by $\frac{180}{\pi}$.

$$-\frac{5\pi}{4}\text{ radians} = \left(-\frac{5\pi}{4}\right)\left(\frac{180}{\pi}\right)\text{ degrees} = -225^\circ$$

   The negative sign indicates clockwise rotation in both systems and is carried along accordingly.

5. $1\text{ radian} = (1)\left(\frac{180}{\pi}\right)\text{ degrees} = \frac{180}{\pi}\text{ degrees}$. This is approximately equal to 57.3°.

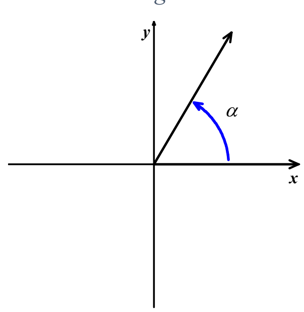
□

In **Example 1.1.2**, we showed that one radian is slightly less than 60°. This may serve as a handy reference for graphing or visualizing angles with radian measures, such as 2 radians or −1 radian.

Standard Position

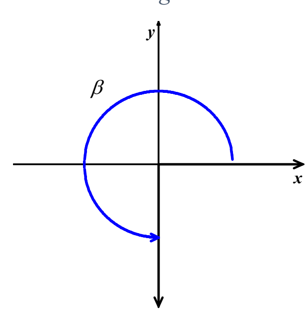
To connect angles with the algebra that has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is in **standard position** if its vertex is the origin and its initial side coincides with the positive side of the *x*-axis. We classify angles in standard position according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a Quadrant I angle. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**.

Figure 1.1. 26



Quadrant I Angle

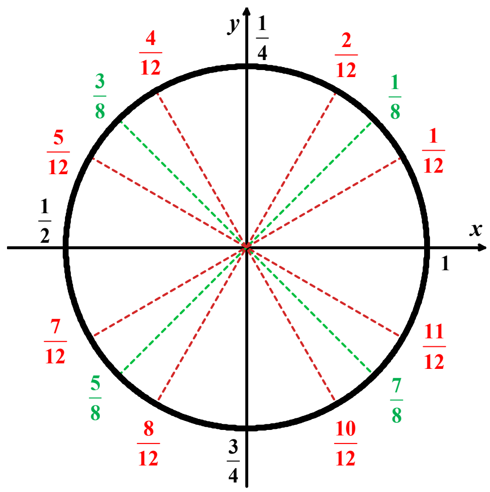
Figure 1.1. 27



Quadrantal Angle

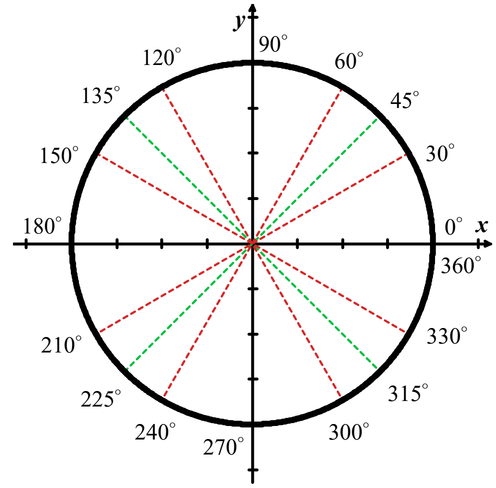
The following angles, in standard position, show locations of terminal sides for angles measured in twelfths and eighths of a revolution, followed by corresponding measures in degrees and radians.

Figure 1.1. 28



Angle Measures in Revolutions

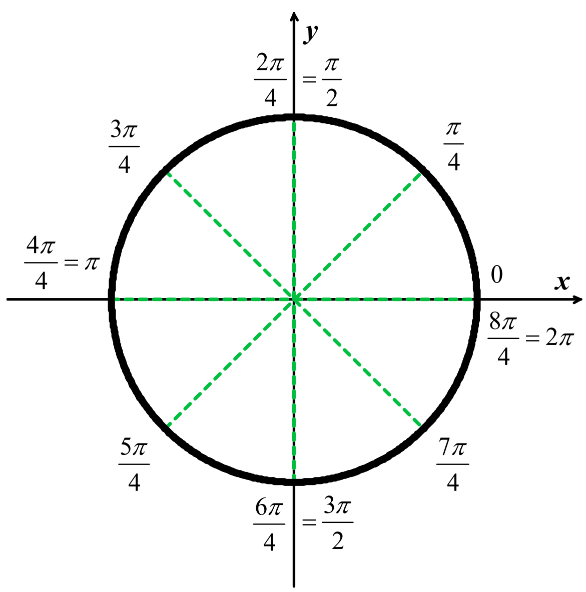
Figure 1.1. 29



Angles in Degrees: Multiply revolutions by 360°.

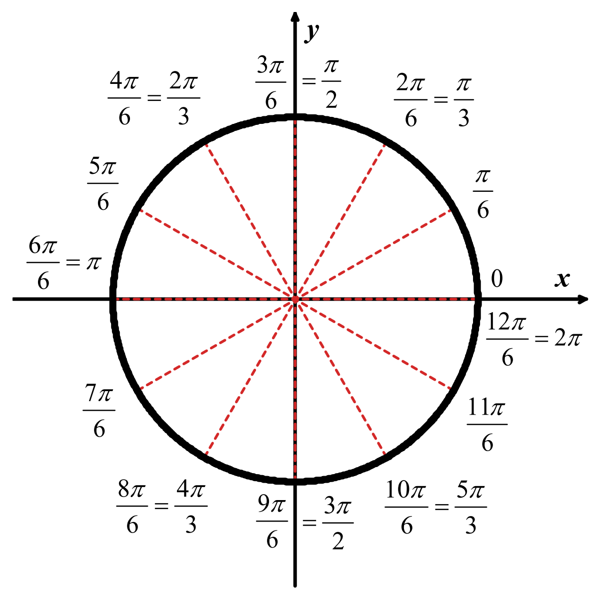
While the reader is likely familiar with degree measure, radian measure is the focus of this textbook in preparation for Calculus. To determine radian equivalents for angles measured in revolutions, we multiply the revolution measure by 2π, as shown below for increments of $1/8$ or $1/12$ revolution.

Figure 1.1. 30



Radian Measure: Increments of $1/8$ Revolution

Figure 1.1. 31



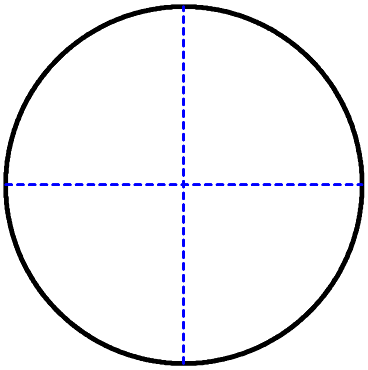
Radian Measure: Increments of $1/12$ Revolution

Before moving on, we note that to sketch the above angles by hand, we could first divide the circle into four equal sectors, or quarter circles, then

- divide each quarter circle into two equal sectors to get portions that are one-eighth of a circle. (Each portion measures 45° or $\pi/4$ radians.)

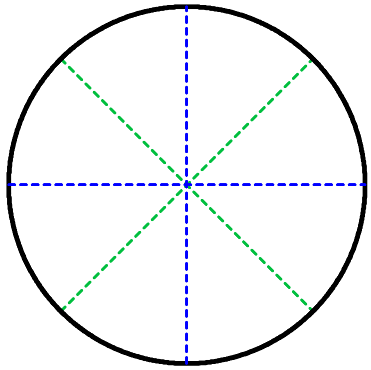
- divide each quarter circle into three equal sectors for portions that are one-twelfth of a circle. (Each portion measures 30° or $\pi/6$ radians.)

Figure 1.1. 32



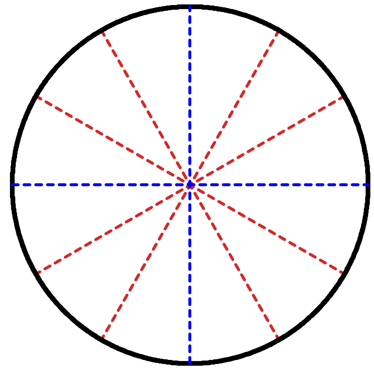
$\frac{1}{4}\text{ Rev} = 90^\circ = \frac{\pi}{2}$

Figure 1.1. 33



$\frac{1}{8}\text{ Rev} = 45^\circ = \frac{\pi}{4}$

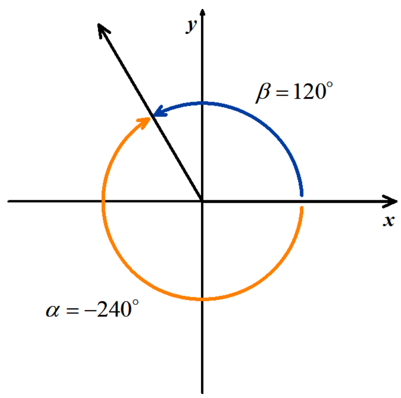
Figure 1.1. 34



$\frac{1}{12}\text{ Rev} = 30^\circ = \frac{\pi}{6}$

Two angles in standard position are called **coterminal** if they share the same terminal side.[5] In the following figure, $\alpha = -240^\circ$ and $\beta = 120^\circ$ are two coterminal Quadrant II angles[6] drawn in standard position. Note that $\beta = \alpha + 360^\circ$, or equivalently $\alpha = \beta - 360^\circ$. We leave it as an exercise for the reader to verify that coterminal angles always differ by a multiple of 360°. More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360^\circ \cdot k$ where k is an integer.[7]

Figure 1.1. 35



[5] Note that by being in standard position they automatically share the same initial side, which is the positive x-axis.
[6] In practice, the distinction between the angle itself and its measure is blurred so that the statement 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42^\circ$'.
[7] Recall that this means $k = 0, \pm 1, \pm 2, \ldots$

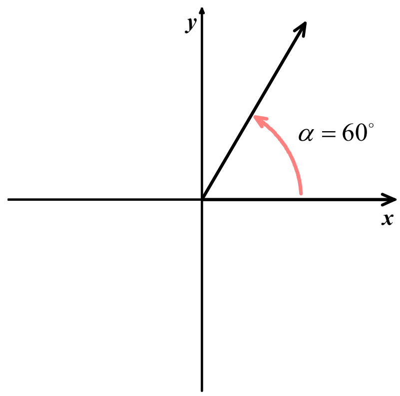
Example 1.1.3. Graph each of the (oriented) angles in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = 60^\circ$ 2. $\beta = -225^\circ$ 3. $\gamma = \frac{9\pi}{4}$ 4. $\varphi = -\frac{5\pi}{2}$

Solution.

1. To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{60^\circ}{360^\circ} = \frac{1}{6}$, or $\frac{2}{12}$, of a revolution. (To locate the terminal side by hand, we divide the first quadrant into three equal 30° pieces. Our 60° angle includes the two pieces closest to the positive x-axis.) We see that α is a Quadrant I angle.

   Figure 1.1. 36

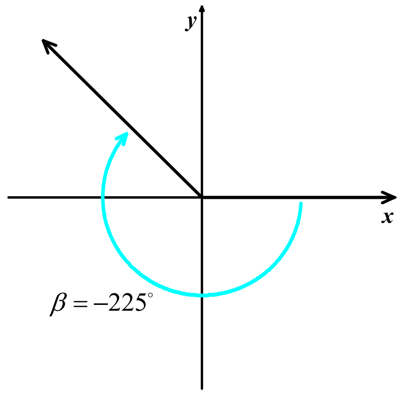
   

   To find angles that are coterminal, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$ for some integer k.

   - When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$.
   - Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$.
   - If we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

2. Since $\beta = -225^\circ$ is negative, we start at the positive x-axis and rotate clockwise $\frac{225^\circ}{360^\circ} = \frac{5}{8}$ of a revolution. (To locate the terminal side by hand, since $-225^\circ = -180^\circ - 45^\circ$, we divide the second quadrant into two equal 45° pieces. Our −225° angle terminates between these two pieces.) We see that β is a Quadrant II angle.

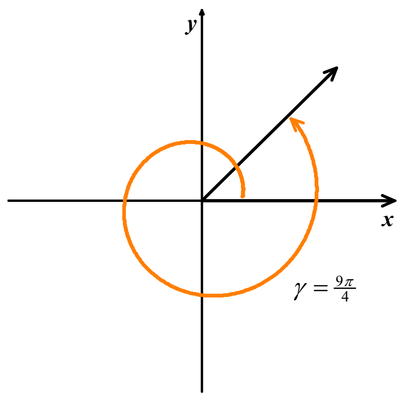
Figure 1.1. 37



To find coterminal angles, we proceed as before and compute $\theta = -225^\circ + 360^\circ \cdot k$ for integer values of k. Letting $k = 1$, $k = -1$, and $k = 2$, we find 135°, −585°, and 495° are all coterminal with −225°.

3. Since $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. (To draw the terminal side by hand, divide the first quadrant into two equal $\frac{\pi}{4}$ pieces.) We have γ as a Quadrant I angle.

Figure 1.1. 38



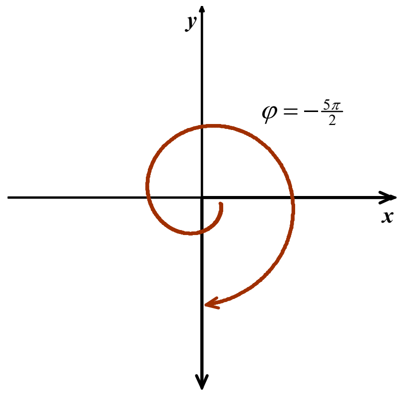
All angles coterminal with γ are of the form $\theta = \gamma + 2\pi k$, where k is an integer. To make arithmetic a bit easier, we note that $2\pi = \frac{8\pi}{4}$, so that $\theta = \frac{9\pi}{4} + \frac{8\pi}{4}k$.

- For $k = -1$, we get $\theta = \frac{9\pi}{4} - \frac{8\pi}{4} = \frac{\pi}{4}$.
- Substituting $k = -2$ gives $\theta = \frac{9\pi}{4} - \frac{16\pi}{4} = -\frac{7\pi}{4}$.

- When we let $k=1$, we get $\theta=\frac{9\pi}{4}+\frac{8\pi}{4}=\frac{17\pi}{4}$.

4. To graph $\varphi=-\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi=\frac{4\pi}{2}$, after one full revolution clockwise we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of φ lies on the negative y-axis, φ is a quadrantal angle.

Figure 1.1. 39



To find coterminal angles, we compute $\theta=-\frac{5\pi}{2}+\frac{4\pi}{2}k$ for a few integers k and obtain, for example, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{2}$.

□

Note that since there are infinitely many integers, any given angle has infinitely many coterminal angles, and the reader is encouraged to plot the few sets of coterminal angles found in **Example 1.1.3** to see this.

Sketching Other Angles

At this point, we have sketched angles in degrees and radians that correspond to revolution measures in increments of eighths and twelfths. In the following example, we will develop a technique for graphing any angle.

Example 1.1.4. Let $\alpha=111^\circ$ and $\beta=\frac{\pi}{5}$.

1. Sketch the angle α.
2. Sketch the angle β.

Solution.

1. We refer to degree measures corresponding to one-eighth revolutions to help us here. To sketch $\alpha=111^\circ$, we first note that $90^\circ<\alpha<180^\circ$. If we divide this range in half, we observe that

$90^\circ < \alpha < 135^\circ$. After one more division, we get $90^\circ < \alpha < 112.5^\circ$. We note that, in fact, 111° is only slightly less than 112.5°, and so 112.5° is a good approximation for graphing. While 112.5° is not shown in the figure on the left, it is midway between 90° and 135° and is included in our sketch of α to the right.

Figure 1.1. 40

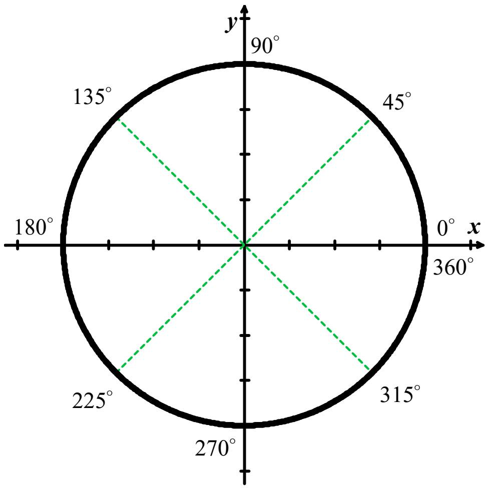
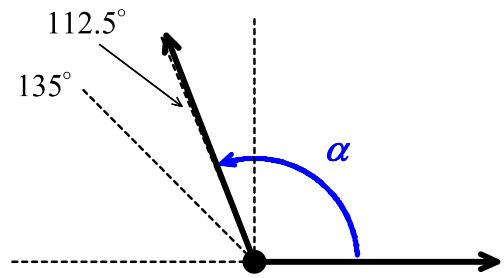


Figure 1.1. 41



2. For $\beta = \frac{\pi}{5}$, we find $0 < \beta < \frac{\pi}{2}$. After dividing the range in half, we get $0 < \beta < \frac{\pi}{4}$, followed by $\frac{\pi}{8} < \beta < \frac{\pi}{4}$, and lastly $\frac{3\pi}{16} < \beta < \frac{\pi}{4}$. The locations of $\frac{\pi}{8}$ and $\frac{3\pi}{16}$ are included in the actual sketch of the angle $\beta = \frac{\pi}{5}$ below, to the right.

Figure 1.1. 42

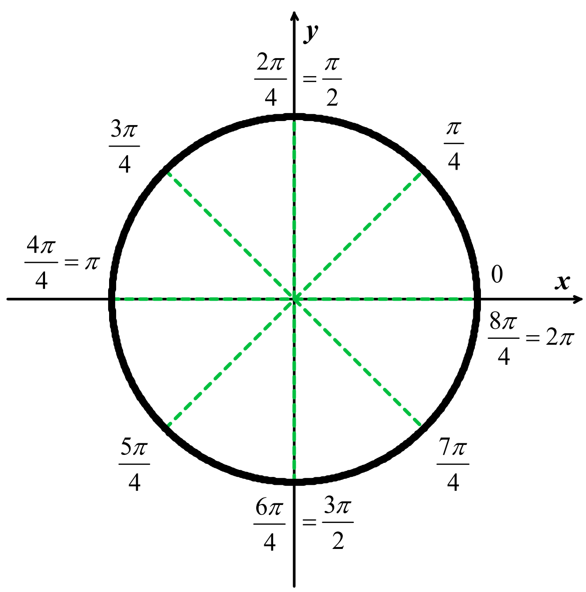
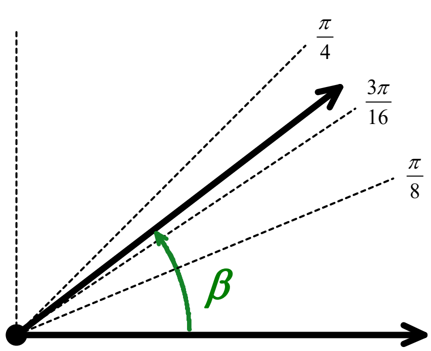


Figure 1.1. 43



□

1.1 Exercises

In Exercises 1 – 8, convert the angle from degree measure into radian measure, giving the exact value in terms of π. These problems should be worked without the aid of a calculator.

1. $0°$
2. $240°$
3. $135°$
4. $-270°$
5. $-315°$
6. $150°$
7. $45°$
8. $-225°$

In Exercises 9 – 16, convert the angle from radian measure into degree measure.

9. π
10. $-\frac{2\pi}{3}$
11. $\frac{7\pi}{6}$
12. $\frac{11\pi}{6}$
13. $\frac{\pi}{3}$
14. $\frac{5\pi}{3}$
15. $-\frac{\pi}{6}$
16. $\frac{\pi}{2}$

In Exercises 17 – 39, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

17. $330°$
18. $-135°$
19. $120°$
20. $405°$
21. $-270°$
22. $300°$
23. $-150°$
24. $135°$
25. $\frac{5\pi}{6}$
26. $-\frac{11\pi}{3}$
27. $\frac{5\pi}{4}$
28. $\frac{3\pi}{4}$
29. $-\frac{\pi}{3}$
30. $\frac{7\pi}{2}$
31. $\frac{\pi}{4}$
32. $-\frac{\pi}{2}$
33. $\frac{7\pi}{6}$
34. $-\frac{5\pi}{3}$
35. 3π
36. -2π
37. $-\frac{\pi}{4}$
38. $\frac{15\pi}{4}$
39. $-\frac{13\pi}{6}$

In Exercises 40 – 47, sketch the angle in standard position. You may refer to **Example 1.1.4**.

40. $100°$
41. $40°$
42. $150°$
43. $80°$
44. $\frac{\pi}{7}$
45. $\frac{\pi}{11}$
46. $-\frac{3\pi}{11}$
47. $\frac{7\pi}{5}$

In Exercises 48 – 51, find a supplementary angle for the given angle.

48. 102 °

49. 39°

50. $\frac{5\pi}{6}$

51. $\frac{\pi}{2}$

In Exercises 52 – 55, find a complementary angle for the given angle.

52. 11°

53. 39°

54. $\frac{\pi}{6}$

55. $\frac{3\pi}{7}$

56. Sketch the oriented angles $\frac{\pi}{2}$ and 100° on the same graph, labeling each.

57. Sketch the oriented angles $-\frac{\pi}{4}$ and −40° on the same graph, labeling each.

58. Sketch the oriented angles $-\frac{2\pi}{3}$ and 240° on the same graph, labeling each.

1.2 Right Triangle Trigonometry

Learning Objectives

- **Define the trigonometric ratios.**
- **Identify the trigonometric ratio values for 30, 45, and 60 degree angles.**
- **Solve right triangles and related applications.**

As we shall see in the sections to come, many applications in Trigonometry involve finding the measures of the interior angles and the lengths of the sides of right triangles. Recall that a **right triangle** is a triangle containing one right angle, which means the remaining two angles are acute angles. In this section, we define trigonometric ratios. Noting that two right triangles are similar if they have one congruent acute angle, we use properties of similar triangles to establish trigonometric ratio values for three special angles: 30°, 45°, and 60°. We then use trigonometric ratios to find lengths of sides of right triangles.

Similar Triangles

We begin with a definition from Geometry. Recall that any two triangles are **similar** if they have the same shape or, more specifically, if their corresponding angles are congruent. Additionally, two triangles are similar if and only if their corresponding sides are proportional. In the following triangles, $\angle A \cong \angle R$, $\angle B \cong \angle S$ and $\angle C \cong \angle T$. Thus, triangle ABC is similar to triangle RST and $\frac{AB}{RS}=\frac{BC}{ST}=\frac{CA}{TR}$.

Figure 1.2. 1

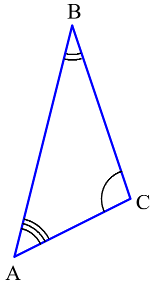
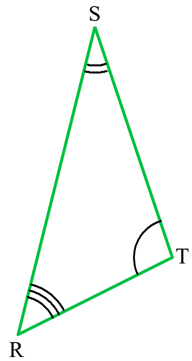


Figure 1.2. 2

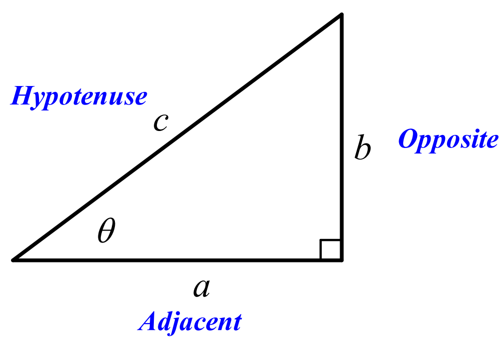


Since $\frac{AB}{RS}=\frac{BC}{ST}$, after multiplying both sides by $\frac{RS}{BC}$, we find $\frac{AB}{BC}=\frac{RS}{ST}$. Similarly, $\frac{AB}{CA}=\frac{RS}{TR}$ and $\frac{BC}{CA}=\frac{ST}{TR}$. So, in any two similar triangles, the ratios of corresponding side lengths are equivalent. This correspondence between side lengths in triangles sharing common angles leads us to trigonometric ratios.

Trigonometric Ratios

The trigonometric ratios introduced in this section will lead to **Section 1.3** where we find trigonometric function values for angles. To get started with ratios, we consider a right triangle and name one of its acute angles θ. The longest side of this right triangle is the side opposite the right angle and is called the **hypotenuse**, shown to have length c in the following illustration. Of the two shorter sides, the side closer to the angle θ is referred to as the side **adjacent** to θ while the side to which θ opens is the **opposite** side. For the angle θ identified below, we have labeled the adjacent side as having length a, and the opposite side with length b. Note that the letters assigned to the side lengths are arbitrary.

Figure 1.2. 3



The six trigonometric ratios are defined below.

Definition 1.2. Suppose θ is an acute angle in a right triangle. If the length of the hypotenuse of the triangle is c, the side adjacent to θ is a, and the side opposite θ is b, then

- The **sine** of θ, denoted $\sin(\theta)$, is $\sin(\theta)=\dfrac{\text{opposite}}{\text{hypotenuse}}=\dfrac{b}{c}$.
- The **cosine** of θ, denoted $\cos(\theta)$, is $\cos(\theta)=\dfrac{\text{adjacent}}{\text{hypotenuse}}=\dfrac{a}{c}$.
- The **tangent** of θ, denoted $\tan(\theta)$, is $\tan(\theta)=\dfrac{\text{opposite}}{\text{adjacent}}=\dfrac{b}{a}$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is $\csc(\theta)=\dfrac{\text{hypotenuse}}{\text{opposite}}=\dfrac{c}{b}$.
- The **secant** of θ, denoted $\sec(\theta)$, is $\sec(\theta)=\dfrac{\text{hypotenuse}}{\text{adjacent}}=\dfrac{c}{a}$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is $\cot(\theta)=\dfrac{\text{adjacent}}{\text{opposite}}=\dfrac{a}{b}$.

Note that it is common practice to write these ratios without parentheses, such as $\sin\theta$ instead of $\sin(\theta)$.

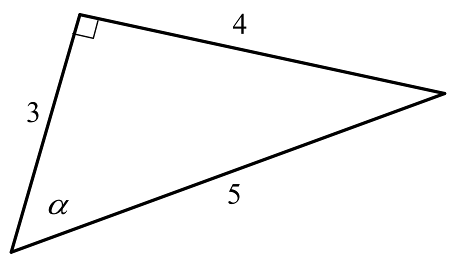
The following are important properties of the trigonometric ratios.

1. For all right triangles with the same acute angle θ, because they are similar, the values of the trigonometric ratios of θ will be equal. This property of equivalent proportions of corresponding sides within similar triangles will be demonstrated in **Example 1.2.2**.
2. Cosecant, secant, and cotangent ratios are reciprocals of the ratios for sine, cosine, and tangent, respectively.[8] Thus, if we know the sine, cosine, and tangent ratios for an angle, we can easily determine the remaining trigonometric ratios. In particular,

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Example 1.2.1. Use the following triangle to evaluate $\sin(\alpha)$, $\cos(\alpha)$, $\tan(\alpha)$, $\csc(\alpha)$, $\sec(\alpha)$, and $\cot(\alpha)$.

Figure 1.2. 4



Solution.

From the definitions of trigonometric ratios,

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

The reciprocals of these ratios result in the remaining trigonometric ratio values:

[8] Don't confuse reciprocal with inverse. We will talk about inverses later on.

$$\csc(\alpha)=\frac{1}{\sin(\alpha)}=\frac{5}{4}$$
$$\sec(\alpha)=\frac{1}{\cos(\alpha)}=\frac{5}{3}$$
$$\cot(\alpha)=\frac{1}{\tan(\alpha)}=\frac{3}{4}$$

□

Example 1.2.2. Verify that the following triangles are similar. Then evaluate the trigonometric ratio values for the angle in the second triangle that corresponds to α. Refer to the trigonometric ratio values we found for α in **Example 1.2.1**. What can you deduce about trigonometric ratios?

Figure 1.2. 5

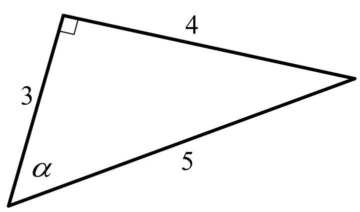
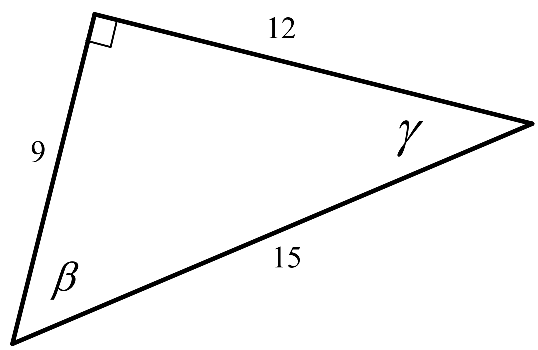


Figure 1.2. 6



Solution.

The side lengths of the second triangle are proportional to the corresponding side lengths of the first triangle by a scale factor of 3:

$$\frac{9}{3}=\frac{12}{4}=\frac{15}{5}=3$$

Thus, the triangles are similar, with the angle β being equal in measure to α. To evaluate the trigonometric ratio values for β, we save a bit of writing by using the abbreviations 'opp', 'adj', and 'hyp' in place of 'opposite', 'adjacent', and 'hypotenuse', respectively. The trigonometric ratio values for this similar triangle will be

$$\sin(\beta)=\frac{\text{opp}}{\text{hyp}}=\frac{12}{15}=\frac{3\cdot 4}{3\cdot 5}=\frac{4}{5}$$
$$\cos(\beta)=\frac{\text{adj}}{\text{hyp}}=\frac{9}{15}=\frac{3\cdot 3}{3\cdot 5}=\frac{3}{5}$$
$$\tan(\beta)=\frac{\text{opp}}{\text{adj}}=\frac{12}{9}=\frac{3\cdot 4}{3\cdot 3}=\frac{4}{3}$$

Using reciprocal properties, the remaining three values are

$$\csc(\beta)=\frac{15}{12}=\frac{3\cdot 5}{3\cdot 4}=\frac{5}{4}$$
$$\sec(\beta)=\frac{15}{9}=\frac{3\cdot 5}{3\cdot 3}=\frac{5}{3}$$
$$\cot(\beta)=\frac{9}{12}=\frac{3\cdot 3}{3\cdot 4}=\frac{3}{4}$$

Referring to **Example 1.2.1**, we note that the trigonometric ratio values are identical for these two similar triangles. We conclude that trigonometric ratios are independent of the size of the triangle.

□

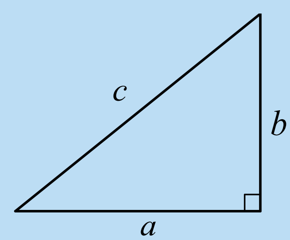
Pythagorean Theorem

The Pythagorean Theorem will be useful in our next task: determining trigonometric ratio values for 30°, 45°, and 60° angles.

Theorem 1.1. The Pythagorean Theorem: In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of its legs.

In the triangle to the right, $c^2=a^2+b^2$.

Figure 1.2. 7



Solving the above equation for c, we get $c=\pm\sqrt{a^2+b^2}$. However, since c is a length and must be positive, $c=\sqrt{a^2+b^2}$. We can also use the Pythagorean Theorem to solve for the length of a leg since, for example, $a^2=c^2-b^2$ gives us $a=\sqrt{c^2-b^2}$.

Ratios of 30°–60°–90° Triangles

We begin by finding the values of trigonometric ratios for 30°. We sketch a 30°–60°–90° triangle with hypotenuse of length c and envision this triangle as being half of a 60°–60°–60° equilateral triangle.

Figure 1.2. 8

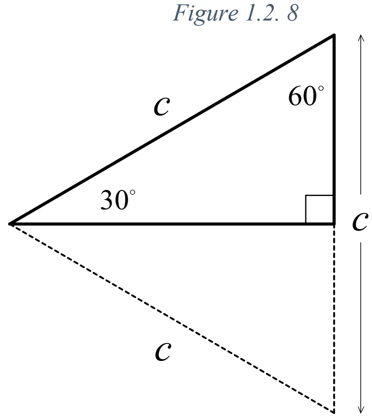
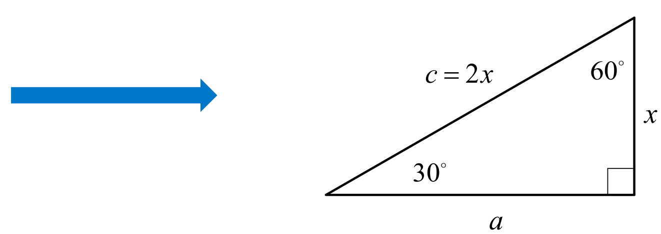


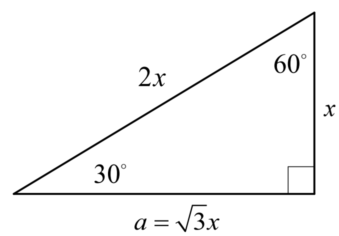
Figure 1.2. 9



Noting that the altitude of the equilateral triangle bisects its base, and assigning the variable x to the length of the shortest side of the 30°–60°–90° triangle (the side opposite the 30° angle), we find the hypotenuse has length $c = 2x$. To find the length of the side adjacent to the 30° angle, a, we apply the Pythagorean Theorem.

$$\begin{aligned}(2x)^2 &= a^2 + x^2\\ 4x^2 &= a^2 + x^2\\ 3x^2 &= a^2\\ a &= \sqrt{3}x\end{aligned}$$

Figure 1.2. 10



Using the resulting side lengths, along with the definitions of the trigonometric ratios, we have

$$\sin(30^\circ) = \frac{x}{2x} = \frac{1}{2}$$
$$\cos(30^\circ) = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$
$$\tan(30^\circ) = \frac{x}{\sqrt{3}x} = \frac{1}{\sqrt{3}}$$

Note that we may choose to rationalize the denominator of $\tan(30^\circ)$ to get $\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Taking the reciprocals of these values results in the remaining three trigonometric ratio values:

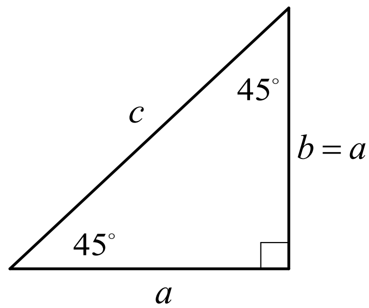
$$\csc(30^\circ) = \frac{1}{\sin(30^\circ)} = 2$$
$$\sec(30^\circ) = \frac{1}{\cos(30^\circ)} = \frac{2}{\sqrt{3}}$$
$$\cot(30^\circ) = \frac{1}{\tan(30^\circ)} = \sqrt{3}$$

We note that these trigonometric ratio values apply to any 30° angle. The reader is encouraged to determine the trigonometric ratio values for 60° angles.

Ratios of 45°–45°–90° Triangles

To find the values of the trigonometric ratios for 45°, we sketch a 45°–45°–90° triangle with hypotenuse c, and note that this is a right isosceles triangle so the two smaller sides must be equal in length.

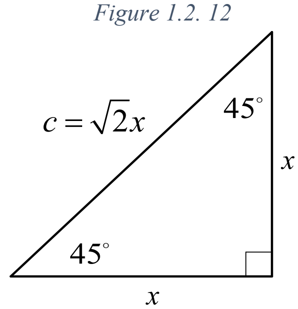
Figure 1.2. 11



Assigning the variable x to the length of a smaller side, we use the Pythagorean Theorem to solve for c.

$$c^2 = x^2 + x^2$$
$$c^2 = 2x^2$$
$$c = \sqrt{2}x$$

Figure 1.2. 12



The resulting trigonometric function values for 45° are

$$\sin\left(45^\circ\right) = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \qquad \csc\left(45^\circ\right) = \frac{1}{\sin\left(45^\circ\right)} = \sqrt{2}$$

$$\cos\left(45^\circ\right) = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \qquad \sec\left(45^\circ\right) = \frac{1}{\cos\left(45^\circ\right)} = \sqrt{2}$$

$$\tan\left(45^\circ\right) = \frac{x}{x} = 1 \qquad \cot\left(45^\circ\right) = \frac{1}{\tan\left(45^\circ\right)} = 1$$

Note that we may choose to rationalize the denominator of $\sin\left(45^\circ\right)$ to get $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Before moving on, we take another look at the triangles from above and their corresponding side ratios.

Ratios for 30°–60°–90° and 45°–45°–90° Triangles

Figure 1.2. 13

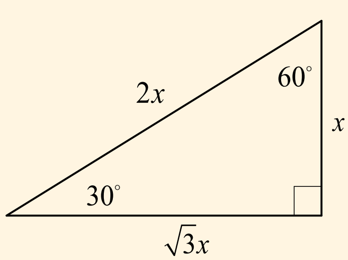
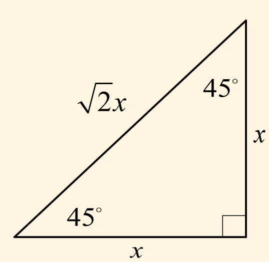


Figure 1.2. 14



In the following table, we summarize the trigonometric ratio values for 30°, 45°, and 60° angles, which we will refer to as **standard angles**. Note that we may choose to rationalize denominators, or not. It is often easier to work with non-rationalized denominators, but you should be familiar with seeing these trigonometric ratios in rationalized format since that is frequently how they are displayed. The following table includes both rationalized and non-rationalized values.

Trigonometric Ratios for Standard Angles: 30°, 45°, and 60°

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

Solving Standard Right Triangles

We will use the values in the table, as well as the ratios represented in the 30°–60°–90° and 45°–45°–90° triangles to determine missing angles and sides in the next several examples. This is sometimes referred to as **solving right triangles**.

Example 1.2.3. A right triangle has one angle of 60° and a hypotenuse of length 20. Find the unknown side lengths and missing angle measure.

Solution. We begin by finding the measure of the missing angle. The sum of the angles of a triangle is 180°, so the missing angle measure is $180^\circ - 60^\circ - 90^\circ = 30^\circ$. We assign variables for the missing side lengths: a for the side adjacent to the 60° angle, and b for the opposite side. Then, noting that this is a 30°–60°–90° triangle, we use the ratios established previously to solve for missing sides.

Figure 1.2. 15

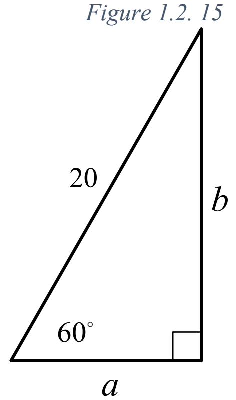
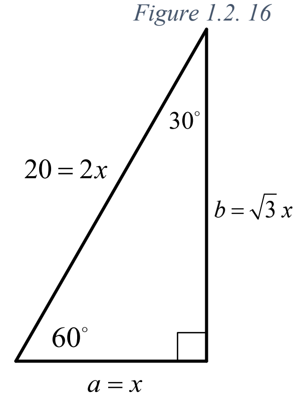
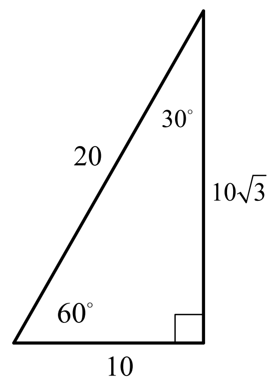


Figure 1.2. 16



From $20 = 2x$, we find $x = 10$, and so $a = 10$. Then $b = \sqrt{3}\,x = \sqrt{3}\cdot 10$, or $10\sqrt{3}$. The triangle with all of its angles and side lengths is recorded below.

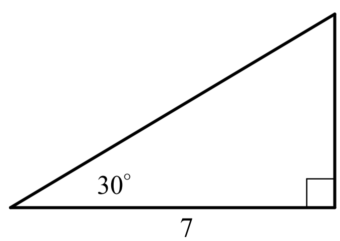
Figure 1.2. 17



□

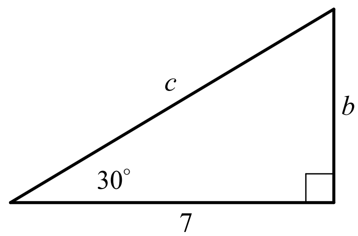
Example 1.2.4. Find the measure of the missing angle and the lengths of the missing sides in the following right triangle.

Figure 1.2. 18



Solution. We begin by finding the measure of the missing angle: $180^\circ - 90^\circ - 30^\circ = 60^\circ$. We continue by labeling the missing side lengths: c for the hypotenuse and b for the side opposite the 30° angle.

Figure 1.2. 19



To solve for the side lengths, we use trigonometric ratio values. First of all, for c we have

$\cos\left(30^\circ\right) = \frac{7}{c}$, from which

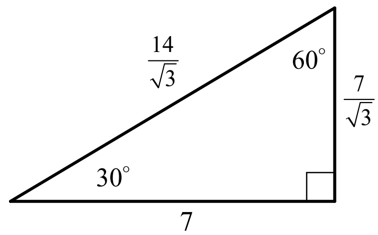
$$\begin{aligned} c &= \frac{7}{\cos\left(30^\circ\right)} \\ &= \frac{7}{\sqrt{3}/2} \\ &= \frac{14}{\sqrt{3}} \end{aligned}$$

For length b, $\tan\left(30^\circ\right) = \frac{b}{7}$ so that

$$\begin{aligned} b &= 7\tan\left(30^\circ\right) \\ &= 7 \cdot \frac{1}{\sqrt{3}} \\ &= \frac{7}{\sqrt{3}} \end{aligned}$$

Below we have the triangle with all angles and sides labeled.

Figure 1.2. 20



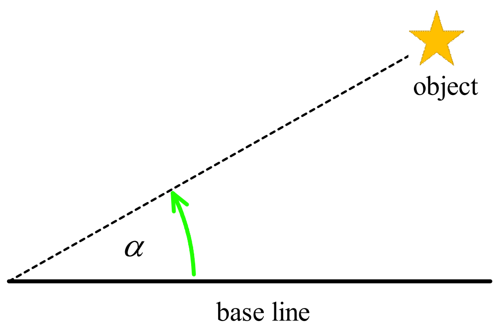
□

Note that we could have solved **Example 1.2.4** using the method from **Example 1.2.3**, just as **Example 1.2.3** could be solved with trigonometric ratio values.

Applications

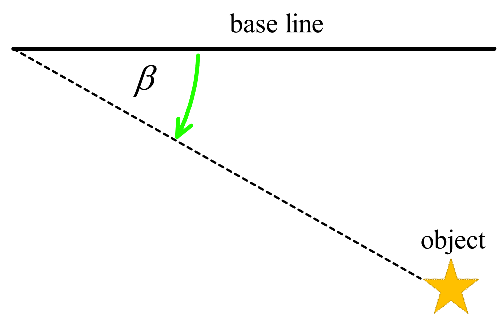
Right triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure. The following example uses trigonometric ratios as well as the concept of an 'angle of inclination'. The **angle of inclination**, also known as the **angle of elevation**, of an object refers to the angle whose initial side is some kind of horizontal base line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. This is represented schematically below.

Figure 1.2. 21



α is angle of inclination (elevation).

Figure 1.2. 22



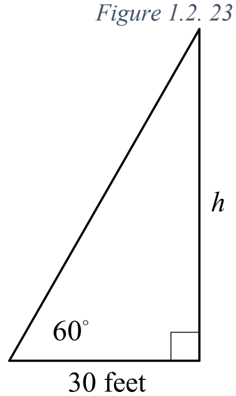
β is angle of depression.

An **angle of depression**, shown to the right, is the angle from the horizontal base line to a terminal side below the base line. This can be thought of from the perspective on an observer who begins by looking horizontally out. If the observer proceeds by looking upward, the angle is one of elevation. If the observer looks downward, the angle is one of depression.

Example 1.2.5. The angle of inclination from a point on the ground 30 feet away from the base of a water tower to the top of the water tower is 60°. Find the height of the water tower to the nearest foot.

Solution. We can represent the problem situation using a right triangle as shown.

Figure 1.2. 23



If we let h denote the height of the tower, then we have $\tan\left(60^\circ\right)=\frac{h}{30}$. From this we get

$$\begin{aligned} h &= 30\tan\left(60^\circ\right) \\ &= 30\sqrt{3} \\ &\approx 51.96 \end{aligned}$$

Hence, the water tower is approximately 52 feet tall.

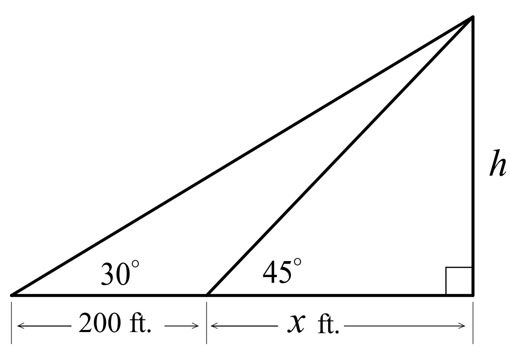
□

We can extend the idea of the last problem to determine the height an object, even if we cannot quite reach its base, as you see in the next example.

Example 1.2.6. In order to determine the height of a California redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination are 45° and 30°, respectively, how tall is the tree to the nearest foot.

Solution. Sketching the problem situation below, we find ourselves with two unknowns: the height h of the tree and the distance x from the base of the tree to the first observation point.

Figure 1.2. 24



Using trigonometric ratios, we get a pair of equations: $\tan\left(45^\circ\right)=\frac{h}{x}$ and $\tan\left(30^\circ\right)=\frac{h}{x+200}$. Since $\tan\left(45^\circ\right)=1$, the first equation gives $\frac{h}{x}=1$, or $x=h$. Substituting this into the second equation gives

$$\begin{aligned} \tan\left(30^\circ\right) &= \frac{h}{h+200} \\ \frac{1}{\sqrt{3}} &= \frac{h}{h+200} \end{aligned}$$

We proceed to solve for h.

$$h\sqrt{3} = h + 200$$
$$h\sqrt{3} - h = 200$$
$$h\left(\sqrt{3} - 1\right) = 200$$
$$h = \frac{200}{\sqrt{3} - 1} \approx 273.205$$

Hence, the tree is approximately 273 feet tall.

□

In the real world, it will be rare to find angles of exactly 30°, 45° or 60°. For other angles, approximate trigonometric values may be found with a calculator. There are three general types of calculators: arithmetic, scientific and graphing. However, some scientific calculators have a Direct Algebraic Logic (D.A.L.) input method that allows them to operate like a graphing calculator. For calculating trigonometric values, you will need a scientific or a graphing calculator. You will need to check that the calculator is set to the correct angle measure unit, i.e. degrees or radians. Refer to your calculator's instruction guide for help with this. For most calculators the following steps apply in finding sine, cosine or tangent trigonometric ratios:

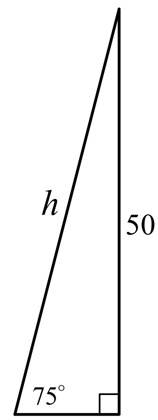
- In a scientific calculator, punch in the angle measure followed by the desired ratio: 'sin', 'cos' or 'tan', respectively.
- In a graphing or D.A.L. calculator, punch in the desired trigonometric ratio, 'sin', 'cos' or 'tan', followed by the angle and then 'enter' or '='.

Of course, if you are interested in cosecant, secant or cotangent ratios, you must find the value of the sine, cosine or tangent, respectively, followed by the x^{-1} button in a scientific calculator, or the x^{-1} button, then 'enter' or '=', in a graphing or D.A.L. calculator.

Example 1.2.7. To what length must an adjustable ladder be set so that it reaches a windowsill 50 feet above the ground with the ladder resting against the building at an angle of 75° with the ground?

Solution. We know that the angle of inclination, or elevation, is 75° and that the opposite side is 50 feet in length. The length of the hypotenuse, h, will give us the necessary length for the ladder to reach a height of 50 feet. Using the ratio for sine of 75°, we have

Figure 1.2. 25



$$\sin\left(75^\circ\right) = \frac{50}{h}$$
$$h = \frac{50}{\sin\left(75^\circ\right)}$$
$$h \approx \frac{50}{0.9659258}$$
$$h \approx 51.7638$$

We have found that the height of the ladder should be exactly $\frac{50}{\sin\left(75^\circ\right)}$ feet. Noting that trigonometric ratio values for 75° are not included in the table for standard angles, a calculator allows us to find an approximate value for $\sin\left(75^\circ\right)$. We find the approximate height is 51.8 feet.

□

1.2 Exercises

1. For each of the given right triangles, label the adjacent side, opposite side and hypotenuse for the indicated angle.

Figure Ex1.2. 1

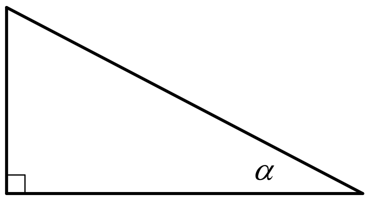
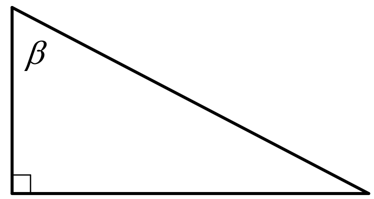


Figure Ex1.2. 2



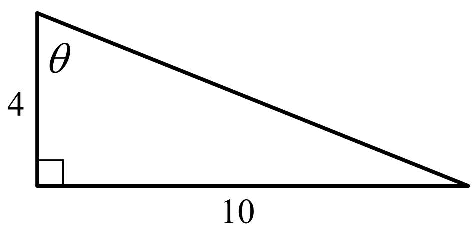
2. The tangent of an angle compares which sides of a right triangle?

3. What is the relationship between the two acute angles in a right triangle?

In Exercises 4 – 7, use the given right triangle to evaluate $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\csc(\theta)$, $\sec(\theta)$ and $\cot(\theta)$. Give exact values.

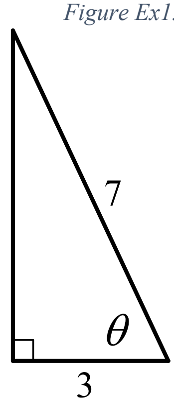
4.

Figure Ex1.2. 3



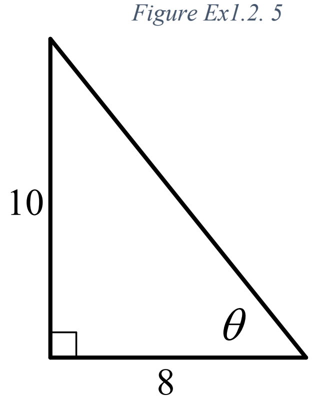
5.

Figure Ex1.2. 4



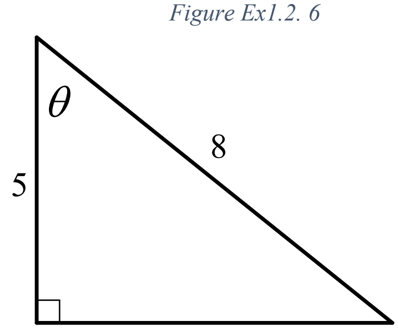
6.

Figure Ex1.2. 5



7.

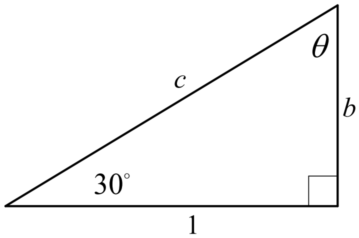
Figure Ex1.2. 6



In Exercises 8 – 15, find the measure of the missing angle and the lengths of the missing sides. Give exact values.

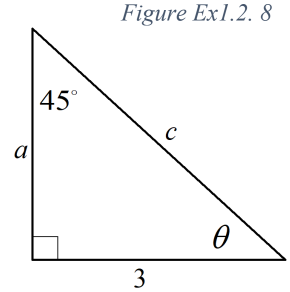
8. Find θ, b and c.

Figure Ex1.2. 7



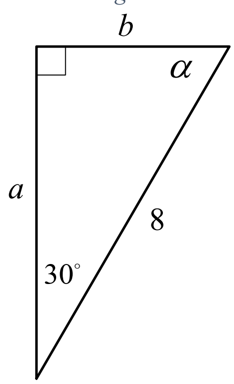
9. Find θ, a and c.

Figure Ex1.2. 8



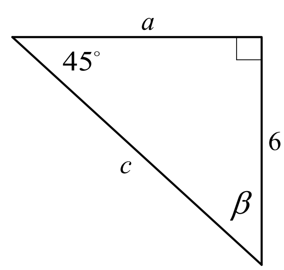
10. Find α, a and b.

Figure Ex1.2. 9



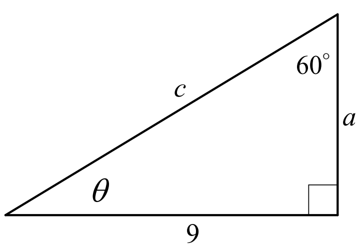
11. Find β, a and c.

Figure Ex1.2. 10



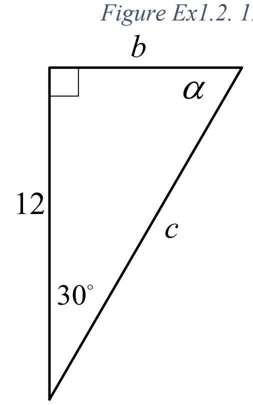
12. Find θ, a and c.

Figure Ex1.2. 11



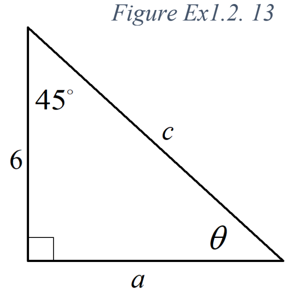
13. Find α, b and c.

Figure Ex1.2. 12



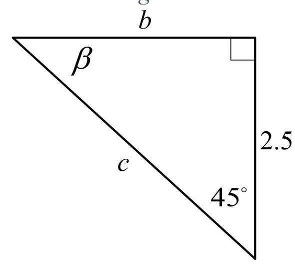
14. Find θ, a and c.

Figure Ex1.2. 13



15. Find β, b and c.

Figure Ex1.2. 14



In Exercises 16 – 27, assume that θ is an acute angle in a right triangle. Find the exact value for the requested side length.

16. If $\theta = 30^\circ$ and the side adjacent to θ has length 4, how long is the hypotenuse?

17. If $\theta = 45^\circ$ and the hypotenuse has length 5280, how long is the side adjacent to θ?

18. If $\theta = 60^\circ$ and the side opposite θ has length 117, how long is the hypotenuse?

19. If $\theta = 30^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?

20. If $\theta = 45^\circ$ and the hypotenuse has length 10, how long is the side adjacent to θ?

21. If $\theta = 60^\circ$ and the side opposite θ has length 306, how long is the side adjacent to θ?

22. If $\theta = 30^\circ$ and the side opposite θ has length 4, how long is the side adjacent to θ?

23. If $\theta = 45^\circ$ and the hypotenuse has length 8, how long is the side opposite θ?

24. If $\theta = 60^\circ$ and the side adjacent to θ has length 2, how long is the side opposite θ?

25. If $\theta = 30^\circ$ and the side opposite θ has length 14, how long is the hypotenuse?

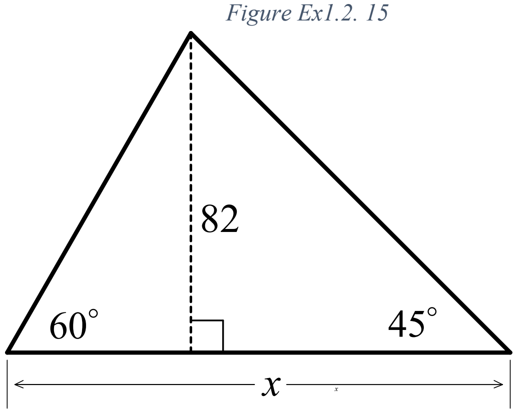
26. If $\theta = 45^\circ$ and the hypotenuse has length 4, how long is the side adjacent to θ?

27. If $\theta = 60^\circ$ and the side adjacent to θ has length 31, how long is the side opposite θ?

In Exercises 28 – 31, find the exact value for x.

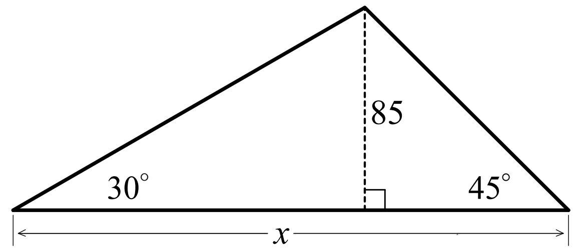
28.

Figure Ex1.2. 15



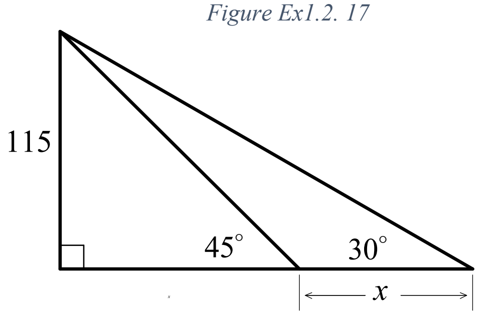
29.

Figure Ex1.2. 16



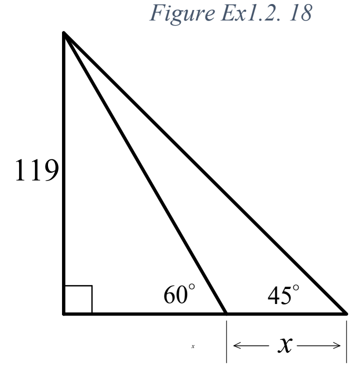
30.

Figure Ex1.2. 17



31.

Figure Ex1.2. 18



32. A tree standing vertically on level ground casts a 120 foot long shadow. The angle of elevation from the end of the shadow to the top of the tree is 21.4°. Find the height of the tree to the nearest foot.[9]

33. The broadcast tower for radio station WSAZ (Home of "Algebra in the Morning with Carl and Jeff") has two enormous flashing red lights on it, one at the very top and one a few feet below the top. From a point 5000 feet away from the base of the tower, on level ground, the angle of elevation to the top light is 7.970° and to the second light is 7.125°. Find the distance between the lights to the nearest foot.

34. From a fire tower 200 feet above level ground in the Sasquatch National Forest, a ranger spots a fire off in the distance. The angle of depression to the fire is 2.5°. How far away from the base of the tower is the fire? Round to the nearest foot.

35. The ranger from the previous problem sees a Sasquatch running directly from the fire toward the fire tower. The ranger takes two sightings. At the first sighting, the angle of depression from the tower to

[9] Research the term *umbra versa* and see what it has to do with the shadow in this problem.

the Sasquatch is 6°. The second sighting, taken just 10 seconds later, gives the angle of depression as 6.5°. With the fire tower being 200 feet above level ground, determine how far the Sasquatch travelled in those 10 seconds. Round your answer to the nearest foot. How fast is this in miles per hour? Round your answer to the nearest mile per hour. If the Sasquatch keeps up this pace how long will it take for the Sasquatch to reach the fire tower from his location at the second sighting? Round your answer to the nearest minute.

36. When Rachel stands 30 feet away from a tree in her yard, the angle of elevation to the top of the tree is 50° and the angle of depression to the base of the tree is 10°. What is the height of the tree? Round your answer to the nearest foot.

37. From the observation deck of the lighthouse at Sasquatch Point, 50 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of 8.2° and the second sighting had an angle of depression of 25.9°. How far had the boat traveled between sightings? Round your answer to the nearest foot.

38. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut, it makes a 43° angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground? Round your answers to the nearest foot.

39. A 33 foot ladder leans against the Student Center so that the angle between the ground and the ladder is 80°. How high, to the nearest tenth of a foot, does the ladder reach up the side of the Student Center?

40. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building to the nearest hundredth of a foot.

41. Assuming that a 370 foot tall giant redwood grows vertically, if Dale walks away from the tree to a point where the angle of elevation to the top of the tree is 60°, how far from the base of the tree is he? Round to the nearest hundredth of a foot.

42. Let α and β be the two acute angles of a right triangle. (Thus α and β are complementary angles.) Show that $\sin(\alpha)=\cos(\beta)$ and $\sin(\beta)=\cos(\alpha)$. The fact that co-functions of complementary angles are equal in this case is not an accident and we will look at a more general result later.

43. Let α and β be the two acute angles of a right triangle. (Thus α and β are complementary angles.) Show that $\sec(\alpha)=\csc(\beta)$ and $\tan(\alpha)=\cot(\beta)$.

1.3 The Unit Circle

Learning Objectives

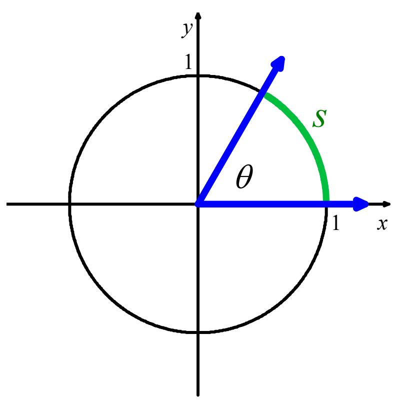
- **Sketch oriented arcs on the Unit Circle.**
- **Determine sine and cosine values of an angle from a point on the Unit Circle.**
- **Use reference angles in determining the sine and cosine of a given angle.**
- **Know the sine and cosine values for 0°, 30°, 45°, 60°, and 90°, or for the equivalent radian measure.**
- **Know and apply the Pythagorean identity.**
- **Know the signs of the sine and cosine in each quadrant.**

In the previous section, we defined the trigonometric ratios of acute angles within right triangles and identified the trigonometric ratio values for the standard angles 30°, 45° and 60°. In this section, we extend the definition of trigonometric ratios for sine and cosine to include all angles. We make good use of the Unit Circle for this task, and thus begin with a definition of the Unit Circle.

Oriented Arcs on the Unit Circle

The **Unit Circle** is a circle with radius 1 that is centered at the origin when drawn in the Cartesian plane. The equation that yields the Unit Circle is $x^2 + y^2 = 1$. The following sketch of the Unit Circle includes a central angle θ, drawn in standard position and subtended by an arc of length s units.

Figure 1.3. 1

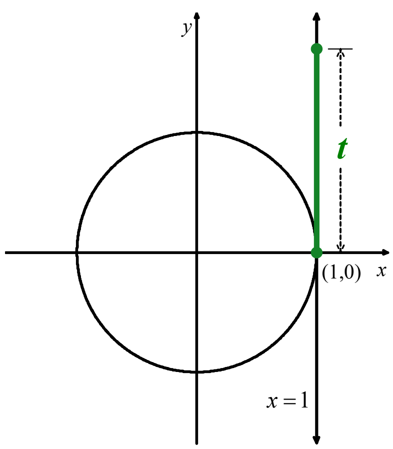


We found in **Section 1.1** that $s = r\,\theta$. In this special case where $r = 1$, that relationship leads us to conclude that $s = \theta$. (Again, we are blurring the distinction between an angle and its measure.) Thus, s

is the same length as the measure of the angle θ, and we remind ourselves that θ is measured in radians. In order to identify real numbers with oriented angles, we make good use of this fact by essentially 'wrapping' the real number line around the Unit Circle and associating to each real number t an oriented arc on the Unit Circle with initial point $(1,0)$. To demonstrate this, consider the Unit Circle and a real number line drawn vertically at $x=1$ with the positive direction being upward.

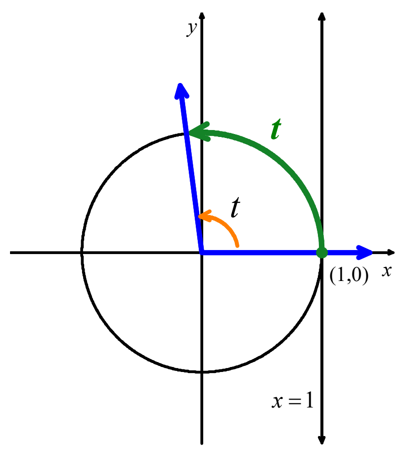
Consider any real number t where $t>0$ and 'wrap' the interval $[0,t]$ on the vertical real number line around the Unit Circle in a counter-clockwise fashion. The resulting arc has length of t units and is oriented in the counter-clockwise direction. The corresponding angle is also oriented in the counter-clockwise direction and has radian measure equal to t.

Figure 1.3. 2



Interval $[0,t]$, $t>0$

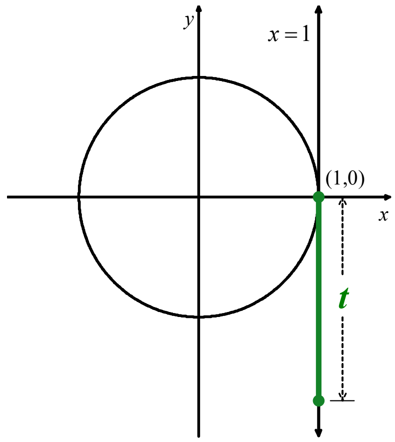
Figure 1.3. 3



After wrapping interval $[0,t]$ around Unit Circle

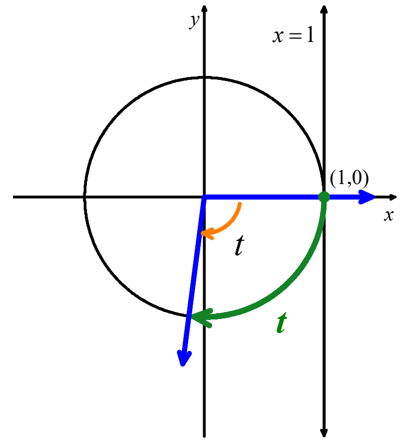
If $t<0$, we wrap the interval $[t,0]$ clockwise around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the oriented angle determined by this arc has a negative radian measure equal to t.

Figure 1.3. 4



Interval $[t,0]$, $t<0$

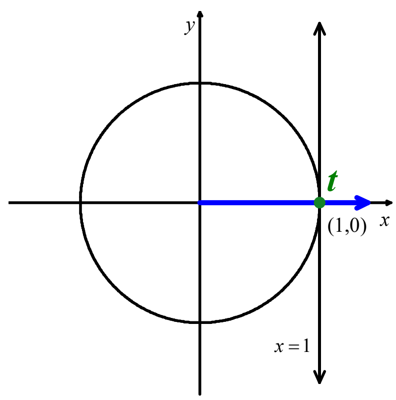
Figure 1.3. 5



After wrapping $[t,0]$ around Unit Circle

If $t=0$, we are at the point $(1,0)$ on the *x*-axis that corresponds to an angle with radian measure 0.

Figure 1.3. 6



$t=0$

Thus, we identify each real number t with a corresponding angle having radian measure of t.

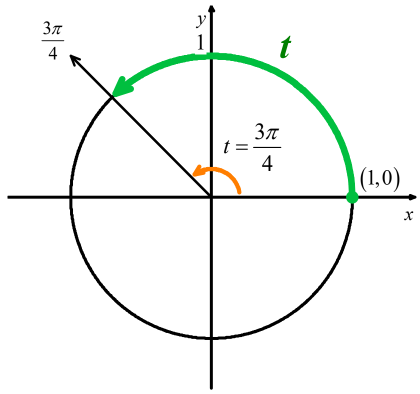
Example 1.3.1. Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

1. $t=\frac{3\pi}{4}$ 2. $t=-2\pi$ 3. $t=-2$ 4. $t=21$

Solution.

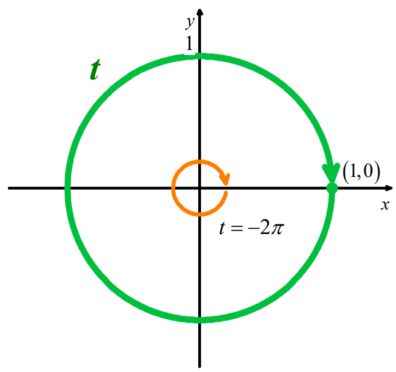
1. The arc associated with $t=\frac{3\pi}{4}$ is the arc on the Unit Circle that subtends the angle $\frac{3\pi}{4}$, measured in radians. Since $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc that begins at the point $(1,0)$ and proceeds counter-clockwise up to midway through Quadrant II.

Figure 1.3. 7



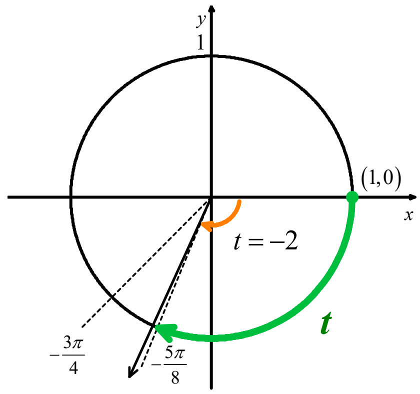
2. Since one revolution is 2π radians, and $t=-2\pi$ is negative, we graph the arc that begins at $(1,0)$ and proceed clockwise for one full revolution.

Figure 1.3. 8



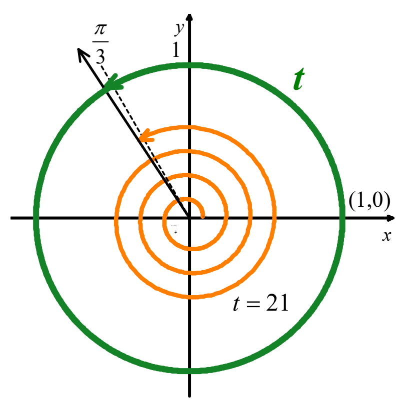
3. Like $t=-2\pi$, $t=-2$ is negative, so we begin our arc at $(1,0)$ and proceed clockwise around the Unit Circle. With $\pi \approx 3.14$ and $\frac{\pi}{2} \approx 1.57$, we find rotating 2 radians clockwise from the point $(1,0)$ lands us in Quadrant III between $-\frac{\pi}{2}$ and $-\pi$. To more accurately place the endpoint, we proceed as we did in **Example 1.1.4**, successively halving the angle measure until we find $-\frac{5\pi}{8} \approx -1.96$, which tells us our arc extends, clockwise, approximately a quarter of the way into Quadrant III.

Figure 1.3. 9



4. Since 21 is positive, the arc corresponding to $t = 21$ begins at $(1,0)$ and proceeds counter-clockwise. As 21 is much greater than 2π, we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate $\frac{21}{2\pi}$ as 3.34 which tells us we complete 3 revolutions counter-clockwise with 0.34, or just slightly more than $\frac{1}{3}$ of a revolution, remaining. In other words, the terminal side of the angle that measures 21 radians in standard position is in Quadrant II, slightly past $\frac{\pi}{3}$ radians. In the following diagram, the arc is wrapping around the Unit Circle 3 times before completing its partial rotation into the second quadrant. This is difficult to display graphically but observing the corresponding angle rotations may help.

Figure 1.3. 10



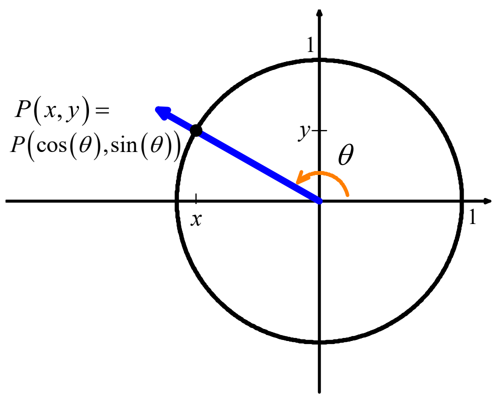
□

We use this association of any real number with the radian measure of an angle subtended by an arc on the Unit Circle, where the length of the arc is equivalent to the angle measure, to define the sine and cosine values for all angles.

The Sine and Cosine as Trigonometric Functions

We have already defined the sine and cosine as ratios for acute angles within right triangles. We now define the sine and cosine for any angle measure. Consider an angle θ in standard position, and let $P(x, y)$ denote the point where the terminal side of θ intersects the Unit Circle. We assign the cosine of θ as the *x*-coordinate of P and the sine of θ as the *y*-coordinate of P: $x = \cos(\theta)$ and $y = \sin(\theta)$.

Figure 1.3. 11



To check that this agrees with our earlier definition of sine and cosine, consider an acute angle θ. A vertical line segment from the point P to the *x*-axis results in a right triangle, as shown below. The right triangle has a hypotenuse of 1, since the radius of the Unit Circle is 1. From the *x*- and *y*-coordinates of P, we find the side of the triangle adjacent to θ has length x and the side opposite θ has length y.

Figure 1.3. 12

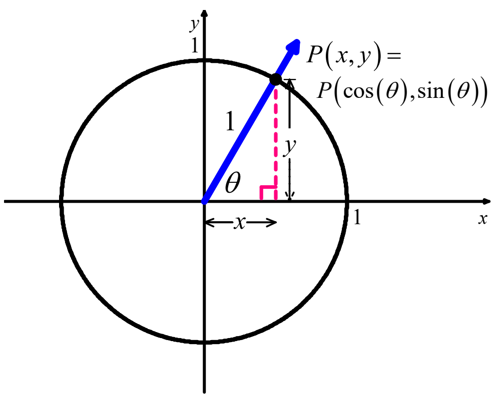
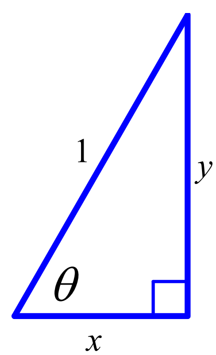


Figure 1.3. 13



Using the right triangle definition of sine and cosine from **Section 1.2**, we find

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

This confirms that our new definition for $\sin(\theta)$ as the y-coordinate and $\cos(\theta)$ as the x-coordinate of a point on the Unit Circle agrees with the right triangle definition. The reader is encouraged to verify that this new definition, that matches an angle with its sine and cosine, satisfies the definition of a **function**: for each angle θ, there is only one associated value of $\sin(\theta)$ and only one associated value of $\cos(\theta)$. Not only does this new definition allow us to find the sine and cosine of any angle, it also means $\sin(\theta)$ and $\cos(\theta)$ are functions. Although we usually denote a function with just one letter, like f, here we use the three letters 'sin' and 'cos' as the names of these functions. We may also write $f(\theta)=\sin(\theta)$ or $g(\theta)=\cos(\theta)$.

The **domain** of both $\sin(\theta)$ and $\cos(\theta)$ is $(-\infty,\infty)$ since θ can be assigned the value of any real number as its radian measure. The terminal side of the angle θ intersects the Unit Circle in a unique point, $(x,y)=(\cos(\theta),\sin(\theta))$. Noting that both x and y values of points on the Unit Circle take on all values between −1 and +1, the **range** of $\sin(\theta)$ and $\cos(\theta)$ is $[-1,1]$.

Recall that any angle that is not labeled as being in degrees is, by default, assumed to be in radians. In the following example, the angles in part 2 and part 4 are radian measures: $\theta=-\pi$ radians and $\theta=\frac{\pi}{6}$ radians, respectively.

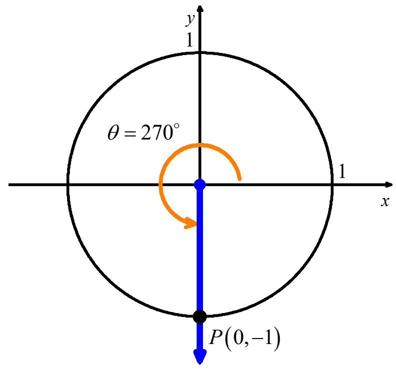
Example 1.3.2. Find the sine and cosine of the following angles.

1. $\theta=270^\circ$ 2. $\theta=-\pi$ 3. $\theta=45^\circ$ 4. $\theta=\frac{\pi}{6}$ 5. $\theta=60^\circ$

Solution.

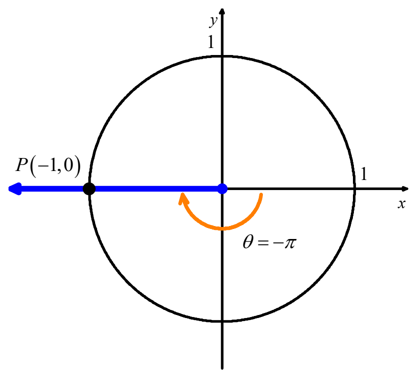
1. To find $\sin(270^\circ)$ and $\cos(270^\circ)$, we plot the angle $\theta=270^\circ$ in standard position and find the point on the terminal side of θ that lies on the Unit Circle. Since 270° represents $\frac{3}{4}$ of a counter-clockwise rotation, the terminal side of θ lies along the negative y-axis. Hence, the point we seek is $(0,-1)$ so that $\sin(270^\circ)=-1$ and $\cos(270^\circ)=0$.

Figure 1.3. 14



2. The angle $\theta = -\pi$ represents one-half of a clockwise rotation so its terminal side lies on the negative x-axis. The point on the Unit Circle that lies on the negative x-axis is $(-1,0)$, from which $\sin(-\pi) = 0$ and $\cos(-\pi) = -1$.

Figure 1.3. 15



3. When we sketch $\theta = 45^\circ$ in standard position, we see that its terminal side does not lie along any of the coordinate axes. We let $P(x, y)$ denote the point on the terminal side of θ that lies on the Unit Circle. By definition, $x = \cos(\theta)$ and $y = \sin(\theta)$. If we drop a perpendicular line segment from P to the x-axis, we obtain a 45°–45°–90° right isosceles triangle whose legs have lengths x and y units. From the properties of isosceles triangles, it follows that $y = x$.

Figure 1.3. 16

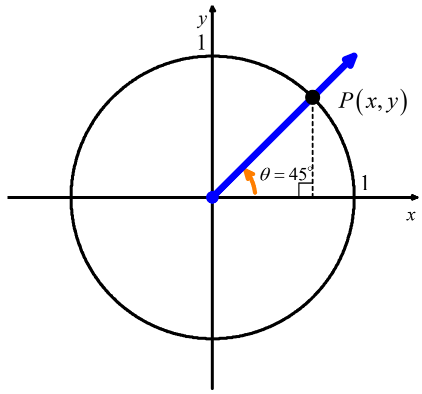
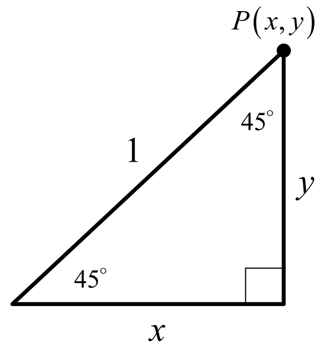


Figure 1.3. 17



$P(x,y)$ lies on the Unit Circle, so $x^2 + y^2 = 1$. Substituting $y = x$ into this equation yields $2x^2 = 1$, or

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

Now, $P(x,y)$ lies in the first quadrant where $x > 0$, so $x = \frac{\sqrt{2}}{2}$. Since $y = x$, we can also conclude that $y = \frac{\sqrt{2}}{2}$. Finally, we have $\sin(45^\circ) = \frac{\sqrt{2}}{2}$ and $\cos(45^\circ) = \frac{\sqrt{2}}{2}$.

4. For $\theta = \frac{\pi}{6}$, as before, the terminal side does not lie on either of the coordinate axes so we proceed using a triangle approach. Letting $P(x,y)$ denote the point on the terminal side of θ that lies on the Unit Circle, we drop a perpendicular line segment from P to the *x*-axis to form a 30°–60°–90° right triangle.

Figure 1.3. 18

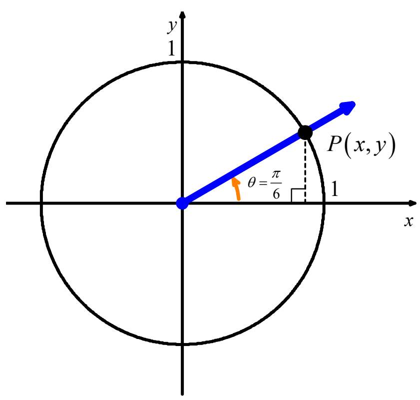
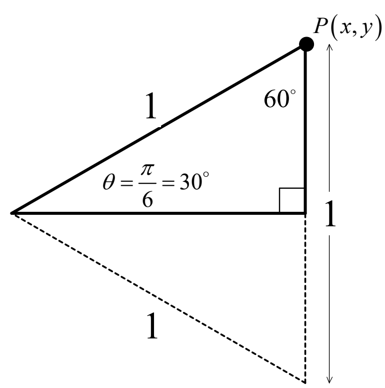


Figure 1.3. 19



Noting that we have half of an equilateral triangle with sides of length 1, we find $y=\frac{1}{2}$ so that $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$. Since $P(x,y)$ lies on the Unit Circle, we substitute $y=\frac{1}{2}$ into $x^2+y^2=1$ to get $x^2=\frac{3}{4}$, or $x=\pm\frac{\sqrt{3}}{2}$. In the first quadrant $x>0$, so $x=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$.

5. Plotting $\theta=60^\circ$ in standard position, we find θ is not a quadrantal angle and set about using a triangle approach. Once again, we get a 30°–60°–90° right triangle and, after computations similar to part 4 of this example, we find $y=\sin\left(60^\circ\right)=\frac{\sqrt{3}}{2}$ and $x=\cos\left(60^\circ\right)=\frac{1}{2}$.

Figure 1.3. 20

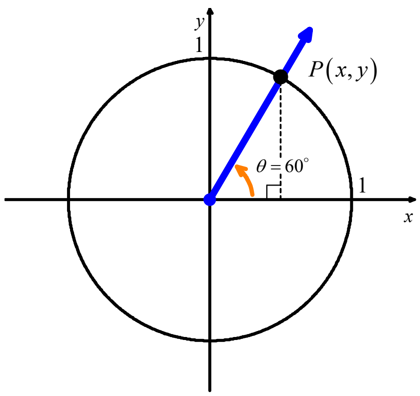
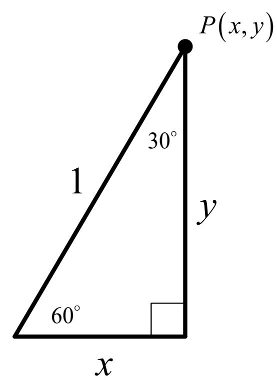


Figure 1.3. 21



□

It is not by accident that the last three angles in **Example 1.3.2** are 30°, 45°, and 60°, or $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, respectively. In **Section 1.2** we used right triangles to obtain these same sine and cosine values for 30°, 45°, and 60°.

The Unit Circle approach to calculating trigonometric function values allows us to expand the domain imposed by acute angles within a right triangle to include negative angles and other angles outside the interval $\left(0^\circ, 90^\circ\right)$. Reference angles can be helpful in determining the sine and cosine of angles outside the first quadrant.

Reference Angles

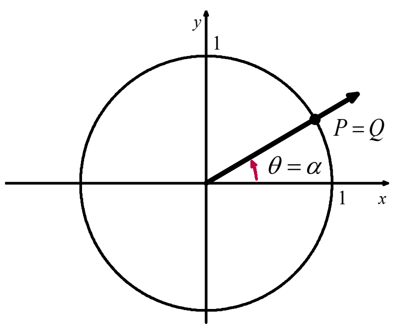
When the sine and cosine of an angle are not readily apparent, a reference angle may be used to determine the magnitude of the sine and cosine.

Definition 1.3. For a non-quadrantal angle θ, the **reference angle** for θ (often denoted α) is the acute angle made between the terminal side of θ and the x-axis.

- If θ is a Quadrant I or IV angle, the reference angle α is the angle between the terminal side of θ and the positive x-axis.
- If θ is a Quadrant II or III angle, the reference angle α is the angle between the terminal side of θ and the negative x-axis.

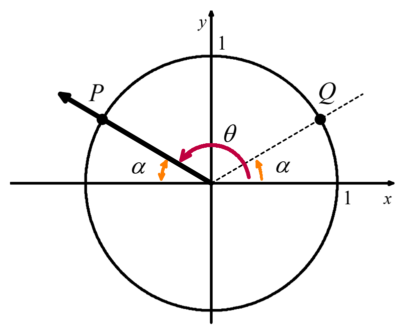
If we let P denote the point $(\cos(\theta), \sin(\theta))$, then P lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the x-axis, y-axis, and origin, regardless of where the terminal side of θ lies, there is a point Q symmetric with P that determines θ's reference angle α, as seen in the following illustration.

Figure 1.3. 22



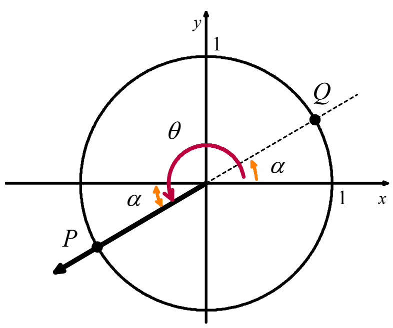
Reference angle α for a Quadrant I angle

Figure 1.3. 23



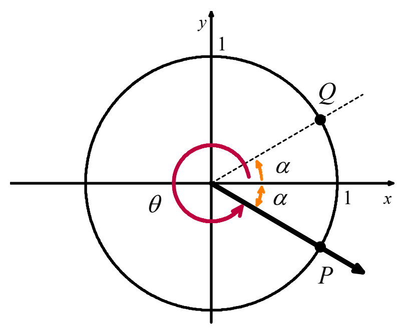
Reference angle α for a Quadrant II angle

Figure 1.3. 24



Reference angle α for a Quadrant III angle

Figure 1.3. 25



Reference angle α for a Quadrant IV angle

Note that in the above illustrations, θ may be any angle whose terminal side is coincident with the terminal side of the angle θ indicted in each sketch. We may use reference angles to determine values of $\sin(\theta)$ and $\cos(\theta)$, as long as we know $\sin(\alpha)$ and $\cos(\alpha)$, along with the quadrant in which the terminal side of θ lies.

Using Reference Angles to Determine $\sin(\theta)$ and $\cos(\theta)$

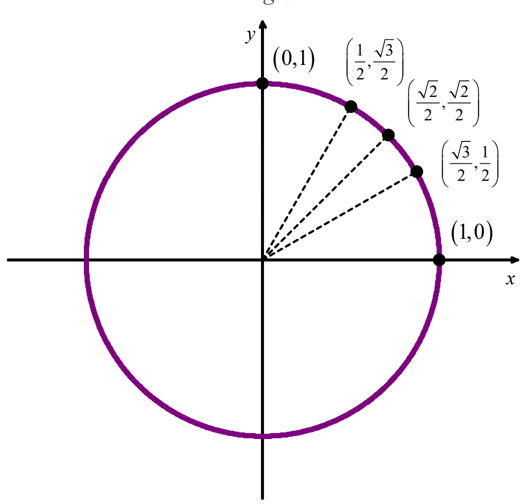
Suppose α is the reference angle for θ. Then

- $\sin(\theta)=\sin(\alpha)$ for θ in Quadrant I or II,
- $\sin(\theta)=-\sin(\alpha)$ for θ in Quadrant III or IV,
- $\cos(\theta)=\cos(\alpha)$ for θ in Quadrant I or IV,
- $\cos(\theta)=-\cos(\alpha)$ for θ in Quadrant II or III.

Notice that, in general, $\sin(\theta)=\pm\sin(\alpha)$ and $\cos(\theta)=\pm\cos(\alpha)$, where the sign, + or −, is determined by the quadrant in which the terminal side of θ lies.

It is important to know the sine and cosine values for the standard angles we introduced in **Section 1.2**, as well as the quadrantal angles 0° and 90°, or 0 and $\frac{\pi}{2}$ radians, respectively.

Figure 1.3. 26



Sine and Cosine Values of Standard Angles

θ degrees	θ radians	$\sin(\theta)$	$\cos(\theta)$
0°	0	0	1
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
90°	$\frac{\pi}{2}$	1	0

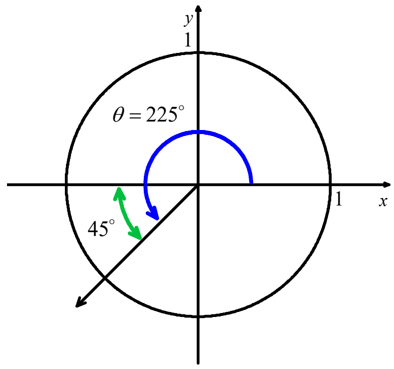
Example 1.3.3. Find the sine and cosine of the following angles.

1. $\theta=225^\circ$ 2. $\theta=\frac{11\pi}{6}$ 3. $\theta=-\frac{5\pi}{4}$ 4. $\theta=\frac{7\pi}{3}$

Solution.

1. We begin by plotting $\theta=225^\circ$ in standard position, and find that its terminal side overshoots the negative x-axis to land in Quadrant III.

Figure 1.3. 27

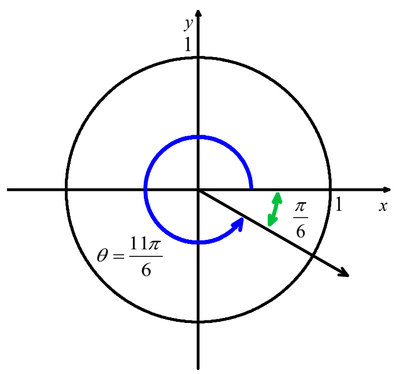


Hence, we obtain a reference angle by subtracting $225^{\circ} - 180^{\circ} = 45^{\circ}$. Since θ is a Quadrant III angle, we have $\sin(\theta) < 0$ and $\cos(\theta) < 0$. Thus, we use the reference angle of 45° to get

$$\sin(225^{\circ}) = -\sin(45^{\circ}) = -\frac{\sqrt{2}}{2} \qquad \cos(225^{\circ}) = -\cos(45^{\circ}) = -\frac{\sqrt{2}}{2}$$

2. The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive x-axis.

Figure 1.3. 28

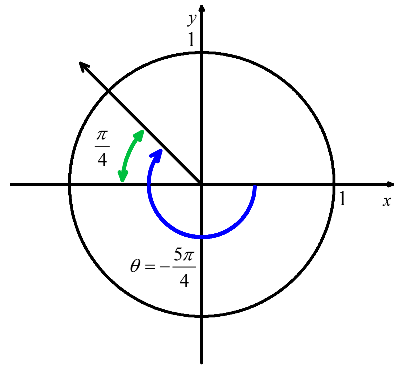


To find the reference angle, we subtract $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Since θ is a Quadrant IV angle, $\sin(\theta) < 0$ and $\cos(\theta) > 0$, so we use the reference angle of $\frac{\pi}{6}$ to find

$$\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2} \qquad \cos\left(\frac{11\pi}{6}\right) = +\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

3. To plot $\theta = -\frac{5\pi}{4}$, we rotate clockwise an angle of $\frac{5\pi}{4}$ from the positive x-axis.

Figure 1.3. 29

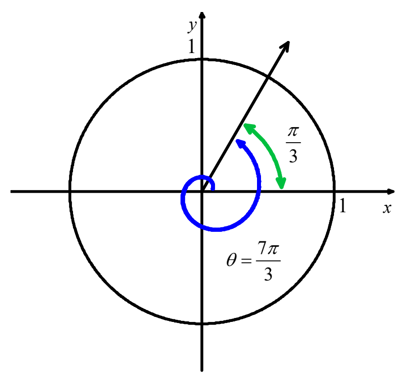


The terminal side of θ lies in Quadrant II, making an angle of $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative x-axis. Since θ is a Quadrant II angle, using the reference angle of $\frac{\pi}{4}$ gives us

$$\sin\left(-\frac{5\pi}{4}\right) = +\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

4. Since the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians.

Figure 1.3. 30



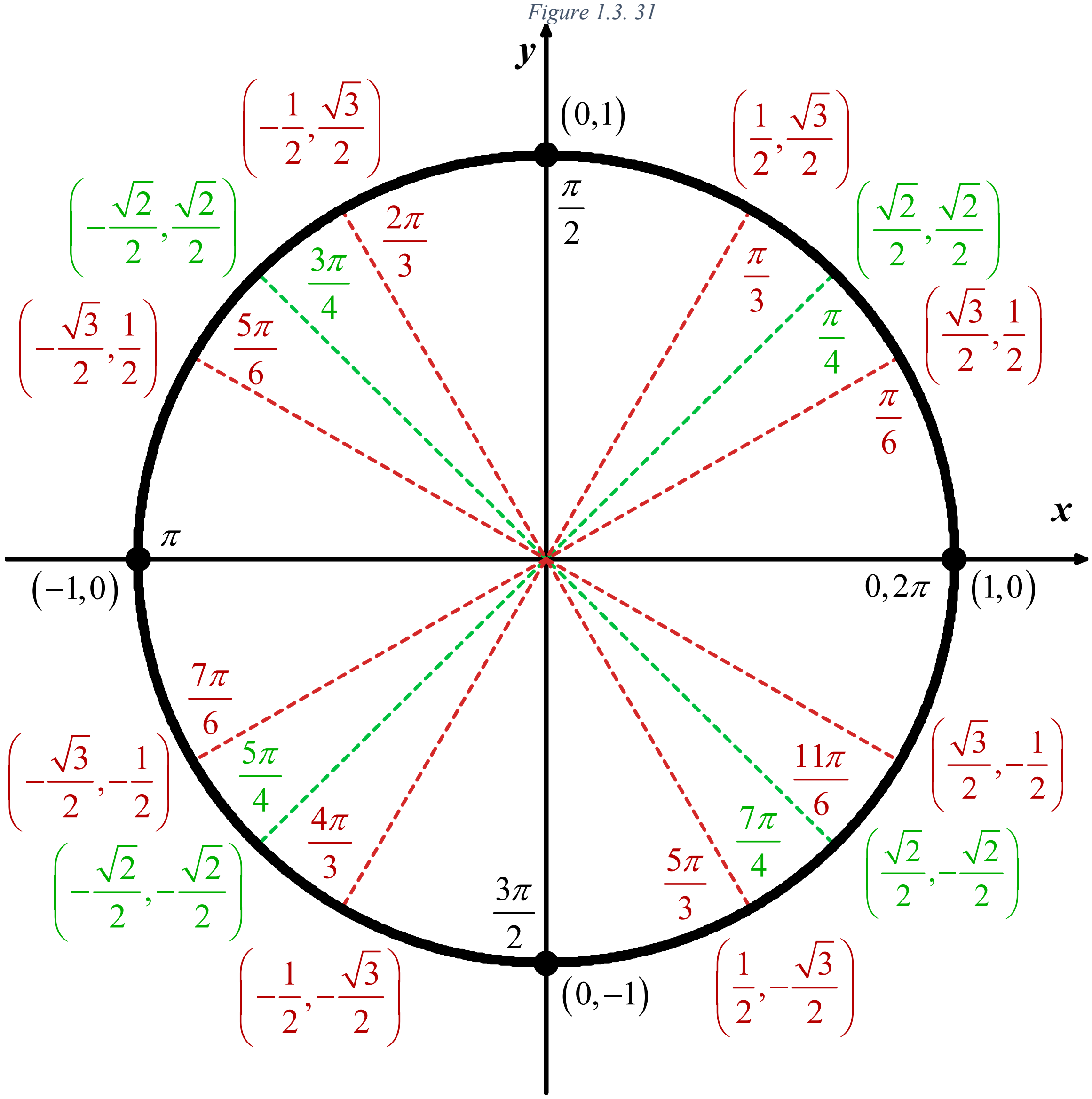
Since θ and $\frac{\pi}{3}$ are coterminal,

$$\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

□

The reader may have noticed that, when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot, as long as the reference angle is a standard angle. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle. The Unit Circle follows, with (x, y) coordinates labeled at increments of $\frac{\pi}{6}$, $\frac{\pi}{4}$ or $\frac{\pi}{3}$ radians. Since $(x, y) = (\cos(\theta), \sin(\theta))$, this graphic provides a handy reference for determining the sine and cosine of an angle terminating at one of these positions.

Figure 1.3. 31

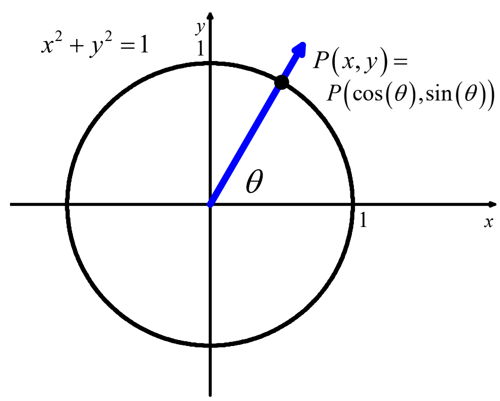


Important Points on the Unit Circle

The Pythagorean Identity

You may have noticed that the sine and cosine values for a given angle are related, and you may have guessed that if you know one you can find the other. The Pythagorean identity gives us the tools to do just that. To arrive at the Pythagorean identity, we note that the point $P(x,y)=(\cos(\theta),\sin(\theta))$ lies on the Unit Circle and $x^2+y^2=1$.

Figure 1.3. 32



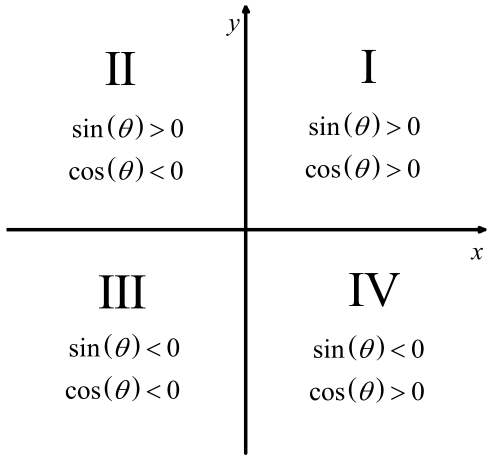
If we substitute $x=\cos(\theta)$ and $y=\sin(\theta)$ into $x^2+y^2=1$, we get $(\cos(\theta))^2+(\sin(\theta))^2=1$.

An unfortunate convention, from a function notation perspective, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. We will follow this convention. Thus, our identity results in the following theorem, one of the most important results in Trigonometry.

Theorem 1.2. The Pythagorean Identity: For any angle θ, $\sin^2(\theta)+\cos^2(\theta)=1$.

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the distance formula and the equation for a circle are derived. The word 'identity' reminds us that, regardless of the angle θ, the equation in **Theorem 1.2** is always true. If one of $\sin(\theta)$ or $\cos(\theta)$ is known, **Theorem 1.2** can be used to determine the other, up to a $(\pm)$ sign. If, in addition, we know where the terminal side of θ lies when in standard position, we can remove the ambiguity of the sign and completely determine the missing value. The following illustration summarizes the signs of sine and cosine for an angle θ with terminal side lying in one of the four quadrants.

Figure 1.3. 33



Example 1.3.4. Using the given information about θ, find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta)=\frac{3}{5}$, find $\cos(\theta)$.
2. If $\pi<\theta<\frac{3\pi}{2}$ with $\cos(\theta)=-\frac{1}{\sqrt{5}}$, find $\sin(\theta)$.
3. If $\sin(\theta)=1$, find $\cos(\theta)$.

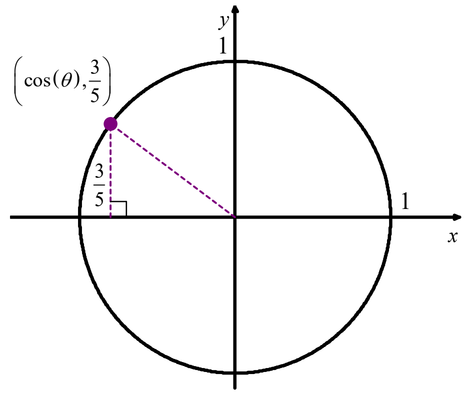
Solution.

1. When we substitute $\sin(\theta)=\frac{3}{5}$ into the Pythagorean identity, $\sin^2(\theta)+\cos^2(\theta)=1$, we obtain

$$\left(\frac{3}{5}\right)^2+\cos^2(\theta)=1$$
$$\frac{9}{25}+\cos^2(\theta)=1$$
$$\cos^2(\theta)=\frac{16}{25}$$
$$\cos(\theta)=\pm\frac{4}{5}$$

Since θ is a Quadrant II angle, its terminal side lies in Quadrant II where $x=\cos(\theta)$ is negative. Hence, $\cos(\theta)=-\frac{4}{5}$.

Figure 1.3. 34



2. Substituting $\cos(\theta) = -\frac{1}{\sqrt{5}}$ into $\sin^2(\theta) + \cos^2(\theta) = 1$, we get

$$\sin^2(\theta) + \left(-\frac{1}{\sqrt{5}}\right)^2 = 1$$
$$\sin^2(\theta) + \frac{1}{5} = 1$$
$$\sin^2(\theta) = \frac{4}{5}$$
$$\sin(\theta) = \pm\frac{2}{\sqrt{5}}$$

   We are given that $\pi < \theta < \frac{3\pi}{2}$, so we know θ is a Quadrant III angle. Since $y = \sin(\theta)$ is negative in Quadrant III, we conclude that $\sin(\theta) = -\frac{2}{\sqrt{5}}$.

3. When we substitute $\sin(\theta) = 1$ into $\sin^2(\theta) + \cos^2(\theta) = 1$, we find $\cos(\theta) = 0$.

□

Symmetry

Another tool that helps in determining sines and cosines of angles is the symmetry inherent in the Unit Circle. Suppose we wish to know the sine and cosine of $\theta = \frac{5\pi}{6}$. We plot θ in standard position and, as usual, let $P(x, y)$ denote the point on the terminal side of θ that lies on the Unit Circle. Note that the terminal side of θ lies $\frac{\pi}{6}$ radians short of one half revolution.

Figure 1.3. 35

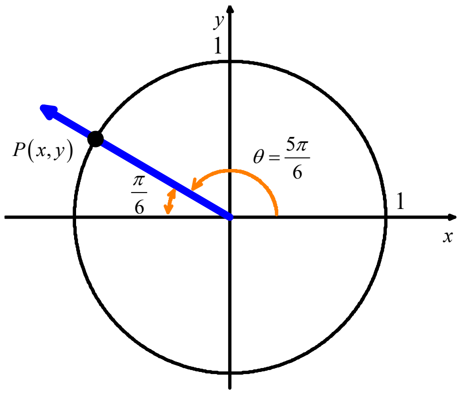
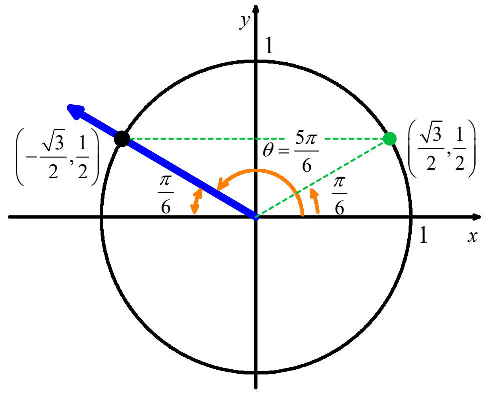


Figure 1.3. 36



From **Example 1.3.2**, we know that $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$. This means that the point on the terminal side of the angle $\frac{\pi}{6}$, when plotted in standard position, is $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. From the figures, it is clear that the point $P(x,y)$ can be obtained by reflecting the point $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ about the *y*-axis. Hence, $\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$ and $\cos\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$.

The next example summarizes all the important ideas discussed in this section.

Example 1.3.5. Suppose α is an acute angle with $\cos(\alpha)=\frac{5}{13}$.

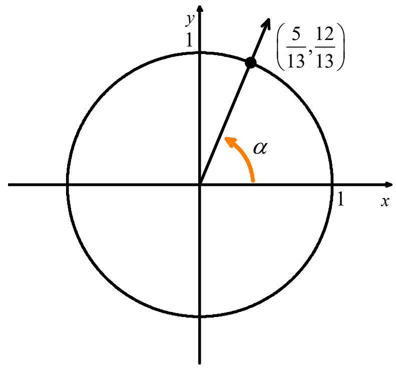
1. Find $\sin(\alpha)$ and use this to plot α in standard position.
2. Find the sine and cosine of the following angles:

(a) $\theta=\pi+\alpha$ (b) $\theta=2\pi-\alpha$ (c) $\theta=3\pi-\alpha$ (d) $\theta=\frac{\pi}{2}+\alpha$

Solution.

1. Proceeding as in **Example 1.3.4**, we substitute $\cos(\alpha)=\frac{5}{13}$ into $\sin^2(\alpha)+\cos^2(\alpha)=1$ and find $\sin(\alpha)=\pm\frac{12}{13}$. Since α is an acute (and therefore Quadrant I) angle, $\sin(\alpha)$ is positive. Hence, $\sin(\alpha)=\frac{12}{13}$. To plot α in standard position, we begin our rotation from the positive *x*-axis to the ray that contains the point $(\cos(\alpha),\sin(\alpha))=\left(\frac{5}{13},\frac{12}{13}\right)$.

Figure 1.3. 37



2. (a) To find the sine and cosine of $\theta = \pi + \alpha$, we first plot θ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotations: a rotation of π radians followed by a rotation of α radians.[10] We see that α is the reference angle for θ, so we have $\sin(\theta) = \pm\sin(\alpha) = \pm\frac{12}{13}$ and $\cos(\theta) = \pm\cos(\alpha) = \pm\frac{5}{13}$. Since the terminal side of θ lies in Quadrant III, both $\sin(\theta)$ and $\cos(\theta)$ are negative. Thus, $\sin(\theta) = -\frac{12}{13}$ and $\cos(\theta) = -\frac{5}{13}$.

Figure 1.3. 38

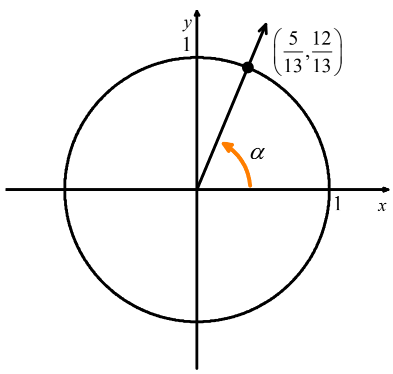
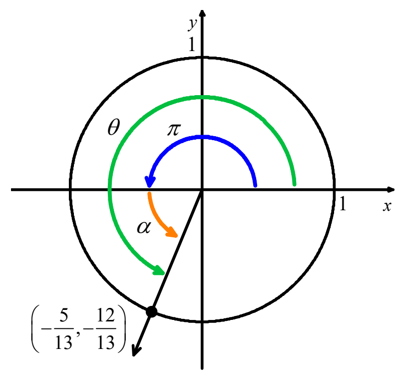


Figure 1.3. 39



$\theta = \pi + \alpha$

(b) Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot θ by visualizing one complete revolution counter-clockwise followed by a clockwise revolution, or 'backing up' of α radians. We see that α is θ's reference angle, and since θ is a Quadrant IV angle, we find $\sin(\theta) = -\frac{12}{13}$ and $\cos(\theta) = \frac{5}{13}$.

---

[10] Since $\pi + \alpha = \alpha + \pi$, θ may be plotted by reversing the order of rotations here. Try it!

Figure 1.3. 40

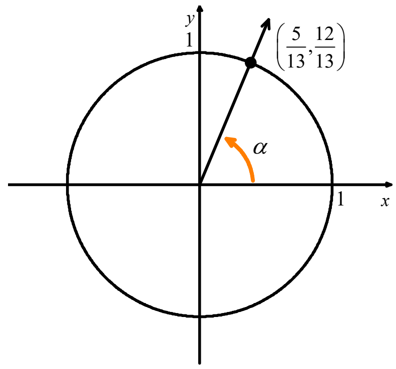
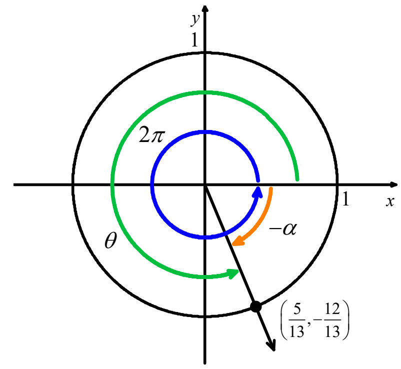


Figure 1.3. 41



$\theta = 2\pi - \alpha$

(c) Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle 3π represents one and a half revolutions counter-clockwise, so when we 'back up' α radians, we end up in Quadrant II. The result is $\sin(\theta) = \frac{12}{13}$ and $\cos(\theta) = -\frac{5}{13}$.

Figure 1.3. 42

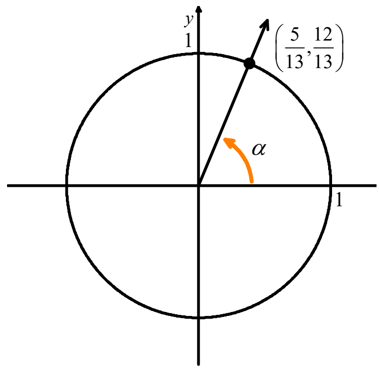
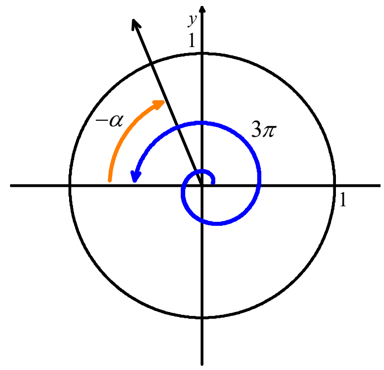
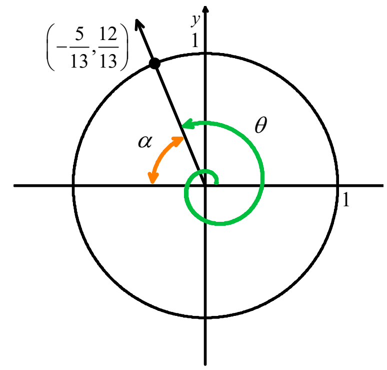


Figure 1.3. 43



$3\pi + (-\alpha)$

Figure 1.3. 44



$\theta = 3\pi - \alpha$

(d) To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with α radians. The reference angle here is not α. (It is important to see why this is the case. Take a moment to think about it before reading on.)

Figure 1.3. 45

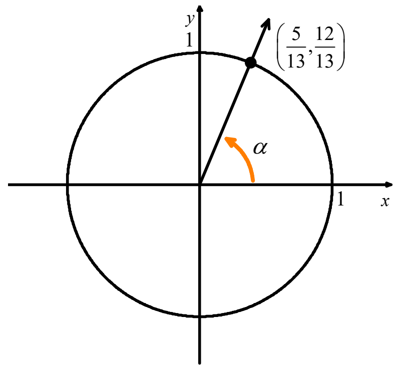
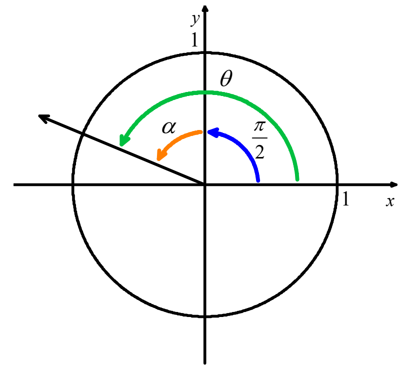


Figure 1.3. 46



$$\theta=\frac{\pi}{2}+\alpha$$

Let $P(x,y)$ be the point on the terminal side of θ that lies on the Unit Circle so that $x=\cos(\theta)$ and $y=\sin(\theta)$. To find values for x and y, we use similar triangles.

- By drawing a perpendicular line segment from P to the *x*-axis, we have a right triangle with sides of lengths 1, $|x|$, and $|y|$ (absolute value since lengths must be positive). Note that the angle opposite the side of length $|y|$ has measure $\frac{\pi}{2}-\alpha$.
- Drawing a perpendicular line segment from the point $\left(\frac{5}{13},\frac{12}{13}\right)$ to the *y*-axis, the resulting right triangle has sides of lengths 1, $\frac{5}{13}$, and $\frac{12}{13}$, with angle $\frac{\pi}{2}-\alpha$.

Figure 1.3. 47

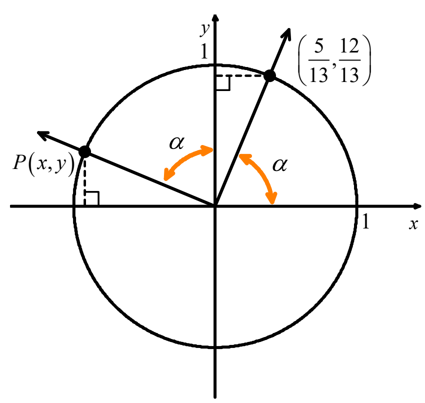


Figure 1.3. 48

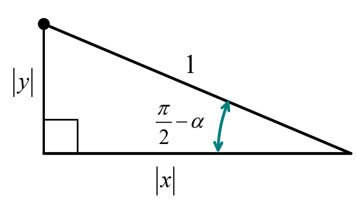
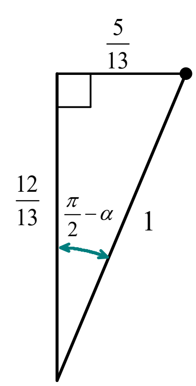


Figure 1.3. 49



We have similar triangles, from which we find that $|x|=\frac{12}{13}$ and $|y|=\frac{5}{13}$. Since P is in Quadrant II, $x=\cos(\theta)=-\frac{12}{13}$ and $y=\sin(\theta)=\frac{5}{13}$.

□

1.3 Exercises

In Exercises 1 – 5, sketch the oriented arc on the Unit Circle that corresponds to the given real number.

1. $t = \frac{5\pi}{6}$
2. $t = -\pi$
3. $t = 6$
4. $t = -2$
5. $t = 12$

In Exercises 6 – 9, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6. $\sin(t) < 0$ and $\cos(t) < 0$
7. $\sin(t) > 0$ and $\cos(t) > 0$
8. $\sin(t) > 0$ and $\cos(t) < 0$
9. $\sin(t) < 0$ and $\cos(t) > 0$

10. Use the numbers 0, 1, 2, 3, and 4 to complete the following table of sine and cosine values for common angles. (This exercise serves as a memory tool for remembering these values.)

θ	$\sin(\theta)$	$\cos(\theta)$
0	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{2}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$

In Exercises 11 – 30, find the exact value of the sine and cosine of the given angle.

11. $\theta = 0$
12. $\theta = \frac{\pi}{4}$
13. $\theta = \frac{\pi}{3}$
14. $\theta = \frac{\pi}{2}$
15. $\theta = \frac{2\pi}{3}$
16. $\theta = \frac{3\pi}{4}$
17. $\theta = \pi$
18. $\theta = \frac{7\pi}{6}$
19. $\theta = \frac{5\pi}{4}$
20. $\theta = \frac{4\pi}{3}$
21. $\theta = \frac{3\pi}{2}$
22. $\theta = \frac{5\pi}{3}$
23. $\theta = \frac{7\pi}{4}$
24. $\theta = \frac{23\pi}{6}$
25. $\theta = -\frac{13\pi}{2}$
26. $\theta = -\frac{43\pi}{6}$

27. $\theta = -\dfrac{3\pi}{4}$ 28. $\theta = -\dfrac{\pi}{6}$ 29. $\theta = \dfrac{10\pi}{3}$ 30. $\theta = 117\pi$

In Exercises 31 – 42, use the results developed throughout this section to find the requested value.

31. If $\cos(\theta) = \dfrac{4}{5}$ and $\sin(\theta) < 0$, find $\sin(\theta)$.

32. If $\sin(\theta) = -\dfrac{2}{5}$ and $\cos(\theta) < 0$, find $\cos(\theta)$.

33. If $\sin(\theta) = -\dfrac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?

34. If $\cos(\theta) = \dfrac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?

35. If $\sin(\theta) = \dfrac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?

36. If $\cos(\theta) = -\dfrac{2}{11}$ with θ in Quadrant III, what is $\sin(\theta)$?

37. If $\sin(\theta) = -\dfrac{2}{3}$ with θ in Quadrant III, what is $\cos(\theta)$?

38. If $\cos(\theta) = \dfrac{28}{53}$ with θ in Quadrant IV, what is $\sin(\theta)$?

39. If $\sin(\theta) = \dfrac{2\sqrt{5}}{5}$ and $\dfrac{\pi}{2} < \theta < \pi$, what is $\cos(\theta)$?

40. If $\cos(\theta) = \dfrac{\sqrt{10}}{10}$ and $2\pi < \theta < \dfrac{5\pi}{2}$, what is $\sin(\theta)$?

41. If $\sin(\theta) = -0.42$ and $\pi < \theta < \dfrac{3\pi}{2}$, what is $\cos(\theta)$?

42. If $\cos(\theta) = -0.98$ and $\dfrac{\pi}{2} < \theta < \pi$, what is $\sin(\theta)$?

1.4 The Six Trigonometric Functions

Learning Objectives

- **Determine the values of the six trigonometric functions from a point on the Unit Circle.**
- **Know and apply the quotient and reciprocal identities.**
- **Find angles that satisfy trigonometric equations.**
- **Use reference angles in determining trigonometric function values.**

In this section, we return to the definition of sine and cosine of any angle and extend that definition to include the remaining four trigonometric functions: tangent, cosecant, secant and cotangent.

The Six Trigonometric Functions

Definition 1.4. Suppose θ is an angle plotted in standard position and $P(x,y)$ is the point on the terminal side of θ that lies on the Unit Circle. The trigonometric functions are defined as follows.

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta)=y$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta)=x$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta)=\dfrac{y}{x}$, provided $x\neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta)=\dfrac{1}{y}$, provided $y\neq 0$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta)=\dfrac{1}{x}$, provided $x\neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta)=\dfrac{x}{y}$, provided $y\neq 0$.

In **Section 1.3**, we defined $\sin(\theta)$ and $\cos(\theta)$ for angles θ using the coordinate values of points on the Unit Circle. Since $\sin(\theta)=y$ and $\cos(\theta)=x$ by this definition, it is customary to rephrase the remaining four trigonometric functions in terms of sine and cosine. To do so, we simply replace y with $\sin(\theta)$ and x with $\cos(\theta)$ in the preceding definition. This results in our reference to the tangent, cosecant, secant and cotangent as **quotient identities** or **reciprocal identities**.

Quotient Identities

- $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$, provided $x = \cos(\theta) \neq 0$.
- $\cot(\theta) = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)}$, provided $y = \sin(\theta) \neq 0$.

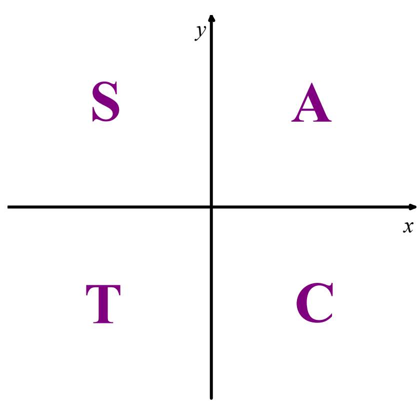
Reciprocal Identities

- $\csc(\theta) = \frac{1}{y} = \frac{1}{\sin(\theta)}$, provided $y = \sin(\theta) \neq 0$.
- $\sec(\theta) = \frac{1}{x} = \frac{1}{\cos(\theta)}$, provided $x = \cos(\theta) \neq 0$.
- The cotangent may also be thought of as the reciprocal of the tangent:

  $\cot(\theta) = \frac{x}{y} = \frac{1}{y/x} = \frac{1}{\tan(\theta)}$, provided $\tan(\theta) \neq 0$ or, equivalently, $y \neq 0$.

Note that, of the six trigonometric functions, only sine and cosine are defined for all angles. Before using the quotient and reciprocal identities in an example, the following mnemonic may help with remembering the signs of sine, cosine, and tangent values in each quadrant. We assign the first letter of each word in the phrase "**A**ll **S**tudents **T**ake **C**alculus" to Quadrants I, II, III, and IV, respectively. Now we observe that sine, cosine, and tangent are **A**ll positive in Quadrant I; the **S**ine alone is positive in Quadrant II; the **T**angent alone is positive in Quadrant III; the **C**osine alone is positive in Quadrant IV.

Figure 1.4. 1



Example 1.4.1. Find the indicated value, if it exists.

1. $\sec(60^\circ)$
2. $\csc\left(\frac{7\pi}{4}\right)$
3. $\cot(3)$
4. $\tan(\theta)$, where θ is any angle coterminal with $\frac{3\pi}{2}$
5. $\cos(\theta)$, where $\csc(\theta) = -\sqrt{5}$ and θ is a Quadrant IV angle
6. $\sin(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$

Solution.

1. For $\sec(60^\circ)$, the reciprocal identity for secant will help us out.

$$\begin{aligned}\sec(60^\circ) &= \frac{1}{\cos(60^\circ)} \\ &= \frac{1}{1/2} \\ &= 2\end{aligned}$$

2. To find $\csc\left(\frac{7\pi}{4}\right)$, we apply the reciprocal identity for cosecant ant note that $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

$$\begin{aligned}\csc\left(\frac{7\pi}{4}\right) &= \frac{1}{\sin\left(\frac{7\pi}{4}\right)} \\ &= \frac{1}{-\sqrt{2}/2} \\ &= -\sqrt{2}\end{aligned}$$

3. In determining the value of $\cot(3)$, since the reference angle for $\theta = 3$ radians is not one of the standard angles from **Section 1.3**, we resort to the calculator for a decimal approximation. We use the quotient identity for cotangent and check that our calculator is in radian mode.

$$\begin{aligned}\cot(3) &= \frac{\cos(3)}{\sin(3)} \\ &\approx -7.015\end{aligned}$$

Noting that $\cot(\theta) = \frac{1}{\tan(\theta)}$, this problem could also be solved as follows.[11]

[11] Cosecant, secant, and cotangent are not available on most calculators so it is necessary to convert to expressions involving sine, cosine, and/or tangent before finding an approximate value with a calculator.

$$\cot(3) = \frac{1}{\tan(3)} \approx -7.015$$

4. To determine $\tan(\theta)$, where θ is coterminal with $\frac{3\pi}{2}$, we find $\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$ and $\cos(\theta) = \cos\left(\frac{3\pi}{2}\right) = 0$. Computing $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ results in $\frac{-1}{0}$, so $\tan(\theta)$ is undefined.

5. We are given that $\csc(\theta) = -\sqrt{5}$ and we are looking for the value of $\cos(\theta)$. From $\csc(\theta) = \frac{1}{\sin(\theta)}$, it follows that $\sin(\theta) = -\frac{1}{\sqrt{5}}$. We apply the Pythagorean identity as follows.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\left(-\frac{1}{\sqrt{5}}\right)^2 + \cos^2(\theta) &= 1 \\
\cos^2(\theta) &= \frac{4}{5} \\
\cos(\theta) &= \pm\frac{2}{\sqrt{5}}
\end{aligned}$$

The problem states that θ is a Quadrant IV angle, so $\cos(\theta) > 0$ and we find $\cos(\theta) = \frac{2}{\sqrt{5}}$.

6. Given that $\tan(\theta) = 3$, we must determine the value of $\sin(\theta)$. From the quotient identity, we know $\frac{\sin(\theta)}{\cos(\theta)} = 3$, but be careful! We can NOT assume any values for $\sin(\theta)$ or $\cos(\theta)$. We CAN assume that $\sin(\theta) = 3\cos(\theta)$. It follows that $\frac{1}{3}\sin(\theta) = \cos(\theta)$ and we use this equality, along with the Pythagorean identity, $\sin^2(\theta) + \cos^2(\theta) = 1$, to determine $\sin(\theta)$.

$$\begin{aligned}
\sin^2(\theta) + \left(\frac{1}{3}\sin(\theta)\right)^2 &= 1 \\
\sin^2(\theta) + \frac{1}{9}\sin^2(\theta) &= 1 \\
\frac{10}{9}\sin^2(\theta) &= 1 \\
\sin^2(\theta) &= \frac{9}{10} \\
\sin(\theta) &= \pm\frac{3}{\sqrt{10}}
\end{aligned}$$

Since the problem states that $\pi < \theta < \frac{3\pi}{2}$, θ is in Quadrant III where $\sin(\theta) < 0$. Thus, $\sin(\theta) = -\frac{3}{\sqrt{10}}$.

□

While the quotient and reciprocal identities allow us to always convert problems involving tangent, cotangent, secant, and cosecant to problems involving sine and cosine, it is not always convenient to do so.[12] The trigonometric function values of standard angles are summarized in the following table. Note that $\tan(\theta)$, $\cot(\theta)$, $\sec(\theta)$, and $\csc(\theta)$ can be derived from $\sin(\theta)$ and/or $\cos(\theta)$ using the quotient and reciprocal identities.

Trigonometric Function Values of Standard Angles

θ degrees	θ radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\cot(\theta)$	$\sec(\theta)$	$\csc(\theta)$
0°	0	0	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1	1	1	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1

Before working with tangent and cotangent values, we apply table values for sine and cosine in a search for angles that satisfy given trigonometric equations.

Finding Angles that Satisfy Sine and Cosine Equations

Our next example asks us to solve some very basic trigonometric equations.[13]

---

[12] As we shall see shortly, when solving equations involving secant and cosecant, we usually convert back to sines and cosines. However, when solving for tangent or cotangent, we usually stick with what we're dealt.

[13] We will study trigonometric equations more formally in **Chapter 4**. Enjoy these relatively straightforward exercises while they last!

Example 1.4.2. Find all angles that satisfy the given equation.

1. $\cos(\theta)=\frac{1}{2}$ 2. $\sin(\theta)=-\frac{1}{2}$ 3. $\cos(\theta)=0$

Solution. Since there is no indication whether to use degrees or radians, we will default to using radian measure in each of these problems. This choice will be justified later in the text when we study what is known as Analytic Trigonometry. In those sections to come, radian measure will be the only appropriate angle measure so it is worth the time to become fluent in radians now.

1. If $\cos(\theta)=\frac{1}{2}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $x=\frac{1}{2}$. This means θ is a Quadrant I or Quadrant IV angle with reference angle $\frac{\pi}{3}$.

Figure 1.4. 2

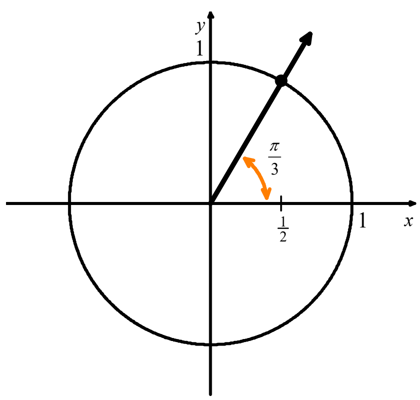
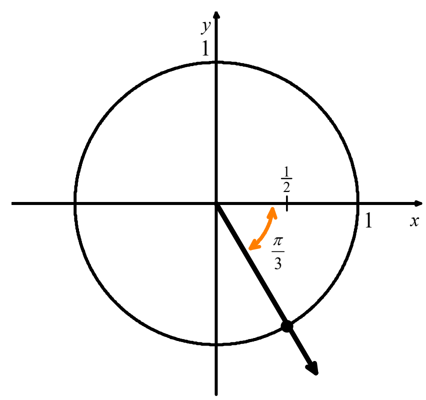


Figure 1.4. 3



One solution in Quadrant I is $\theta=\frac{\pi}{3}$, and since all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$, we find $\theta=\frac{\pi}{3}+2\pi k$ for integers k.[14] Proceeding similarly for the Quadrant IV case, we find a solution to $\cos(\theta)=\frac{1}{2}$ is $\frac{5\pi}{3}$, so our answer in this quadrant is $\theta=\frac{5\pi}{3}+2\pi k$ for integers k.

2. If $\sin(\theta)=-\frac{1}{2}$, then when θ is plotted in standard position its terminal side intersects with the Unit Circle at $y=-\frac{1}{2}$. From this, we determine θ is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$.

[14] Recall in **Section 1.1**, two angles in radian measure are coterminal if and only if they differ by an integer multiple of 2π. Hence, to describe all angles coterminal with a given angle, we add $2\pi k$ for integers $k=0,\pm1,\pm2,\ldots$.

Figure 1.4. 4

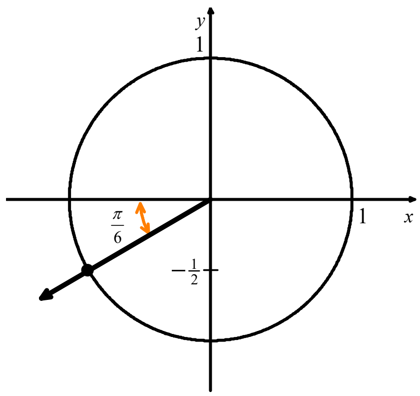
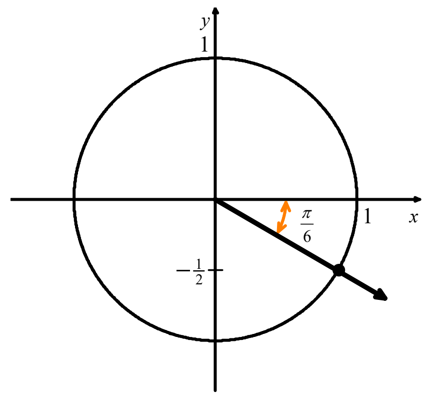


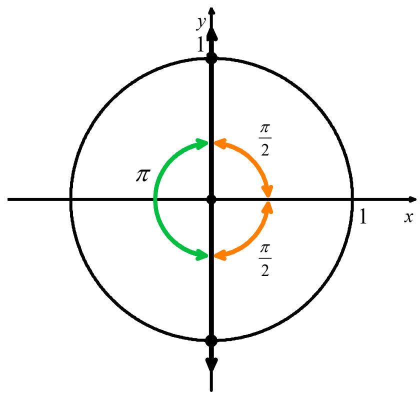
Figure 1.4. 5



In Quadrant III one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of 2π : $\theta = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all solutions here are of the form $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k .

3. The angles with $\cos(\theta) = 0$ are quadrantal angles whose terminal sides, when plotted in standard position, lie along the y-axis.

Figure 1.4. 6



While, technically speaking, $\frac{\pi}{2}$ is not a reference angle, we can still use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\theta = \frac{\pi}{2} + 2\pi k$ and $\theta = \frac{3\pi}{2} + 2\pi k$ for integers k . While this solution is correct, it can be shortened to $\theta = \frac{\pi}{2} + \pi k$ for integers k . (Can you see why this works from the diagram?)

□

One of the key concepts to take from the previous example is that, in general, solutions to trigonometric equations consist of infinitely many answers. To get a feel for the answers, the reader is encouraged to follow our mantra 'When in doubt, write it out!' For example, in part 2 of **Example 1.4.2**, another

Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers could have been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. While on the surface this family may look different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k, we leave it to the reader to show they represent the same list of angles.

Finding Angles that Satisfy Other Trigonometric Equations

Before solving equations that contain $\tan(\theta)$, $\csc(\theta)$, $\sec(\theta)$, or $\cot(\theta)$, we return to the reference angle for θ, as identified in **Section 1.3**. By coupling the quotient and reciprocal identities with our use of reference angles in determining $\sin(\theta)$ and $\cos(\theta)$, we have the following.

Using Reference Angles to Determine Trigonometric Function Values

Suppose α is the reference angle for θ. Then,

- $\sin(\theta) = \sin(\alpha)$ and $\csc(\theta) = \csc(\alpha)$ in Quadrants I & II,
- $\sin(\theta) = -\sin(\alpha)$ and $\csc(\theta) = -\csc(\alpha)$ in Quadrants III & IV,
- $\cos(\theta) = \cos(\alpha)$ and $\sec(\theta) = \sec(\alpha)$ in Quadrants I & IV,
- $\cos(\theta) = -\cos(\alpha)$ and $\sec(\theta) = -\sec(\alpha)$ in Quadrants II & III,
- $\tan(\theta) = \tan(\alpha)$ and $\cot(\theta) = \cot(\alpha)$ in Quadrants I & III,
- $\tan(\theta) = -\tan(\alpha)$ and $\cot(\theta) = -\cot(\alpha)$ in Quadrants II & IV.

Notice that, in general, $\sin(\theta) = \pm\sin(\alpha)$, $\cos(\theta) = \pm\cos(\alpha)$, $\tan(\theta) = \pm\tan(\alpha)$, $\csc(\theta) = \pm\csc(\alpha)$, $\sec(\theta) = \pm\sec(\alpha)$, and $\cot(\theta) = \pm\cot(\alpha)$, where the sign, + or −, is determined by the quadrant in which the terminal side of θ lies. The following example makes good use of reference angles.

Example 1.4.3. Find all angles that satisfy the given equation.

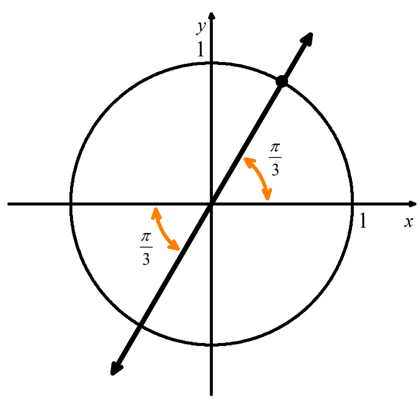
1. $\sec(\theta) = 2$
2. $\tan(\theta) = \sqrt{3}$
3. $\cot(\theta) = -1$

Solution.

1. To solve $\sec(\theta)=2$, we convert to cosines and get $\frac{1}{\cos(\theta)}=2$, or $\cos(\theta)=\frac{1}{2}$. This is the same equation we solved in **Example 1.4.2**, so we know the answer is $\theta=\frac{\pi}{3}+2\pi k$ or $\theta=\frac{5\pi}{3}+2\pi k$ for integers k.

2. From the table 'Trigonometric Function Values of Standard Angles', we see $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$. It follows that the solutions to $\tan(\theta)=\sqrt{3}$ must have a reference angle of $\frac{\pi}{3}$. Our next task is to determine in which quadrants the solutions to this equation lie. Since the tangent is defined as the ratio $\frac{y}{x}$ of points (x,y) on the Unit Circle with $x\neq 0$, the tangent is positive when x and y have the same sign (i.e. when they are both positive or both negative.) This happens in Quadrants I and III.

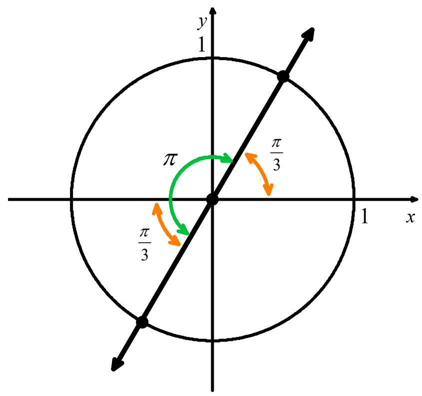
Figure 1.4. 7



In Quadrant I we get the solutions $\theta=\frac{\pi}{3}+2\pi k$ for integers k, and for Quadrant III we get $\theta=\frac{4\pi}{3}+2\pi k$ for integers k. While these descriptions of the solutions are correct, they can be combined as $\theta=\frac{\pi}{3}+\pi k$ for integers k. The latter form of the solution is best understood by looking at the geometry of the situation, below.[15]

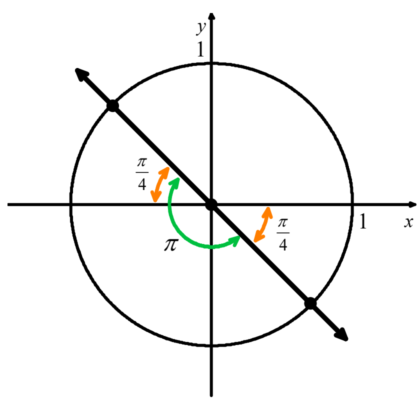
[15] See **Example 1.4.2**, part 3, for another example of this kind of simplification of the solution.

Figure 1.4. 8



3. We find that $\frac{\pi}{4}$ has a cotangent of 1. This tells us the solutions to $\cot(\theta) = -1$ have a reference angle of $\frac{\pi}{4}$. To find the quadrants in which our solutions lie, we note that $\cot(\theta) = \frac{x}{y}$ for a point (x, y) on the Unit Circle where $y \neq 0$. If $\cot(\theta)$ is negative, then x and y must have different signs (i.e. one positive and one negative.) Hence, our solutions lie in Quadrants II and IV. Our Quadrant II solution is $\theta = \frac{3\pi}{4} + 2\pi k$, and for Quadrant IV we get $\theta = \frac{7\pi}{4} + 2\pi k$, for integers k. Can these be combined? Indeed, they can! One such way to capture all the solutions is $\theta = \frac{3\pi}{4} + \pi k$ for integers k.

Figure 1.4. 9



□

Suppose we are asked to solve an equation such as $\sin(t) = -\frac{1}{2}$. Recall our introduction of t in **Section 1.3** as a real number. The real number t may be associated with an oriented arc on the Unit Circle that subtends an angle of t radians. Indeed, we solve $\sin(t) = -\frac{1}{2}$ in the same exact manner[16] as we did in **Example 1.4.2** part 2. Our solution, which is only cosmetically different in that the variable used is t rather than θ, is $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers k. As we progress in our study of the trigonometric functions, keep in mind that any properties developed that regard them as functions of angles in radian measure apply equally well if the inputs are regarded as real numbers.

[16] Well, to be pedantic, we would technically be using reference numbers or reference arcs instead of reference angles, but the idea is the same.

1.4 Exercises

In Exercises 1 – 20, find the exact value or state that it is undefined.

1. $\tan\left(\frac{\pi}{4}\right)$
2. $\sec\left(\frac{\pi}{6}\right)$
3. $\csc\left(\frac{5\pi}{6}\right)$
4. $\cot\left(\frac{4\pi}{3}\right)$
5. $\tan\left(-\frac{11\pi}{6}\right)$
6. $\sec\left(-\frac{3\pi}{2}\right)$
7. $\csc\left(-\frac{\pi}{3}\right)$
8. $\cot\left(\frac{13\pi}{2}\right)$
9. $\tan(117\pi)$
10. $\sec\left(-\frac{5\pi}{3}\right)$
11. $\csc(3\pi)$
12. $\cot(-5\pi)$
13. $\tan\left(\frac{31\pi}{2}\right)$
14. $\sec\left(\frac{\pi}{4}\right)$
15. $\csc\left(-\frac{7\pi}{4}\right)$
16. $\cot\left(\frac{7\pi}{6}\right)$
17. $\tan\left(\frac{2\pi}{3}\right)$
18. $\sec(-7\pi)$
19. $\csc\left(\frac{\pi}{2}\right)$
20. $\cot\left(\frac{3\pi}{4}\right)$

In Exercises 21 – 44, find all angles θ that satisfy the given equation. Give exact values in radians.

21. $\sin(\theta)=\frac{1}{2}$
22. $\cos(\theta)=-\frac{\sqrt{3}}{2}$
23. $\sin(\theta)=0$
24. $\cos(\theta)=\frac{\sqrt{2}}{2}$
25. $\sin(\theta)=\frac{\sqrt{3}}{2}$
26. $\cos(\theta)=-1$
27. $\sin(\theta)=-1$
28. $\cos(\theta)=\frac{\sqrt{3}}{2}$
29. $\cos(\theta)=-1.001$
30. $\tan(\theta)=\sqrt{3}$
31. $\sec(\theta)=2$
32. $\csc(\theta)=-1$
33. $\cot(\theta)=\frac{\sqrt{3}}{3}$
34. $\tan(\theta)=0$
35. $\sec(\theta)=1$
36. $\csc(\theta)=2$
37. $\cot(\theta)=0$
38. $\tan(\theta)=-1$
39. $\sec(\theta)=0$
40. $\csc(\theta)=-\frac{1}{2}$
41. $\sec(\theta)=-1$
42. $\tan(\theta)=-\sqrt{3}$
43. $\csc(\theta)=-2$
44. $\cot(\theta)=-1$

In Exercises 45 – 60, solve the equation for the indicated angle/real number. Give exact values in radians.

45. $\cos(t)=0$
46. $\sin(t)=-\frac{\sqrt{2}}{2}$
47. $\cos(x)=3$
48. $\cos(\beta)=\frac{1}{2}$

49. $\sin(t) = -2$

50. $\cos(t) = 1$

51. $\sin(x) = 1$

52. $\cos(x) = -\frac{\sqrt{2}}{2}$

53. $\cot(\beta) = 1$

54. $\tan(\gamma) = \frac{\sqrt{3}}{3}$

55. $\sec(x) = -\frac{2\sqrt{3}}{3}$

56. $\csc(t) = 0$

57. $\cot(\beta) = -\sqrt{3}$

58. $\tan(x) = -\frac{\sqrt{3}}{3}$

59. $\sec(t) = \frac{2\sqrt{3}}{3}$

60. $\csc(\gamma) = \frac{2\sqrt{3}}{3}$

61. Explain why the fact that $\tan(\theta) = 3 = \frac{3}{1}$ does not necessarily mean $\sin(\theta) = 3$ and $\cos(\theta) = 1$.

1.5 Trigonometric Identities

Learning Objectives

- **Apply Pythagorean identities and Pythagorean conjugates.**
- **Verify that a trigonometric equation is an identity.**

You may recall that an equation is an identity if it holds true for all independent variables for which each side of the equation is defined. We have already seen the following identities in our study of Trigonometry.

Pythagorean Identity: $\sin^2(\theta)+\cos^2(\theta)=1$

Quotient Identities: $\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}$, $\cot(\theta)=\dfrac{\cos(\theta)}{\sin(\theta)}$

Reciprocal Identities: $\csc(\theta)=\dfrac{1}{\sin(\theta)}$, $\sec(\theta)=\dfrac{1}{\cos(\theta)}$, $\cot(\theta)=\dfrac{1}{\tan(\theta)}$

Notice that the Pythagorean identity holds for all real values of θ while, for example, $\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}$ holds for all values of θ for which $\cos(\theta)\neq 0$. These are the angles whose terminal side, in standard position, is not on the *y*-axis. Our next task is to use these identities to derive Pythagorean identities for the remaining four trigonometric functions.

Variations of the Pythagorean Identity

Theorem 1.2 states that, for any angle θ, $\sin^2(\theta)+\cos^2(\theta)=1$. Through manipulating the identity, we will obtain two alternate versions relating tangent and secant, followed by cotangent and cosecant. To obtain an identity relating tangent and secant, we start with $\sin^2(\theta)+\cos^2(\theta)=1$ and, assuming $\cos(\theta)\neq 0$, divide both sides of the equation by $\cos^2(\theta)$.

$$\begin{aligned}\sin^2(\theta)+\cos^2(\theta)&=1\\ \frac{\sin^2(\theta)}{\cos^2(\theta)}+\frac{\cos^2(\theta)}{\cos^2(\theta)}&=\frac{1}{\cos^2(\theta)}\\ \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2+1&=\left(\frac{1}{\cos(\theta)}\right)^2\end{aligned}$$

Applying the quotient and reciprocal identities, we get the Pythagorean identity $\tan^2(\theta)+1=\sec^2(\theta)$.

Next, for an identity relating cotangent and cosecant, we assume $\sin(\theta)\neq 0$ and divide both sides of $\sin^2(\theta)+\cos^2(\theta)=1$ by $\sin^2(\theta)$.

$$\sin^2(\theta)+\cos^2(\theta)=1$$
$$\frac{\sin^2(\theta)}{\sin^2(\theta)}+\frac{\cos^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)}$$
$$1+\left(\frac{\cos(\theta)}{\sin(\theta)}\right)^2=\left(\frac{1}{\sin(\theta)}\right)^2$$

Applying the quotient and reciprocal identities, we get the third Pythagorean identity, $1+\cot^2(\theta)=\csc^2(\theta)$.

The three Pythagorean identities, along with some of their other common forms, are summarized in the following theorem.

Theorem 1.3. The Pythagorean Identity and its Variations:

1. $\mathbf{\sin^2(\theta)+\cos^2(\theta)=1}$

   Alternate Forms: $1-\sin^2(\theta)=\cos^2(\theta)$ and $1-\cos^2(\theta)=\sin^2(\theta)$

2. $\mathbf{\tan^2(\theta)+1=\sec^2(\theta)}$

   Alternate Forms: $\sec^2(\theta)-1=\tan^2(\theta)$ and $\sec^2(\theta)-\tan^2(\theta)=1$

3. $\mathbf{1+\cot^2(\theta)=\csc^2(\theta)}$

   Alternate Forms: $\csc^2(\theta)-1=\cot^2(\theta)$ and $\csc^2(\theta)-\cot^2(\theta)=1$

Trigonometric identities play an important role, both in Trigonometry and Calculus. We use them in this book to find the values of the trigonometric functions of an angle and to solve equations. In Calculus, they are needed to rewrite expressions in a format that enables or simplifies integration.

Verifying that a Trigonometric Equation is an Identity

To verify or prove an identity, we typically start with one side of the equation and reduce it to the other side. In the next few examples, we will demonstrate several techniques to do just that, including: perform algebraic operations; factor; rewrite each trigonometric function in terms of the trigonometric functions that appear on the other side; multiply numerator and denominator by the same expression; apply trigonometric identities; combine two or more of the previous techniques.

Example 1.5.1. Verify the identity $\left(\sec(\theta)-\tan(\theta)\right)\left(\sec(\theta)+\tan(\theta)\right)=1$.

Solution. The left side is the more complicated side, so we begin with the left side. We perform the indicated algebraic operation, then apply a trigonometric identity, bringing us to the right side of the equation.

$$\begin{aligned}
&\left(\sec(\theta)-\tan(\theta)\right)\left(\sec(\theta)+\tan(\theta)\right)\\
&\quad=\sec^2(\theta)+\sec(\theta)\tan(\theta)-\tan(\theta)\sec(\theta)-\tan^2(\theta) && \text{binomial multiplication}\\
&\quad=\sec^2(\theta)-\tan^2(\theta)\\
&\quad=1 && \text{Pythagorean identity}
\end{aligned}$$

We have verified that $\left(\sec(\theta)-\tan(\theta)\right)\left(\sec(\theta)+\tan(\theta)\right)=1$.

□

Example 1.5.2. Verify the identity $6\sec(\theta)\tan(\theta)=\dfrac{3}{1-\sin(\theta)}-\dfrac{3}{1+\sin(\theta)}$.

Solution. The right side is the more complicated side in this equation, so we start with the right side, performing the indicated algebraic operation and applying identities.

$$\begin{aligned}
\frac{3}{1-\sin(\theta)}-\frac{3}{1+\sin(\theta)}&=\frac{3\left(1+\sin(\theta)\right)-3\left(1-\sin(\theta)\right)}{\left(1-\sin(\theta)\right)\left(1+\sin(\theta)\right)} && \text{combine fractions}\\
&=\frac{3+3\sin(\theta)-3+3\sin(\theta)}{1-\sin^2(\theta)}\\
&=\frac{6\sin(\theta)}{1-\sin^2(\theta)}\\
&=\frac{6\sin(\theta)}{\cos^2(\theta)} && \text{Pythagorean identity}\\
&=6\left(\frac{1}{\cos(\theta)}\right)\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\\
&=6\sec(\theta)\tan(\theta) && \text{quotient \& reciprocal identities}
\end{aligned}$$

We have verified that $6\sec(\theta)\tan(\theta)=\dfrac{3}{1-\sin(\theta)}-\dfrac{3}{1+\sin(\theta)}$.

□

Example 1.5.3. Verify the identity $\dfrac{\sin^3(\theta)+\cos^3(\theta)}{\sin^2(\theta)-\cos^2(\theta)}=\dfrac{\sec(\theta)-\sin(\theta)}{\tan(\theta)-1}$.

Solution. We start with the left side and factor both numerator and denominator.

$$\begin{aligned}
\frac{\sin^3(\theta)+\cos^3(\theta)}{\sin^2(\theta)-\cos^2(\theta)} &= \frac{\left(\sin(\theta)+\cos(\theta)\right)\left(\sin^2(\theta)-\sin(\theta)\cos(\theta)+\cos^2(\theta)\right)}{\left(\sin(\theta)-\cos(\theta)\right)\left(\sin(\theta)+\cos(\theta)\right)} && \text{factor} \\
&= \frac{\sin^2(\theta)-\sin(\theta)\cos(\theta)+\cos^2(\theta)}{\sin(\theta)-\cos(\theta)} \\
&= \frac{1-\sin(\theta)\cos(\theta)}{\sin(\theta)-\cos(\theta)} && \text{Pythagorean identity} \\
&= \frac{\dfrac{1}{\cos(\theta)}-\dfrac{\sin(\theta)\cos(\theta)}{\cos(\theta)}}{\dfrac{\sin(\theta)}{\cos(\theta)}-\dfrac{\cos(\theta)}{\cos(\theta)}} && \text{multiply by } \frac{1/\cos(\theta)}{1/\cos(\theta)} \\
&= \frac{\sec(\theta)-\sin(\theta)}{\tan(\theta)-1} && \text{quotient \& reciprocal identities}
\end{aligned}$$

We have verified the identity. Note that, while it requires more steps, we could also verify this identity by beginning with the right side of the equation, as follows.

Alternate Solution. We start with the right side of the equation, applying reciprocal and quotient identities.

$$\begin{aligned}
\frac{\sec(\theta)-\sin(\theta)}{\tan(\theta)-1} &= \frac{\dfrac{1}{\cos(\theta)}-\sin(\theta)}{\dfrac{\sin(\theta)}{\cos(\theta)}-1} \\
&= \frac{1-\sin(\theta)\cos(\theta)}{\sin(\theta)-\cos(\theta)} && \text{multiply by } \frac{\cos(\theta)}{\cos(\theta)} \\
&= \frac{\left(1-\sin(\theta)\cos(\theta)\right)\left(\sin(\theta)+\cos(\theta)\right)}{\left(\sin(\theta)-\cos(\theta)\right)\left(\sin(\theta)+\cos(\theta)\right)} \\
&= \frac{\sin(\theta)+\cos(\theta)-\sin^2(\theta)\cos(\theta)-\sin(\theta)\cos^2(\theta)}{\sin^2(\theta)-\cos^2(\theta)} \\
&= \frac{\sin(\theta)+\cos(\theta)-\left(1-\cos^2(\theta)\right)\cos(\theta)-\sin(\theta)\left(1-\sin^2(\theta)\right)}{\sin^2(\theta)-\cos^2(\theta)} && \text{Pythagorean identity} \\
&= \frac{\sin(\theta)+\cos(\theta)-\cos(\theta)+\cos^3(\theta)-\sin(\theta)+\sin^3(\theta)}{\sin^2(\theta)-\cos^2(\theta)} \\
&= \frac{\sin^3(\theta)+\cos^3(\theta)}{\sin^2(\theta)-\cos^2(\theta)}
\end{aligned}$$

□

Example 1.5.4. Verify the identity $\frac{\sec(\theta)}{1-\tan(\theta)} = \frac{1}{\cos(\theta)-\sin(\theta)}$.

Solution. We start with the left side and rewrite its trigonometric functions in terms of the trigonometric functions that appear on the right.

$$\frac{\sec(\theta)}{1-\tan(\theta)} = \frac{\left(\frac{1}{\cos(\theta)}\right)}{1-\left(\frac{\sin(\theta)}{\cos(\theta)}\right)} \qquad \text{reciprocal identity for secant \& quotient identity for tangent}$$

$$= \frac{\left(\frac{1}{\cos(\theta)}\right)}{1-\left(\frac{\sin(\theta)}{\cos(\theta)}\right)} \cdot \frac{\cos(\theta)}{\cos(\theta)} \qquad \text{multiplication by LCD of numerator \& denominator}$$

$$= \frac{\left(\frac{\cos(\theta)}{\cos(\theta)}\right)}{\cos(\theta)-\left(\frac{\sin(\theta)\cos(\theta)}{\cos(\theta)}\right)}$$

$$= \frac{1}{\cos(\theta)-\sin(\theta)}$$

Once again, we have arrived at the opposite side of the equation, verifying the given identity.

□

Example 1.5.5. Verify the identity $\frac{\sin(\theta)}{1-\cos(\theta)} = \frac{1+\cos(\theta)}{\sin(\theta)}$.

Solution. It is debatable which side of the equation is more complicated. Noting that the denominator on the left side is $1-\cos(\theta)$ while the numerator on the right side is $1+\cos(\theta)$, we select the strategy of starting with the left side and multiplying the numerator and denominator by the quantity $1+\cos(\theta)$.

$$\frac{\sin(\theta)}{1-\cos(\theta)} = \frac{\sin(\theta)}{(1-\cos(\theta))} \cdot \frac{(1+\cos(\theta))}{(1+\cos(\theta))}$$

$$= \frac{\sin(\theta)(1+\cos(\theta))}{1-\cos^2(\theta)} \qquad \text{binomial multiplication}$$

$$= \frac{\sin(\theta)(1+\cos(\theta))}{\sin^2(\theta)} \qquad \text{Pythagorean identity}$$

$$= \frac{1+\cos(\theta)}{\sin(\theta)}$$

After arriving at the right side, we have verified the identity. An alternate solution follows.

Alternate Solution. We start with the right side of the equation.

$$\begin{aligned}\frac{1+\cos(\theta)}{\sin(\theta)} &= \frac{(1+\cos(\theta))(1-\cos(\theta))}{(\sin(\theta))(1-\cos(\theta))}\\ &= \frac{1-\cos^2(\theta)}{(\sin(\theta))(1-\cos(\theta))}\\ &= \frac{\sin^2(\theta)}{(\sin(\theta))(1-\cos(\theta))}\\ &= \frac{\sin(\theta)}{1-\cos(\theta)}\end{aligned}$$

□

In the preceding example, we see that multiplying $1-\cos(\theta)$ by $1+\cos(\theta)$ produces a difference of squares that can be simplified to one term using **Theorem 1.3**. This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1-\sqrt{2}$ by $1+\sqrt{2}$. For this reason, the quantities $1-\cos(\theta)$ and $1+\cos(\theta)$ are called 'Pythagorean conjugates'. The following list includes other Pythagorean conjugates.

Pythagorean Conjugates

- $\mathbf{1-\cos(\theta)}$ and $\mathbf{1+\cos(\theta)}$ since $(1-\cos(\theta))(1+\cos(\theta))=1-\cos^2(\theta)=\sin^2(\theta)$
- $\mathbf{1-\sin(\theta)}$ and $\mathbf{1+\sin(\theta)}$ since $(1-\sin(\theta))(1+\sin(\theta))=1-\sin^2(\theta)=\cos^2(\theta)$
- $\mathbf{\sec(\theta)-1}$ and $\mathbf{\sec(\theta)+1}$ since $(\sec(\theta)-1)(\sec(\theta)+1)=\sec^2(\theta)-1=\tan^2(\theta)$
- $\mathbf{\sec(\theta)-\tan(\theta)}$ and $\mathbf{\sec(\theta)+\tan(\theta)}$
  since $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta))=\sec^2(\theta)-\tan^2(\theta)=1$
- $\mathbf{\csc(\theta)-1}$ and $\mathbf{\csc(\theta)+1}$ since $(\csc(\theta)-1)(\csc(\theta)+1)=\csc^2(\theta)-1=\cot^2(\theta)$
- $\mathbf{\csc(\theta)-\cot(\theta)}$ and $\mathbf{\csc(\theta)+\cot(\theta)}$
  since $(\csc(\theta)-\cot(\theta))(\csc(\theta)+\cot(\theta))=\csc^2(\theta)-\cot^2(\theta)=1$

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling just to become proficient in the basics. Like many things in life, there is no short cut here. There is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies that may be helpful (depending on the situation) follows.

Strategies for Verifying Identities

- Start with the side of the identity that looks easier for you to manipulate.
- Use the quotient and reciprocal identities to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify any resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean identities in **Theorem 1.3** to exchange sines and cosines, tangents and secants, cotangents and cosecants, and simplify sums or differences of squares to one term.
- Multiply numerator and denominator by Pythagorean conjugates in order to take advantage of the Pythagorean identities in **Theorem 1.3**.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity or try writing everything in terms of sines and cosines.

Most importantly, keep in mind that we are not solving equations. To show that an equation is not an identity, all that is needed is to show that the two sides are not equal for just one value of the independent variable. However, to verify identities, we choose one side of the identity and work with that side until it matches the other side. An alternate strategy of manipulating both sides of an equation until they arrive at a common expression may be used by some instructors in verifying identities. Verifying identities is an important skill and we will work with identities again in **Chapter 3**, as more tools become available. Time spent now in developing some proficiency will be useful throughout the course. We finish off this section with a couple of equations that may or may not be identities.

Example 1.5.6. Prove or disprove that the given equation is an identity.

1. $(\sin(\theta)-\cos(\theta))^2 = 1+2\dfrac{\sin^3(\theta)-\sin(\theta)}{\cos(\theta)}$

2. $(\sin(\theta)-\cos(\theta))^2 = \sin^2(\theta)-\cos^2(\theta)$

Solution.

1. To prove that $\left(\sin(\theta)-\cos(\theta)\right)^2 = 1+2\dfrac{\sin^3(\theta)-\sin(\theta)}{\cos(\theta)}$ is an identity, we begin with the 'more complicated' right side, although either side would work.

$$\begin{aligned}
1+2\frac{\sin^3(\theta)-\sin(\theta)}{\cos(\theta)} &= \frac{\cos(\theta)+2\sin^3(\theta)-2\sin(\theta)}{\cos(\theta)} \\
&= \frac{\cos(\theta)-2\sin(\theta)\left(-\sin^2(\theta)+1\right)}{\cos(\theta)} \\
&= \frac{\cos(\theta)-2\sin(\theta)\cos^2(\theta)}{\cos(\theta)} && \text{Pythagorean identity} \\
&= 1-2\sin(\theta)\cos(\theta) \\
&= \sin^2(\theta)+\cos^2(\theta)-2\sin(\theta)\cos(\theta) && \text{Pythagorean identity} \\
&= \sin^2(\theta)-2\sin(\theta)\cos(\theta)+\cos^2(\theta) \\
&= \left(\sin(\theta)-\cos(\theta)\right)^2
\end{aligned}$$

Having verified that the right side equals the left side, we have proved that the equation is an identity.

2. We can show that $\left(\sin(\theta)-\cos(\theta)\right)^2 = \sin^2(\theta)-\cos^2(\theta)$ is not an identity by finding one value of θ for which the left side is not equal to the right side. One such value is $\theta = \dfrac{\pi}{6}$.

Left side:

$$\begin{aligned}
\left(\sin\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{6}\right)\right)^2 &= \left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^2 \\
&= \left(\frac{1-\sqrt{3}}{2}\right)^2 \\
&= \frac{\left(1-\sqrt{3}\right)^2}{4}
\end{aligned}$$

Right side:

$$\begin{aligned}
\sin^2\left(\frac{\pi}{6}\right)-\cos^2\left(\frac{\pi}{6}\right) &= \left(\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2 \\
&= \frac{1}{4}-\frac{3}{4} \\
&= -\frac{1}{2}
\end{aligned}$$

Since $\dfrac{\left(1-\sqrt{3}\right)^2}{4} > 0$ and $-\dfrac{1}{2} < 0$, $\dfrac{\left(1-\sqrt{3}\right)^2}{4} \neq -\dfrac{1}{2}$. Therefore, the equation is not an identity.

□

1.5 Exercises

In Exercises 1 – 47, verify the identity. Assume that all quantities are defined.

1. $\cos(\theta)\sec(\theta)=1$
2. $\tan(\theta)\cos(\theta)=\sin(\theta)$
3. $\sin(\theta)\csc(\theta)=1$
4. $\tan(\theta)\cot(\theta)=1$
5. $\csc(\theta)\cos(\theta)=\cot(\theta)$
6. $\dfrac{\sin(\theta)}{\cos^2(\theta)}=\sec(\theta)\tan(\theta)$
7. $\dfrac{\cos(\theta)}{\sin^2(\theta)}=\csc(\theta)\cot(\theta)$
8. $\dfrac{1+\sin(\theta)}{\cos(\theta)}=\sec(\theta)+\tan(\theta)$
9. $\dfrac{1-\cos(\theta)}{\sin(\theta)}=\csc(\theta)-\cot(\theta)$
10. $\dfrac{\cos(\theta)}{1-\sin^2(\theta)}=\sec(\theta)$
11. $\dfrac{\sin(\theta)}{1-\cos^2(\theta)}=\csc(\theta)$
12. $\dfrac{\sec(\theta)}{1+\tan^2(\theta)}=\cos(\theta)$
13. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)}=\sin(\theta)$
14. $\dfrac{\tan(\theta)}{\sec^2(\theta)-1}=\cot(\theta)$
15. $\dfrac{\cot(\theta)}{\csc^2(\theta)-1}=\tan(\theta)$
16. $4\sin^2(\theta)+4\cos^2(\theta)=4$
17. $9-\sin^2(\theta)-\cos^2(\theta)=8$
18. $\tan^3(\theta)=\tan(\theta)\sec^2(\theta)-\tan(\theta)$
19. $\sin^5(\theta)=\left(1-\cos^2(\theta)\right)^2\sin(\theta)$
20. $\sec^{10}(\theta)=\left(1+\tan^2(\theta)\right)^4\sec^2(\theta)$
21. $\cos^2(\theta)\tan^3(\theta)=\tan(\theta)-\sin(\theta)\cos(\theta)$
22. $\sec^4(\theta)-\sec^2(\theta)=\tan^2(\theta)+\tan^4(\theta)$
23. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1}=\dfrac{1+\sec(\theta)}{1-\sec(\theta)}$
24. $\dfrac{\sin(\theta)+1}{\sin(\theta)-1}=\dfrac{1+\csc(\theta)}{1-\csc(\theta)}$
25. $\dfrac{1-\cot(\theta)}{1+\cot(\theta)}=\dfrac{\tan(\theta)-1}{\tan(\theta)+1}$
26. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)}=\dfrac{\cos(\theta)-\sin(\theta)}{\cos(\theta)+\sin(\theta)}$
27. $\tan(\theta)+\cot(\theta)=\sec(\theta)\csc(\theta)$
28. $\csc(\theta)-\sin(\theta)=\cot(\theta)\cos(\theta)$
29. $\cos(\theta)-\sec(\theta)=-\tan(\theta)\sin(\theta)$
30. $\cos(\theta)\left(\tan(\theta)+\cot(\theta)\right)=\csc(\theta)$

31. $\sin(\theta)\left(\tan(\theta)+\cot(\theta)\right)=\sec(\theta)$

32. $\dfrac{1}{1-\cos(\theta)}+\dfrac{1}{1+\cos(\theta)}=2\csc^2(\theta)$

33. $\dfrac{1}{\sec(\theta)+1}+\dfrac{1}{\sec(\theta)-1}=2\csc(\theta)\cot(\theta)$

34. $\dfrac{1}{\csc(\theta)+1}+\dfrac{1}{\csc(\theta)-1}=2\sec(\theta)\tan(\theta)$

35. $\dfrac{1}{\csc(\theta)-\cot(\theta)}-\dfrac{1}{\csc(\theta)+\cot(\theta)}=2\cot(\theta)$

36. $\dfrac{\cos(\theta)}{1-\tan(\theta)}+\dfrac{\sin(\theta)}{1-\cot(\theta)}=\sin(\theta)+\cos(\theta)$

37. $\dfrac{1}{\sec(\theta)+\tan(\theta)}=\sec(\theta)-\tan(\theta)$

38. $\dfrac{1}{\sec(\theta)-\tan(\theta)}=\sec(\theta)+\tan(\theta)$

39. $\dfrac{1}{\csc(\theta)-\cot(\theta)}=\csc(\theta)+\cot(\theta)$

40. $\dfrac{1}{\csc(\theta)+\cot(\theta)}=\csc(\theta)-\cot(\theta)$

41. $\dfrac{1}{1-\sin(\theta)}=\sec^2(\theta)+\sec(\theta)\tan(\theta)$

42. $\dfrac{1}{1+\sin(\theta)}=\sec^2(\theta)-\sec(\theta)\tan(\theta)$

43. $\dfrac{1}{1-\cos(\theta)}=\csc^2(\theta)+\csc(\theta)\cot(\theta)$

44. $\dfrac{1}{1+\cos(\theta)}=\csc^2(\theta)-\csc(\theta)\cot(\theta)$

45. $\dfrac{\cos(\theta)}{1+\sin(\theta)}=\dfrac{1-\sin(\theta)}{\cos(\theta)}$

46. $\csc(\theta)-\cot(\theta)=\dfrac{\sin(\theta)}{1+\cos(\theta)}$

47. $\dfrac{1-\sin(\theta)}{1+\sin(\theta)}=\left(\sec(\theta)-\tan(\theta)\right)^2$

In Exercises 48 – 51, verify the identity. You may need to review the properties of absolute value and logarithms before proceeding.

48. $\ln\left|\sec(\theta)\right|=-\ln\left|\cos(\theta)\right|$

49. $-\ln\left|\csc(\theta)\right|=\ln\left|\sin(\theta)\right|$

50. $-\ln\left|\sec(\theta)-\tan(\theta)\right|=\ln\left|\sec(\theta)+\tan(\theta)\right|$

51. $-\ln\left|\csc(\theta)+\cot(\theta)\right|=\ln\left|\csc(\theta)-\cot(\theta)\right|$

1.6 Beyond the Unit Circle

Learning Objectives

- **Given any point on the terminal side of an angle in standard position, determine the values of the six trigonometric functions for that angle.**
- **Given the quadrant of an angle and the value of one trigonometric function for that angle, determine the values of the remaining five trigonometric functions.**

Recall that in defining the sine and cosine functions in **Section 1.3**, we assigned to each angle a position on the Unit Circle. Here we broaden our scope to include circles of radius r centered at the origin.

Determining Sine and Cosine

Consider for the moment the acute Quadrant I angle θ, drawn below in standard position.

Figure 1.6. 1

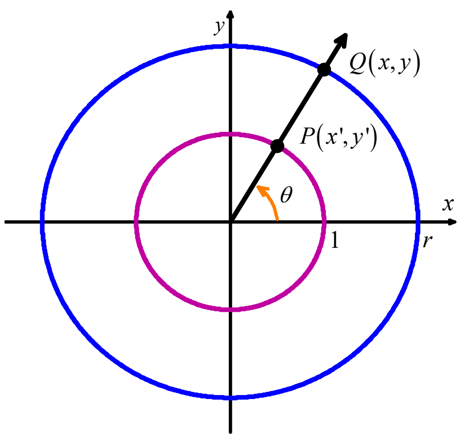
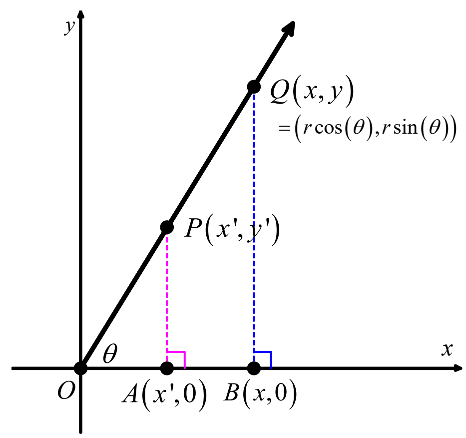


Figure 1.6. 2



Let $Q(x,y)$ be the point on the terminal side of θ that lies on the circle of radius r, centered at the origin, and let $P(x',y')$ be the point on the terminal side of θ that lies on the Unit Circle. Now consider dropping perpendiculars from P and Q to create two triangles $\triangle OPA$ and $\triangle OQB$. These triangles are similar.[17] Thus, it follows that $\frac{x}{x'}=\frac{r}{1}=r$, from which $x=rx'$. We similarly find $y=ry'$. Since, by definition[18], $x'=\cos(\theta)$ and $y'=\sin(\theta)$, we get the coordinates of Q to be $x=r\cos(\theta)$ and

[17] Do you remember why? If not, refer to **Section 1.2**.
[18] See **Section 1.3**.

$y = r\sin(\theta)$. Also, from triangle $\triangle OQB$, $r^2 = x^2 + y^2$ and, since the radius is a positive quantity, $r = \sqrt{x^2 + y^2}$. This means that not only can we describe the coordinates of Q in terms of $\sin(\theta)$ and $\cos(\theta)$, we can also express $\sin(\theta)$ and $\cos(\theta)$ in terms of the coordinates of Q.

By use of reference angles, we obtain these results for all non-quadrantal angles. Moreover, these results hold for the quadrantal angles. Our results are summarized in the following theorem.

Theorem 1.4. Suppose θ is an angle in standard position and $Q(x, y)$ is a point on the terminal side of θ that lies on the circle of radius r, centered at the origin. Then

1. $x^2 + y^2 = r^2$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Rewriting these we have
2. $\sin(\theta) = \dfrac{y}{r} = \dfrac{y}{\sqrt{x^2 + y^2}}$ and $\cos(\theta) = \dfrac{x}{r} = \dfrac{x}{\sqrt{x^2 + y^2}}$

In the case of the Unit Circle we have $r = \sqrt{x^2 + y^2} = 1$, so **Theorem 1.4** reduces to our Unit Circle definitions of $\sin(\theta)$ and $\cos(\theta)$.

Example 1.6.1. Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(4, -2)$. Find $\sin(\theta)$ and $\cos(\theta)$.

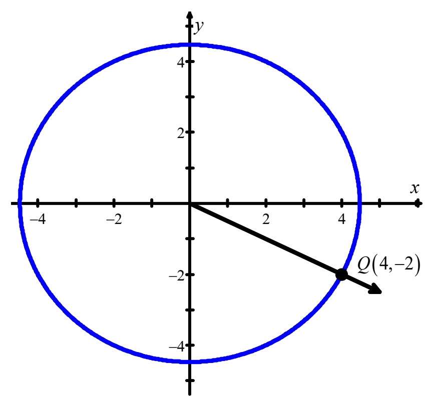
Solution. Using **Theorem 1.4** with $x = 4$ and $y = -2$, we find

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(4)^2 + (-2)^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Thus, $\sin(\theta) = \dfrac{y}{r} = \dfrac{-2}{2\sqrt{5}}$ and $\cos(\theta) = \dfrac{x}{r} = \dfrac{4}{2\sqrt{5}}$.

We simplify to get $\sin(\theta) = -\dfrac{1}{\sqrt{5}}$ and $\cos(\theta) = \dfrac{2}{\sqrt{5}}$.

Figure 1.6. 3



□

Determining the Other Four Trigonometric Functions

We have generalized the sine and cosine functions from coordinates on the Unit Circle to coordinates on circles of radius r. We use the results from **Theorem 1.4**, $\sin(\theta)=\frac{y}{r}$ and $\cos(\theta)=\frac{x}{r}$, where the point (x,y) on the circle of radius r, centered at the origin, lies along the terminal side of an angle θ, plotted in standard position, to determine the remaining four trigonometric functions. Additionally, we make use of the quotient and reciprocal identities as follows.

- $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}=\frac{y/r}{x/r}$ so, we have $\tan(\theta)=\frac{y}{x}$, as long as $x\neq 0$
- $\csc(\theta)=\frac{1}{\sin(\theta)}=\frac{r}{y}$ for $y\neq 0$
- $\sec(\theta)=\frac{1}{\cos(\theta)}=\frac{r}{x}$ for $x\neq 0$
- $\cot(\theta)=\frac{1}{\tan(\theta)}=\frac{x}{y}$ for $y\neq 0$

Theorem 1.5. Suppose $Q(x,y)$ is the point on the terminal side of an angle θ, plotted in standard position, that lies on the circle of radius r, centered at the origin. Then

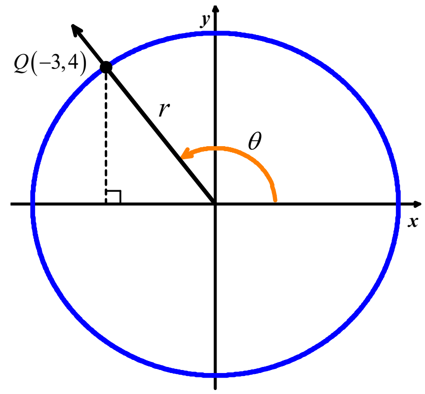
- $\tan(\theta)=\frac{y}{x}$, provided $x\neq 0$.
- $\csc(\theta)=\frac{r}{y}$, provided $y\neq 0$.
- $\sec(\theta)=\frac{r}{x}$, provided $x\neq 0$.
- $\cot(\theta)=\frac{x}{y}$, provided $y\neq 0$.

Keep in mind that $x\neq 0$ means that the terminal side of the angle θ is not on the y-axis, while $y\neq 0$ means that the terminal side of the angle θ is not on the x-axis.

Example 1.6.2. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(-3,4)$. Find the values of the six trigonometric functions of θ.

Solution. The radius of the circle containing the point $Q(-3,4)$ is

Figure 1.6. 4



$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (4)^2} = 5$$

With $x = -3$, $y = 4$ and $r = 5$, we apply **Theorems 1.4** and **1.5** to find the values of the six trigonometric functions of θ.[19]

$$\sin(\theta) = \frac{y}{r} = \frac{4}{5} \qquad \csc(\theta) = \frac{r}{y} = \frac{5}{4}$$

$$\cos(\theta) = \frac{x}{r} = -\frac{3}{5} \qquad \sec(\theta) = \frac{r}{x} = -\frac{5}{3}$$

$$\tan(\theta) = \frac{y}{x} = -\frac{4}{3} \qquad \cot(\theta) = \frac{x}{y} = -\frac{3}{4}$$

□

Example 1.6.3. Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining trigonometric functions of θ.

Solution. We look for a point $Q(x, y)$ that lies on the terminal side of θ when θ is plotted in standard position.[20] We are given that θ is a Quadrant IV angle, so we know $x > 0$ and $y < 0$. Also, $\cot(\theta) = -4 = \frac{x}{y}$. Since $-4 = \frac{4}{-1}$, we may choose[21] $x = 4$ and $y = -1$, from which

---

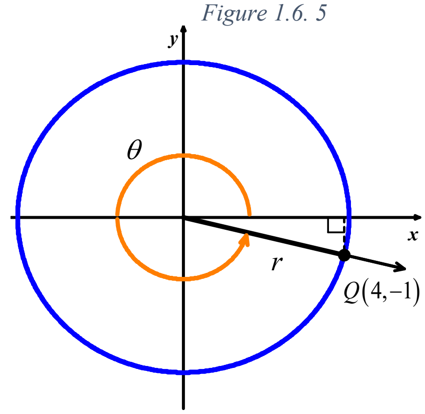
[19] For convenience, the sketch shows $0 \le \theta < 2\pi$. In reality, θ may be any angle, plotted in standard position, that contains the point $Q(-3,4)$ on its terminal side.

[20] Again, θ may be any angle, plotted in standard position, with Q on its terminal side.

[21] We may choose any values x and y so long as $x > 0$, $y < 0$ and $\frac{x}{y} = -4$. For example, we could choose $x = 8$ and $y = -2$. The fact that all such points lie on the terminal side of θ is a consequence of the fact that the terminal side of θ is the portion of the line with slope $-\frac{1}{4}$ that extends from the origin into Quadrant IV.

$$r=\sqrt{x^2+y^2}$$
$$=\sqrt{(4)^2+(-1)^2}$$
$$=\sqrt{17}$$

Figure 1.6. 5



The five remaining trigonometric function values follow.

$$\sin(\theta)=\frac{y}{r}=-\frac{1}{\sqrt{17}} \qquad \csc(\theta)=\frac{r}{y}=\frac{\sqrt{17}}{-1}=-\sqrt{17}$$

$$\cos(\theta)=\frac{x}{r}=\frac{4}{\sqrt{17}} \qquad \sec(\theta)=\frac{r}{x}=\frac{\sqrt{17}}{4}$$

$$\tan(\theta)=\frac{y}{x}=-\frac{1}{4}$$

□

Example 1.6.4. If $\sec(\theta)=\frac{5}{4}$ and $\sin(\theta)<0$, find $\tan(\theta)$.

Solution. Knowing that $\sin(\theta)<0$, we can determine the value of $\sin(\theta)$ by first finding $\cos(\theta)$. We have $\sec(\theta)=\frac{5}{4}=\frac{1}{\cos(\theta)}$, from which $\cos(\theta)=\frac{4}{5}$. Applying the Pythagorean identity results in the following:

$$\sin^2(\theta)+\cos^2(\theta)=1$$
$$\sin^2(\theta)+\left(\frac{4}{5}\right)^2=1$$
$$\sin^2(\theta)=\frac{9}{25}$$
$$\sin(\theta)=\pm\frac{3}{5}$$

Since $\sin(\theta)<0$, we choose $\sin(\theta)=-\frac{3}{5}$ and proceed with finding $\tan(\theta)$.

$$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}=\frac{-3/5}{4/5}$$

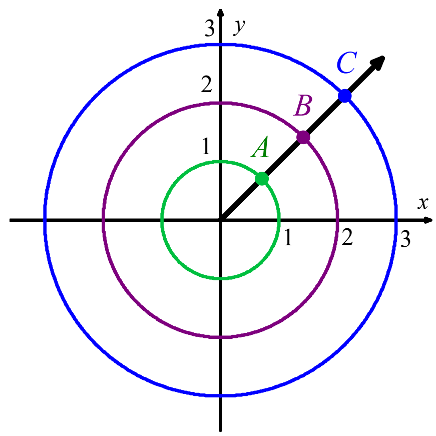
Simplifying, we have $\tan(\theta)=-\frac{3}{4}$.

□

We finish off this section, and chapter, with an illustration that demonstrates the connection between points on the Unit Circle and points 'beyond the Unit Circle'.

Example 1.6.5. In the following diagram, if $A=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, determine the coordinates of points B and C.

Figure 1.6. 6



Solution. If we drop vertical line segments from each of the points A, B and C to the *x*-axis, we have the three right triangles $\triangle OAA'$, $\triangle OBB'$ and $\triangle OCC'$. Since all three of these right triangles share the same angle at the origin, they are similar triangles.

Figure 1.6. 7

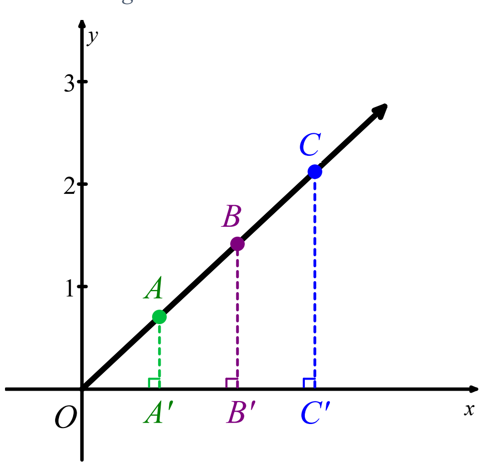
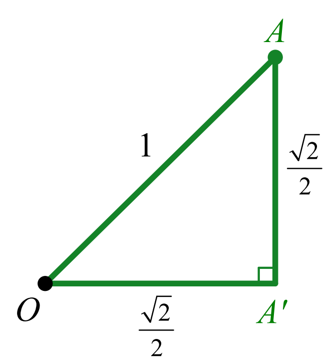


Figure 1.6. 8



From $A=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, we find $OA'=\frac{\sqrt{2}}{2}$ and $A'A=\frac{\sqrt{2}}{2}$. Since $\triangle OAA'$ and $\triangle OBB'$ are similar triangles, $\frac{OA}{OB}=\frac{OA'}{OB'}=\frac{A'A}{B'B}$. With $OA=1$ and $OB=2$, we find $OB'=\sqrt{2}$ and $B'B=\sqrt{2}$. Thus, point

B has coordinates $\left(\sqrt{2}, \sqrt{2}\right)$. Similarly, $\frac{OA}{OC} = \frac{OA'}{OC'} = \frac{A'A}{C'C}$ and $OC = 3$, resulting in point C having coordinates $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

□

1.6 Exercises

In Exercises 1 – 8, let θ be an angle in standard position whose terminal side contains the given point. Find the exact values of the six trigonometric functions of θ.

1. $A(1,5)$
2. $B(3,-1)$
3. $C(-6,-2)$
4. $D(-10,12)$
5. $P(-7,24)$
6. $Q(3,4)$
7. $R(5,-9)$
8. $T(-2,-11)$

In Exercises 9 – 22, use the given information to find the exact values of the remaining trigonometric functions of θ.

9. $\sin(\theta)=\frac{3}{5}$ with θ in Quadrant II
10. $\tan(\theta)=\frac{12}{5}$ with θ in Quadrant III
11. $\csc(\theta)=\frac{25}{24}$ with θ in Quadrant I
12. $\sec(\theta)=7$ with θ in Quadrant IV
13. $\csc(\theta)=-\frac{10\sqrt{91}}{91}$ with θ in Quadrant III
14. $\cot(\theta)=-23$ with θ in Quadrant II
15. $\tan(\theta)=-2$ with θ in Quadrant IV
16. $\sec(\theta)=-4$ with θ in Quadrant II
17. $\cot(\theta)=\sqrt{5}$ with θ in Quandrant III
18. $\cos(\theta)=\frac{1}{3}$ with θ in Quadrant I
19. $\cot(\theta)=2$ with $0<\theta<\frac{\pi}{2}$
20. $\csc(\theta)=5$ with $\frac{\pi}{2}<\theta<\pi$
21. $\tan(\theta)=\sqrt{10}$ with $\pi<\theta<\frac{3\pi}{2}$
22. $\sec(\theta)=2\sqrt{5}$ with $\frac{3\pi}{2}<\theta<2\pi$

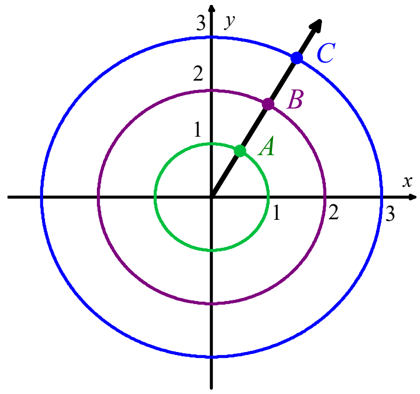
23. If $\sin(\theta)=-\frac{3}{5}$ and $\tan(\theta)<0$, find $\cos(\theta)$

24. If $\cos(\theta)=\frac{2}{3}$ and $\sin(\theta)<0$, find $\cot(\theta)$.

25. If $\csc(\theta)=-2$ and $\cos(\theta)<0$, find $\tan(\theta)$.

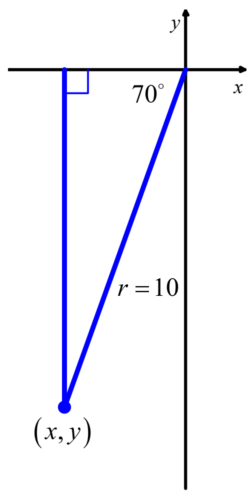
26. In the following diagram, if $A=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, determine the coordinates of points B and C.

Figure Ex1.6. 1



27. In the following diagram, determine the values of x and y. What would the values of x and y be if $r=5$?

Figure Ex1.6. 2



CHAPTER 2
TRIGONOMETRIC GRAPHS AND OTHER APPLICATIONS OF RADIAN MEASURE

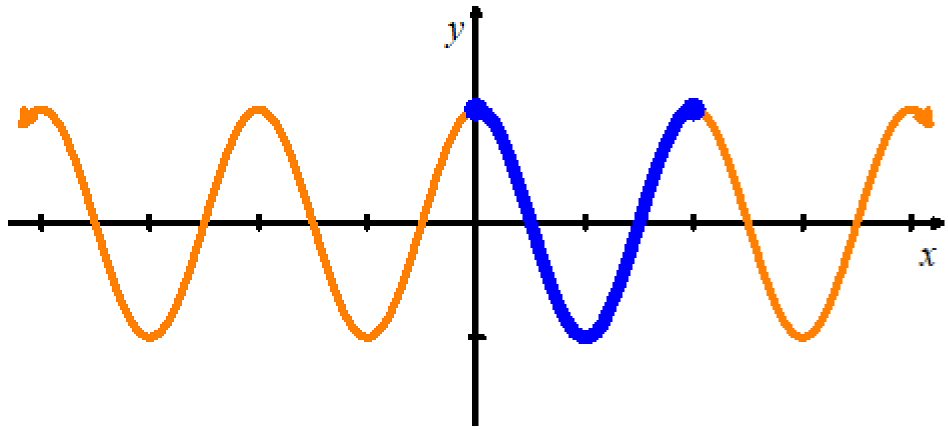


Figure 2.0. 1

Chapter Outline

Introduction

In algebra, we've had a great deal of exposure to graphs of linear, quadratic, polynomial, rational, and even radical functions. We saw patterns and generalized about short- and long-term behaviors of these functions. Trigonometric functions may exhibit a type of behavior – that of periodicity. The graphs examined in this section will demonstrate a recurring pattern that relates to real-life relationships governed by cyclical behavior. You may notice some visual similarities to functions from your past, but be assured – these functions are unique and are necessary components of what comes next in mathematics.

We begin in Section 2.1 with the introduction of the sine and cosine waves – as functions, as a collection of points, and as smooth curves. This discussion leads to the common designation of the sine and cosine as sinusoids. Moving on to Section 2.2, we proceed to visually define the remaining four trigonometric functions and to show how, though not waves, they exhibit related repeating patterns. Finally, in Section 2.3, these visualizations are directed to applied concepts of radian measurement. In addition to allowing us to visualize properties and applications of trigonometric functions and radian measure, Chapter 2 provides many of the tools we will need in applying identities and formulas, and in solving trigonometric equations, in future chapters.

2.1 Graphs of the Sine and Cosine Functions

Learning Objectives

- **Graph sine and cosine functions and their transformations.**
- **Determine whether a sine or cosine function is even or odd.**
- **Identify domain, range, period, horizontal (phase) shift, amplitude, and vertical shift used in graphing sine and cosine functions.**
- **Write an equation of the form $S(x) = A\sin(\omega x - \phi) + B$ or $C(x) = A\cos(\omega x - \phi) + B$ from the graph of a sinusoidal function.**

We now turn our attention to graphing sine and cosine functions in the Cartesian Plane. We use radians exclusively as input values for our graphing of trigonometric functions. As you will see, radian measure combines nicely with arithmetic properties of fractions to simplify graphing these functions. We note that the graphs of both sine and cosine functions are **continuous**, **smooth**, and **periodic**. Geometrically this means the graphs of the sine and cosine functions have no jumps, gaps, holes, vertical asymptotes, corners, or cusps, and meander nicely in a repeating fashion.

Note that $y = \sin(x)$ and $y = \cos(x)$, both trigonometric functions of x, are defined for all real values of x. This follows from our discussion in **Section 1.3** where we found the domain to be $(-\infty, \infty)$ for each function. We also found in **Section 1.3** that the range includes all real numbers between −1 and 1, inclusive, for both $y = \sin(x)$ and $y = \cos(x)$.

Graph of the Sine Function

To graph the sine and cosine functions in the Cartesian Plane, we use x as the independent variable and y as the dependent variable.[1] We graph $y = \sin(x)$ by making a table using some of the common values of x in the interval $[0, 2\pi]$. This generates a portion of the sine graph that we call the **fundamental cycle** of $y = \sin(x)$. So that we do not have to deal with *y*-values such as $\frac{\sqrt{3}}{2}$, to make graphing easier,

[1] The use of *x* and *y* in this context is not to be confused with the *x*- and *y*-coordinates of points on the Unit Circle, which define sine and cosine.

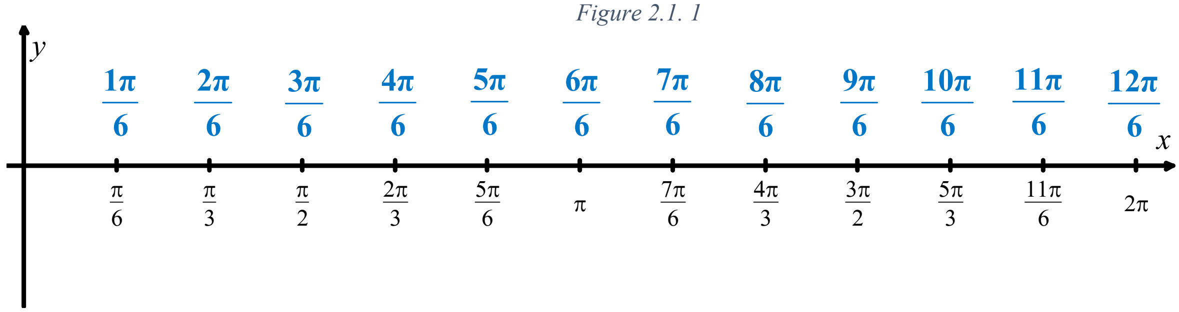
we plot points with y-values of 0, $\pm\frac{1}{2}$, and ± 1. Recall that $\sin(x)=0$ for $x=0$, π, or 2π; $\sin(x)=1$ for $x=\frac{\pi}{2}$; and $\sin(x)=-1$ for $x=\frac{3\pi}{2}$. In the case where $\sin(x)=\pm\frac{1}{2}$, the reference angle for x is $\frac{\pi}{6}$ and the sign is determined by the quadrant in which x terminates. A table of these values follows.

x	$y=\sin(x)$	$(x,\sin(x))$
0	0	$(0,0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6},\frac{1}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2},1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6},\frac{1}{2}\right)$
π	0	$(\pi,0)$

x	$y=\sin(x)$	$(x,\sin(x))$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6},-\frac{1}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2},-1\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6},-\frac{1}{2}\right)$
2π	0	$(2\pi,0)$

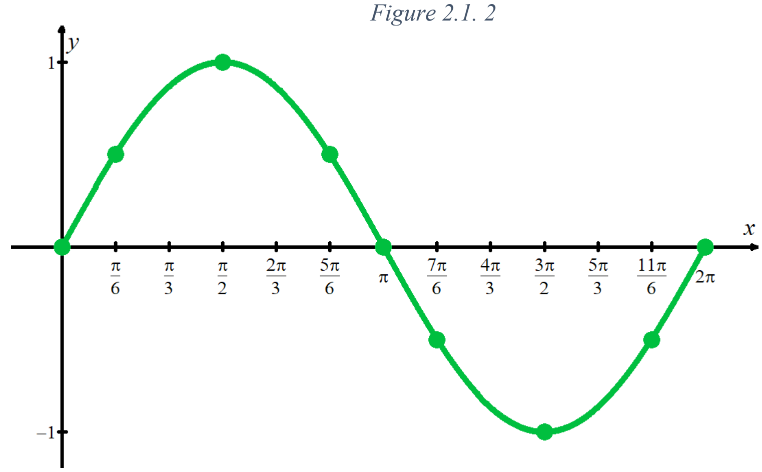
Noting that $y=\sin(x)$ is defined for all real numbers x, we plot the points $(x,\sin(x))$ from the table to guide us in sketching the graph of $y=\sin(x)$ on the interval $[0,2\pi]$. The tick marks on the x-axis result from dividing the interval $[0,2\pi]$ into 12 increments, each representing $\frac{2\pi}{12}=\frac{\pi}{6}$ units, as shown below.

Figure 2.1. 1



After plotting and connecting the points from our table with a smooth, continuous curve, we arrive at the following graph.

Figure 2.1. 2

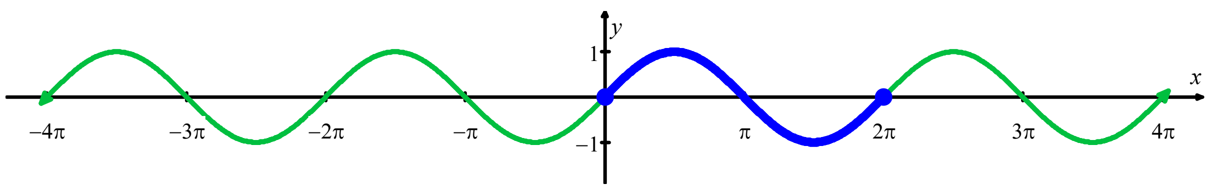


Fundamental Cycle of $y = \sin(x)$

A few things about the graph above are worth mentioning:

1. This graph represents only part of the graph of $y = \sin(x)$. To get the entire graph, imagine copying and pasting this graph end to end infinitely in both directions (left and right) along the x-axis.
2. The vertical scale size was chosen for clarity and aesthetics. Below is a graph that uses the same scale size on both axes. This graph of $y = \sin(x)$ shows several cycles, with the fundamental cycle in blue.

Figure 2.1. 3



$y = \sin(x)$

The graph of $y = \sin(x)$ is usually described as 'wavelike' and, indeed, many of the applications involving the sine and cosine functions feature the modeling of wavelike phenomena. Graphs that follow the patterns of sine curves are referred to as **sinusoidal**.

Graph of the Cosine Function

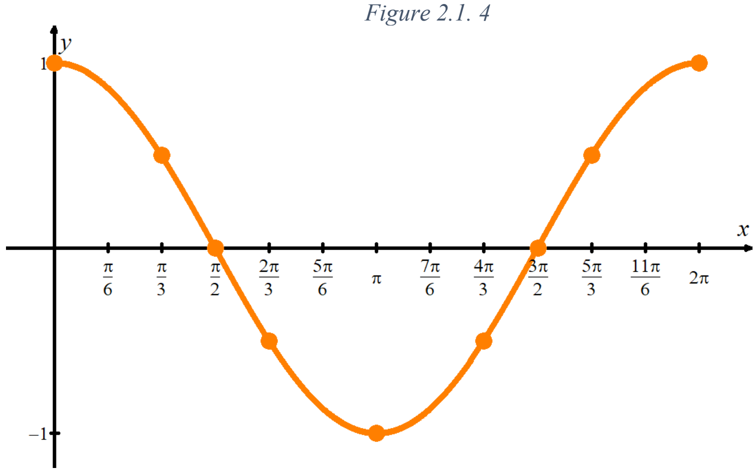
We plot the fundamental cycle of the graph of $y = \cos(x)$ in a similar way. To start, we determine the x-values in $[0, 2\pi]$ that result in y-values of 0, $\pm\frac{1}{2}$, and ± 1. We find $\cos(x) = 0$ for $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$;

$\cos(x)=1$ for $x=0$ or 2π; and $\cos(x)=-1$ for $x=\pi$. When $\cos(x)=\pm\frac{1}{2}$, the reference angle for x is $\frac{\pi}{3}$ and the sign is determined by the quadrant in which x terminates. A table of values follows.

x	$y=\cos(x)$	$(x,\cos(x))$
0	1	$(0,1)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3},\frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2},0\right)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{2\pi}{3},-\frac{1}{2}\right)$
π	-1	$(\pi,-1)$

x	$y=\cos(x)$	$(x,\cos(x))$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{4\pi}{3},-\frac{1}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2},0\right)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\left(\frac{5\pi}{3},\frac{1}{2}\right)$
2π	1	$(2\pi,1)$

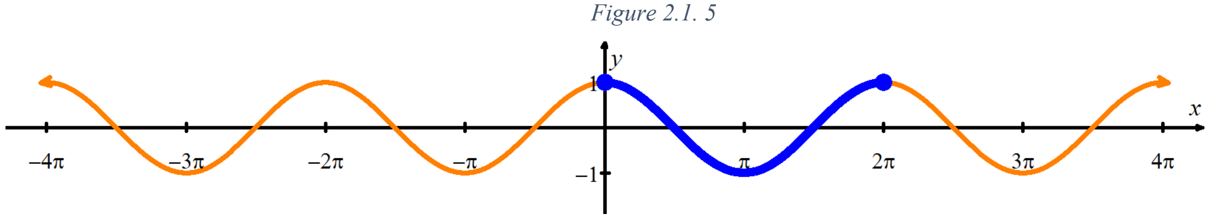
Figure 2.1. 4



Fundamental Cycle of $y=\cos(x)$

As with the graph of $y=\sin(x)$, we provide a graph of $y=\cos(x)$, below, that uses the same scale size on both axes. Again, the fundamental cycle is in blue.

Figure 2.1. 5



$y=\cos(x)$

It is no accident that the graphs of $y = \sin(x)$ and $y = \cos(x)$ are so similar. The graph of $y = \cos(x)$ is a result of the graph of $y = \sin(x)$ being shifted $\frac{\pi}{2}$ units to the left. Try it! Thus, the cosine graph is also a sinusoidal curve.

Period of the Sine and Cosine Functions

Not only can we obtain a graph of the cosine function by shifting the graph of the sine function $\frac{\pi}{2}$ units to the left, we can shift the graph of $y = \cos(x)$ by 2π units to the left and obtain a graph that is equivalent to the original graph of $y = \cos(x)$. The same can be said for shifts of 4π, 6π, 8π, $\ldots$ units to the left. We say that the cosine function is periodic, as defined below.

Definition 2.1. A function f is said to be **periodic** if there is a real number p so that $f(x+p) = f(x)$ for all real numbers x in the domain of f. The smallest such positive number p, if it exists, is called the **period** of f.

We see by the definition of periodic functions that $f(x) = \cos(x)$ is periodic, since $\cos(x + 2\pi k) = \cos(x)$ for any integer k. To determine the period of $f(x) = \cos(x)$, we need to find the smallest positive real number p so that $f(x+p) = f(x)$ for all real numbers x or, said differently, the smallest positive real number p such that $\cos(x+p) = \cos(x)$ for all real numbers x.

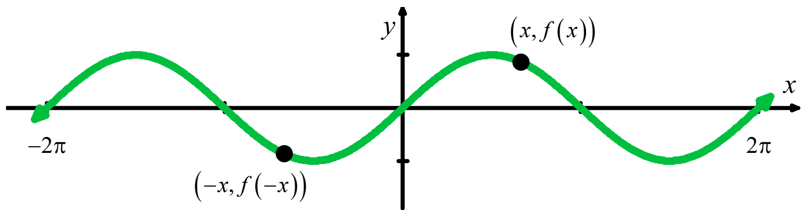
We know that $\cos(x + 2\pi) = \cos(x)$ for all real numbers x but the question remains if any smaller real number will do the trick. Suppose $p > 0$ and $\cos(x+p) = \cos(x)$ for all real numbers x. Then, in particular, $\cos(0+p) = \cos(0)$ so that $\cos(p) = 1$. From this we know that p is a multiple of 2π and, since the smallest positive multiple of 2π is 2π itself, we have the result.

Similarly, we can show $\sin(x)$ is periodic with 2π as its period. Having period 2π essentially means that we can completely understand everything about the functions $\sin(x)$ and $\cos(x)$ by studying one interval of length 2π, say $[0, 2\pi]$.

Even/Odd Properties of the Sine and Cosine Functions

While we will explore the even and odd properties of the sine and cosine functions further in **Section 3.1**, for now we demonstrate it graphically. You may recall that a function is called **even** if its graph is symmetric about the *y*-axis, and **odd** if its graph is symmetric about the origin. You may also recall that, for an even function f, $f(-x) = f(x)$, while for an odd function f, $f(-x) = -f(x)$, for all x-values in the domain of f. Let's look again at graphs of the sine and cosine functions.

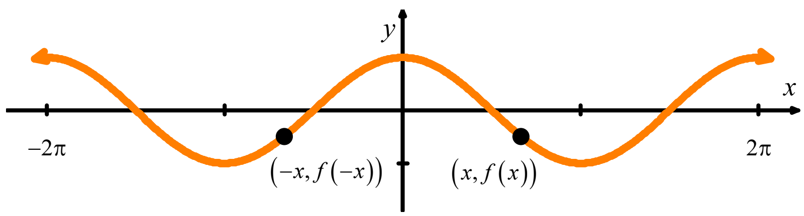
Figure 2.1. 6



$$y = f(x) = \sin(x)$$

The graph of $f(x) = \sin(x)$ is symmetric about the origin. This tells us that the sine function is an odd function. We see that $f(-x) = -f(x)$, or equivalently $\sin(-x) = -\sin(x)$.

Figure 2.1. 7



$$y = f(x) = \cos(x)$$

The graph of $f(x) = \cos(x)$ is symmetric about the *y*-axis. That is, the cosine function is an even function. Noting that $f(-x) = f(x)$, we have $\cos(-x) = \cos(x)$. Following is a summary of properties of the sine and cosine functions.

Properties of the Sine and Cosine Functions

The function $y = \sin(x)$

- has domain $(-\infty, \infty)$
- has range $[-1, 1]$
- is odd
- has period 2π

The function $y = \cos(x)$

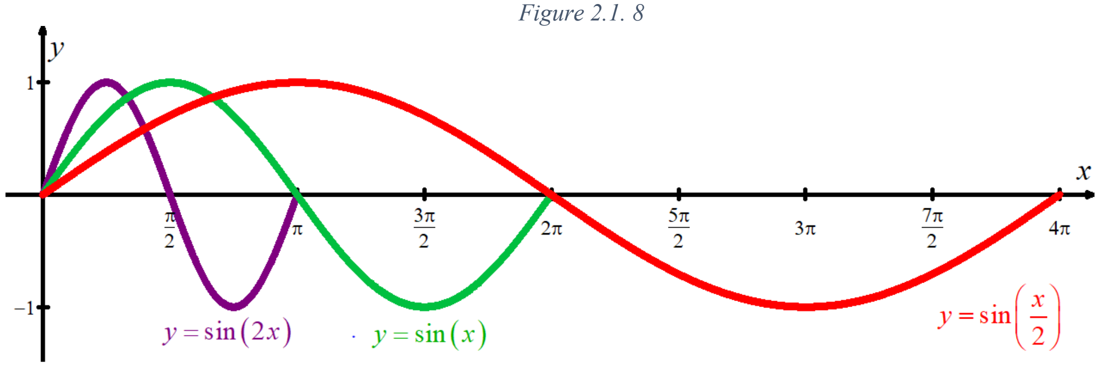
- has domain $(-\infty, \infty)$
- has range $[-1, 1]$
- is even
- has period 2π

Properties of Sinusoids

Now that we know the basic shapes and properties of the graphs of $y=\sin(x)$ and $y=\cos(x)$, we move on to graphing transformations of these functions, referred to as **sinusoids**. Sinusoids may be expressed in the general form $S(x)=A\sin(\omega x-\phi)+B$ or $C(x)=A\cos(\omega x-\phi)+B$, with $\omega>0$, and are characterized by four properties: period, phase shift, amplitude, and vertical shift.

- We have already discussed **period**; that is, how long it takes for the sinusoid to complete one cycle. The standard period of both $y=\sin(x)$ and $y=\cos(x)$ is 2π, but horizontal scalings will change the period. For example, revisiting transformations from a prior algebra course, we find the period of $y=\sin(2x)$ is $\frac{1}{2}(2\pi)=\pi$ while the period of $y=\sin\left(\frac{x}{2}\right)$, is $2(2\pi)=4\pi$.

Figure 2.1. 8



For a closer look at what is going on, the following points occur at key positions[2] in the fundamental cycle of each graph; i.e. points where $y=0$, $y=1$ or $y=-1$.

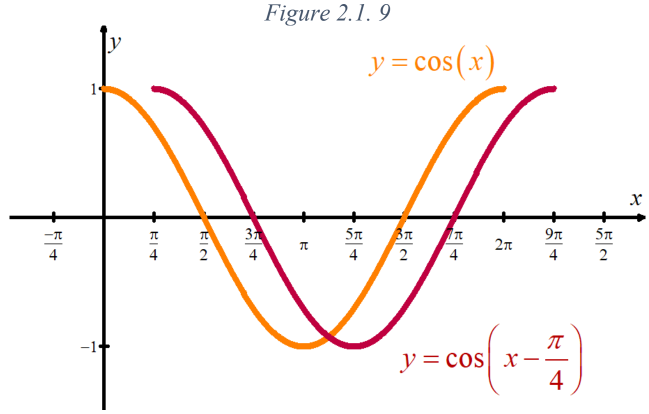
Function	Points at Key Positions				
$y=\sin(x)$	$(0,0)$	$\left(\frac{\pi}{2},1\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-1\right)$	$(2\pi,0)$
$y=\sin\left(\frac{x}{2}\right)$	$(0,0)$	$(\pi,1)$	$(2\pi,0)$	$(3\pi,-1)$	$(4\pi,0)$
$y=\sin(2x)$	$(0,0)$	$\left(\frac{\pi}{4},1\right)$	$\left(\frac{\pi}{2},0\right)$	$\left(\frac{3\pi}{4},-1\right)$	$(\pi,0)$

In general, the period of $S(x)=A\sin(\omega x-\phi)+B$ or $C(x)=A\cos(\omega x-\phi)+B$, $\omega>0$, is $\frac{2\pi}{\omega}$.

[2] These key positions divide one period into four equal pieces, and we will later refer to them as **quarter marks**.

- The **phase shift** of a sinusoid is the horizontal shift experienced by its fundamental cycle. Thinking back to horizontal shifts from algebra, we would expect the graph of the fundamental cycle of $y=\cos\left(x-\frac{\pi}{4}\right)$ to be $\frac{\pi}{4}$ units to the right of the fundamental cycle of $y=\cos(x)$, and that is indeed the case, as shown in the following figure.

Figure 2.1. 9

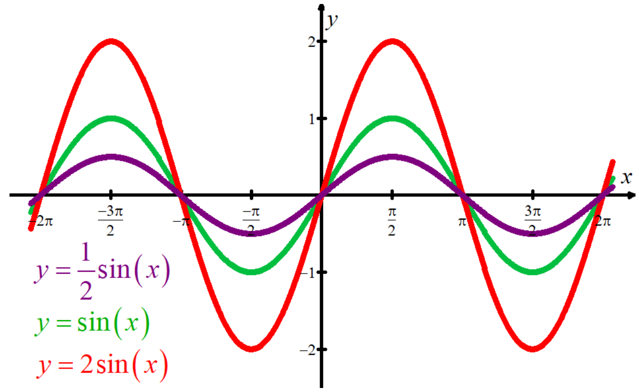


For $S(x)=A\sin(\omega x-\phi)+B$ or $C(x)=A\cos(\omega x-\phi)+B$, the phase shift is determined by the value of ϕ. Specifically, the phase shift is equal to $\frac{\phi}{\omega}$. If ϕ is positive, the graph of the function is shifted to the right (in the positive x-direction) by an amount equal to $\frac{\phi}{\omega}$. If ϕ is negative, the graph is shifted to the left (in the negative x-direction) by an amount equal to $\frac{|\phi|}{\omega}$.

Note that a phase (horizontal) shift of $\frac{\pi}{2}$ to the right takes $y=\cos(x)$ to $y=\sin(x)$, so that $\cos\left(x-\frac{\pi}{2}\right)=\sin(x)$. As mentioned earlier, a phase shift of $\frac{\pi}{2}$ to the left takes $y=\sin(x)$ to $y=\cos(x)$. Thus, by employing the necessary phase shift, the formula for a sinusoid may be written as either $S(x)=A\sin(\omega x-\phi)+B$ or $C(x)=A\cos(\omega x-\phi)+B$.

- The **amplitude** of a sinusoid is a measure of how 'tall' the wave is, and is half the difference between the smallest and largest y-values. The amplitude of the standard sine and cosine functions is 1, but vertical scalings can alter this. In the following figure, the amplitude of $y=\sin(x)$ is 1, the amplitude of $y=\frac{1}{2}\sin(x)$ is $\frac{1}{2}$, and the amplitude of $y=2\sin(x)$ is 2.

Figure 2.1. 10



For a closer look at what is going on, the following points occur at key positions in the fundamental cycle of each graph; i.e. points where $x=0$, $x=\frac{\pi}{2}$, $x=\pi$, $x=\frac{3\pi}{2}$ and $x=2\pi$.

Function	Points at Key Positions				
$y=\sin(x)$	$(0,0)$	$\left(\frac{\pi}{2},1\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-1\right)$	$(2\pi,0)$
$y=\frac{1}{2}\sin(x)$	$(0,0)$	$\left(\frac{\pi}{2},\frac{1}{2}\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-\frac{1}{2}\right)$	$(2\pi,0)$
$y=2\sin(x)$	$(0,0)$	$\left(\frac{\pi}{2},2\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-2\right)$	$(2\pi,0)$

In general, the amplitude of $S(x)=A\sin(\omega x-\phi)+B$ or $C(x)=A\cos(\omega x-\phi)+B$ is $|A|$.

- The **vertical shift** of sine and cosine is assumed to be 0, and this corresponds to a vertical **midline** of the *x*-axis, or $y=0$. For the sinusoids $y=\sin(x)-\frac{3}{2}$ and $y=\cos(x)+2$, the midlines are $y=-\frac{3}{2}$ and $y=2$, respectively.

Figure 2.1. 11

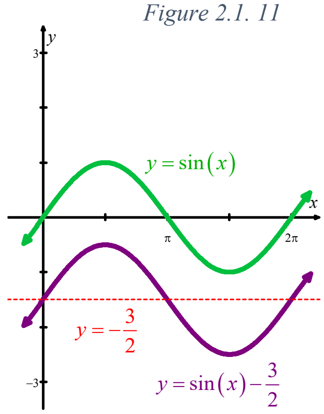
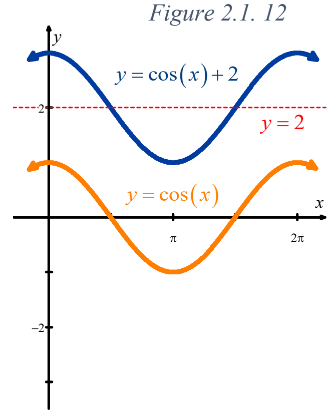


Figure 2.1. 12



In general, for $S(x) = A\sin(\omega x - \phi) + B$ and $C(x) = A\cos(\omega x - \phi) + B$, the vertical shift is B, and the midline is $y = B$.

Determining Period, Phase Shift, Amplitude, and Vertical Shift

For $\omega > 0$, the functions $S(x) = A\sin(\omega x - \phi) + B$ and $C(x) = A\cos(\omega x - \phi) + B$ have

- Period: $\dfrac{2\pi}{\omega}$
- Phase Shift: $\dfrac{\phi}{\omega}$
- Amplitude: $|A|$
- Vertical Shift: B

Before moving on, we note the requirement that $\omega > 0$. Should we be given the formula for a sinusoid in which the coefficient of x is negative, we can use even/odd properties of the sine and cosine to rewrite the formula before proceeding. For example, say we are given $f(x) = -2\sin(-3x) + 1$. Then

$$\begin{aligned} f(x) &= -2\sin(-3x) + 1 \\ &= -2\sin(-(3x)) + 1 \\ &= -2[-\sin(3x)] + 1 \quad \text{since sine is an odd function} \\ &= 2\sin(3x) + 1 \end{aligned}$$

A similar procedure allows us to rewrite the function $g(x) = -3\cos(-2x + \pi)$:

$$\begin{aligned} g(x) &= -3\cos(-2x + \pi) \\ &= -3\cos(-(2x - \pi)) \\ &= -3[\cos(2x - \pi)] \quad \text{since cosine is an even function} \\ &= -3\cos(2x - \pi) \end{aligned}$$

We continue with these two functions, in their revised format, in the following two examples.

Example 2.1.1. Identify the period, phase shift, amplitude, and midline of $f(x) = 2\sin(3x) + 1$.

Solution. We compare the equation $f(x) = 2\sin(3x) + 1$ to $S(x) = A\sin(\omega x - \phi) + B$.

- Since $\omega = 3$, the period of f is $\dfrac{2\pi}{3}$.
- If we rewrite f as $f(x) = 2\sin(3x - 0) + 1$, we see the phase shift is $\dfrac{\phi}{\omega} = \dfrac{0}{3} = 0$.
- The amplitude of f is $|A| = |2| = 2$.
- With $B = 1$, the vertical shift is one unit up. Thus, the midline is $y = 1$.

□

We will return to the function from **Example 2.1.1** in a bit, as well as the function in the following example, to observe these properties graphically.

Example 2.1.2. Identify the period, phase shift, amplitude, and midline of $g(x) = -3\cos(2x - \pi)$.

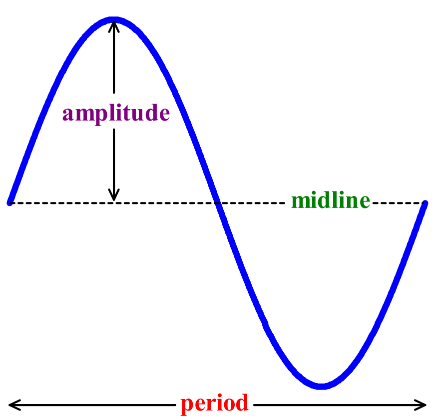
Solution. We compare $g(x) = -3\cos(2x-\pi)$ to $C(x) = A\cos(\omega x - \phi) + B$.

- We first observe that $\omega = 2$, and so the period is $\frac{2\pi}{2} = \pi$.
- The phase shift is $\frac{\phi}{\omega}$. If we rewrite $C(x) = A\cos(\omega x - \phi) + B$ as $C(x) = A\cos\left(\omega\left(x - \frac{\phi}{\omega}\right)\right) + B$, we can identify the phase shift from the formula. We find
$$\begin{aligned} g(x) &= -3\cos(2x-\pi) \\ &= -3\cos\left(2\left(x - \frac{\pi}{2}\right)\right) \end{aligned}$$
Thus, the phase shift is $\frac{\pi}{2}$ units to the right.
- Since $A = -3$, the amplitude is $|-3| = 3$.
- Noting that $B = 0$, there is no vertical shift, so the midline is $y = 0$.

□

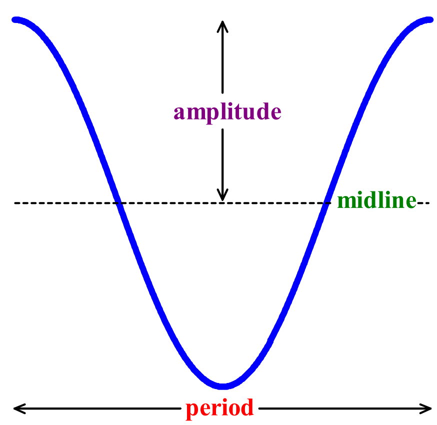
The period, amplitude, and midline are illustrated below. Note that we are assuming $A > 0$. For $A < 0$, each graph would be reflected across its midline.

Figure 2.1. 13



One Cycle of $S(x) = A\sin(\omega x - \phi) + B$

Figure 2.1. 14

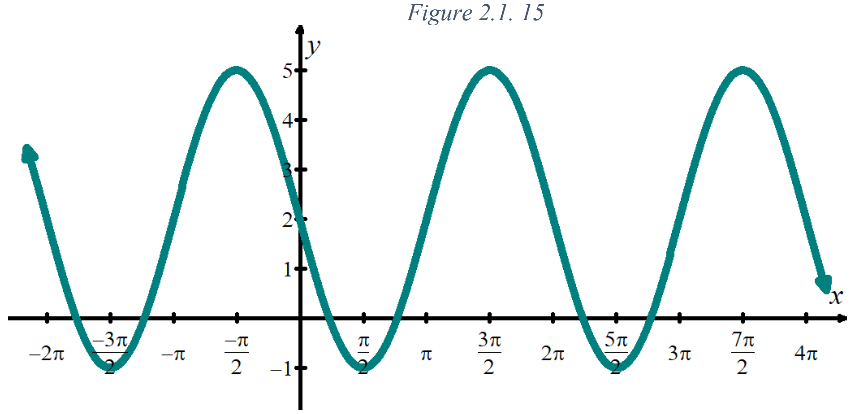


One Cycle of $C(x) = A\cos(\omega x - \phi) + B$

Determining an Equation from the Graph of a Sinusoid

Example 2.1.3. Write an equation for the sinusoid whose graph is shown below.

Figure 2.1. 15

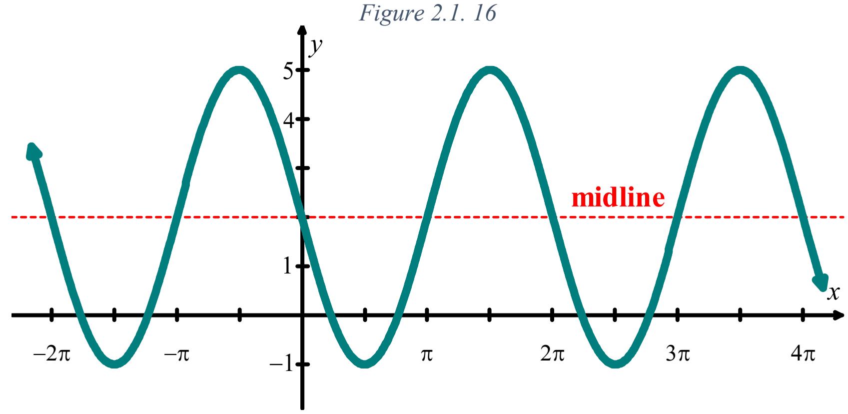


Solution. We identify A, ω, ϕ, and B so that we may fit the data to a function of the form

$S(x) = A\sin(\omega x - \phi) + B$ or $C(x) = A\cos(\omega x - \phi) + B$.

- We start by determining the midline. High points on the graph occur at $y = 5$ while low points occur at $y = -1$. The midline is in the center of these two *y*-values. Thus, $y = 2$ is the midline and the vertical shift is $B = 2$.

Figure 2.1. 16



- The amplitude is the distance from the midline to either a maximum point or a minimum point. From the graph we see the amplitude is $|A| = 5 - 2 = 3$. So, either $A = 3$ or $A = -3$.

- For the period, we measure the length of one cycle, from a peak at $x = -\dfrac{\pi}{2}$ to the next peak at $x = \dfrac{3\pi}{2}$, to find that the period is $2\pi = \dfrac{2\pi}{\omega}$. Thus, $\omega = 1$.

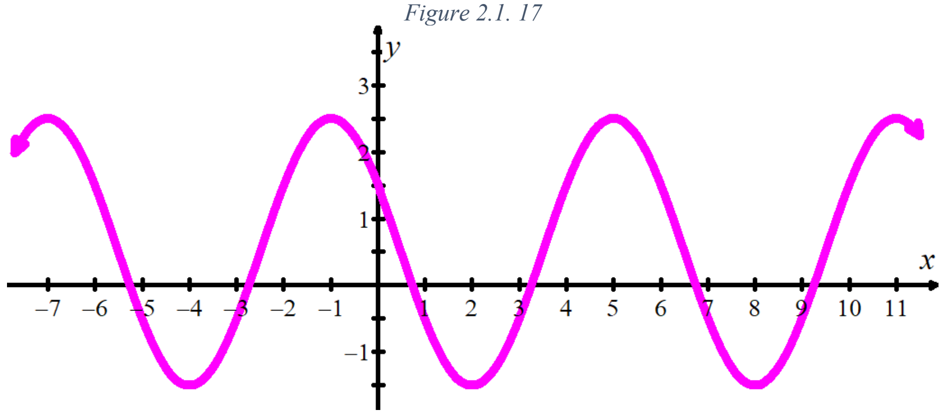
- When $x = 0$, the graph has a point on the midline, as does the graph of $y = \sin(x)$. This tells us that it will be simpler to use the formula $S(x) = A\sin(\omega x - \phi) + B$ with a phase shift of 0. Thus $\frac{\phi}{\omega} = 0$, from which $\phi = 0$.

Returning to the amplitude, the shape of our graph indicates a reflection of the sine graph about its midline, so we choose $A = -3$. The resulting function is $S(x) = -3\sin(x) + 2$.

□

Example 2.1.4. Write an equation of the form $C(x) = A\cos(\omega x - \phi) + B$, and an equation of the form $S(x) = A\sin(\omega x - \phi) + B$, for the sinusoid whose graph is shown below.

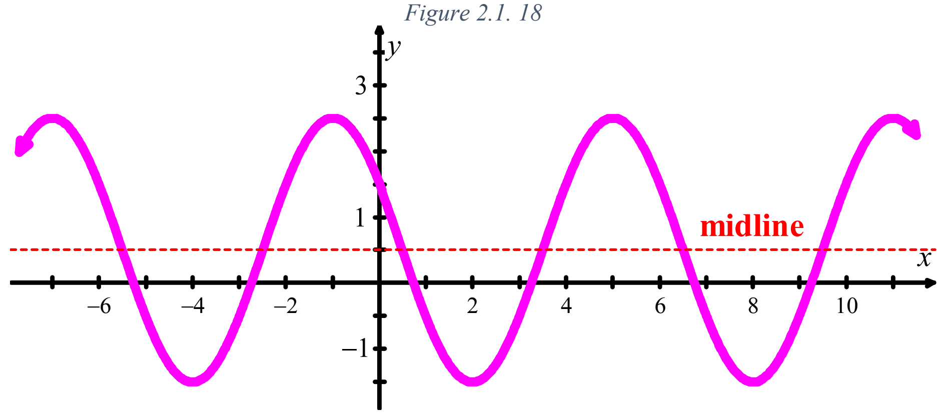
Figure 2.1. 17



Solution. We look for values A, ω, ϕ, and B to establish the equations $C(x) = A\cos(\omega x - \phi) + B$ and $S(x) = A\sin(\omega x - \phi) + B$.

- With high points at $y = \frac{5}{2}$ and low points at $y = -\frac{3}{2}$, the midline is $y = \frac{1}{2}\left(-\frac{3}{2} + \frac{5}{2}\right) = \frac{1}{2}$, so $B = \frac{1}{2}$.

Figure 2.1. 18



- The amplitude is $|A| = \frac{5}{2} - \frac{1}{2} = 2$, so $A = 2$ or $A = -2$. Noting that a phase shift will be required, we select $A = 2$ and make a note to define the phase shift accordingly.
- The period is $5 - (-1) = 6$ using the horizontal distance from one maximum point to the next, or can be found using the horizontal distance between minimum points, such as $8 - 2 = 6$. Since the period is $\frac{2\pi}{\omega} = 6$, we find $\omega = \frac{\pi}{3}$.
- So far, our equations are $C(x) = 2\cos\left(\frac{\pi}{3}x - \phi\right) + \frac{1}{2}$ and $S(x) = 2\sin\left(\frac{\pi}{3}x - \phi\right) + \frac{1}{2}$. We look at the graph to determine the correct phase shift. For the first equation, the cosine graph needs to shift one unit to the right, so we have $\frac{\phi}{\omega} = 1$, from which $\phi = \omega = \frac{\pi}{3}$. For the second equation, the sine graph can be shifted $\frac{5}{2}$ units to the right. We have $\frac{\phi}{\omega} = \frac{5}{2}$, and so $\phi = \frac{5}{2}\omega = \frac{5}{2} \cdot \frac{\pi}{3} = \frac{5\pi}{6}$.

Putting everything together, either the function $C(x) = 2\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + \frac{1}{2}$ or the function $S(x) = 2\sin\left(\frac{\pi}{3}x - \frac{5\pi}{6}\right) + \frac{1}{2}$ will yield the given graph.

□

Graphing Sinusoids

We next look at techniques for graphing sinusoids. While using transformation techniques you learned in a prior algebra course is always a good way to go, we will present two slightly different ways of graphing and leave it to the reader to use the one they prefer. In the first method, we will plot points with simple sine or cosine values. The second method takes advantage of knowing the period and phase shift of the graph.

Example 2.1.5. Graph one cycle of the function $f(x) = 2\sin(3x) + 1$.

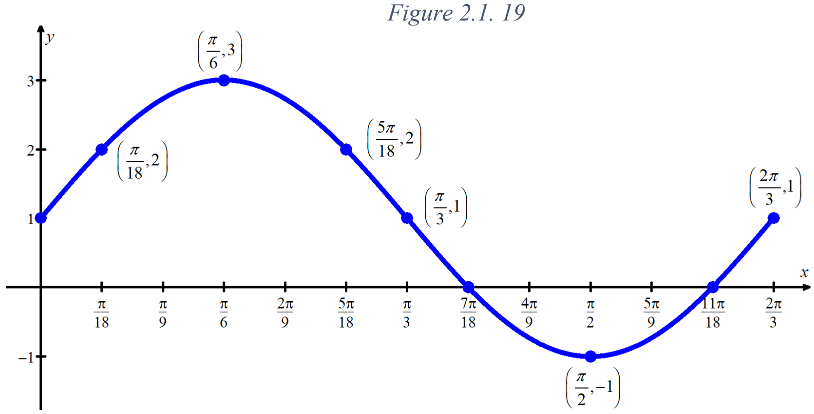
Solution 1. As we did when graphing $y = \sin(x)$, we choose convenient input values to $f(x) = 2\sin(3x) + 1$ that result in $\sin(3x)$ being equal to 0, $\pm\frac{1}{2}$, or ± 1. In other words, we set $3x$ equal to 0, $\frac{\pi}{6}$, $\frac{\pi}{2}$, …, 2π as shown in the following table. From there, we determine the corresponding values for x and y that will guide our sine curve.

$3x$	$x=\frac{3x}{3}$	$y=2\sin(3x)+1$	(x,y)
0	0	$2(0)+1=1$	$(0,1)$
$\frac{\pi}{6}$	$\frac{\pi}{18}$	$2\left(\frac{1}{2}\right)+1=2$	$\left(\frac{\pi}{18},2\right)$
$\frac{\pi}{2}$	$\frac{\pi}{6}$	$2(1)+1=3$	$\left(\frac{\pi}{6},3\right)$
$\frac{5\pi}{6}$	$\frac{5\pi}{18}$	$2\left(\frac{1}{2}\right)+1=2$	$\left(\frac{5\pi}{18},2\right)$
π	$\frac{\pi}{3}$	$2(0)+1=1$	$\left(\frac{\pi}{3},1\right)$

$3x$	$x=\frac{3x}{3}$	$y=2\sin(3x)+1$	(x,y)
$\frac{7\pi}{6}$	$\frac{7\pi}{18}$	$2\left(-\frac{1}{2}\right)+1=0$	$\left(\frac{7\pi}{18},0\right)$
$\frac{3\pi}{2}$	$\frac{\pi}{2}$	$2(-1)+1=-1$	$\left(\frac{\pi}{2},-1\right)$
$\frac{11\pi}{6}$	$\frac{11\pi}{18}$	$2\left(-\frac{1}{2}\right)+1=0$	$\left(\frac{11\pi}{18},0\right)$
2π	$\frac{2\pi}{3}$	$2(0)+1=1$	$\left(\frac{2\pi}{3},1\right)$

Connecting these points with the smooth shape of the sine curve, we have the following:

Figure 2.1. 19



$$y=f(x)=2\sin(3x)+1$$

□

Graphing Sinusoidal Functions by Plotting Points

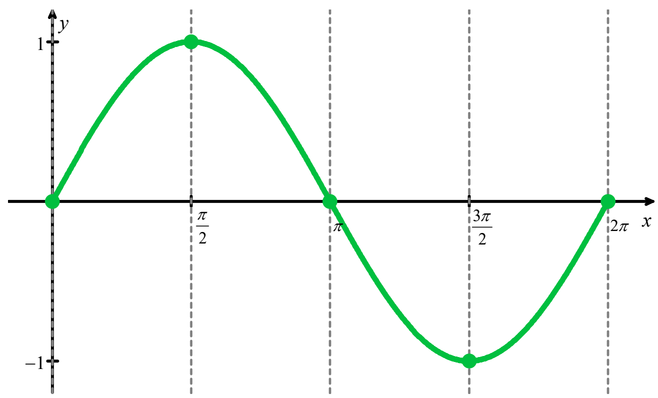
To graph $S(x)=A\sin(\omega x-\phi)+B$ or $C(x)=A\cos(\omega x-\phi)+B$,

1. Choose values for the argument, $\omega x-\phi$, that give sine or cosine values of 0, $\pm\frac{1}{2}$, ± 1.
    - For the sine function, use $\omega x-\phi=0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$.
    - For the cosine function, use $\omega x-\phi=0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, 2\pi$.
2. Find the corresponding *x*- and *y*-values.
3. Plot the points, connect them, and extend in the shape of a sinusoidal curve.

Before proceeding with the second solution technique, note that the period of the sine can be divided into quarter periods of length $\frac{2\pi}{4}=\frac{\pi}{2}$. Using **quarter marks** (vertical dashed lines) to divide the graph of the fundamental cycle of $y=\sin(x)$ into these four quarter periods, we find the following:

- Quarter Period 1: The graph starts on the midline, $y=0$, and ends at a maximum point.
- Quarter Period 2: The graph starts at a maximum point and ends on the midline.
- Quarter Period 3: The graph starts on the midline and ends at a minimum point.
- Quarter Period 4: The graph starts at a minimum point and ends on the midline.

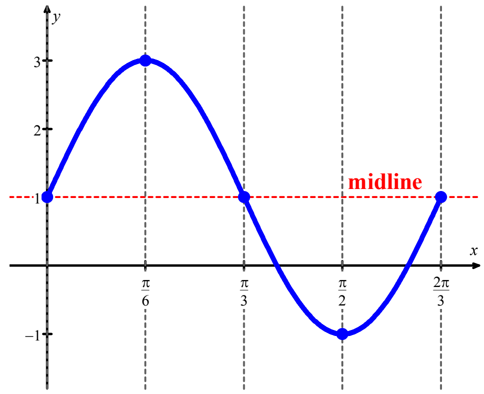
Figure 2.1. 20



$y=\sin(x)$

Solution 2. We first divide the period of $f(x)=2\sin(3x)+1$ into quarter periods. In **Example 2.1.1**, we found that the period of $f(x)=2\sin(3x)+1$ is $\frac{2\pi}{3}$, so the length of these quarter periods is $\frac{1}{4}\cdot\frac{2\pi}{3}=\frac{\pi}{6}$. Starting at $x=0$, since there is no phase shift, we sketch dashed lines for quarter marks at $x=0$, $x=0+\frac{\pi}{6}=\frac{\pi}{6}$, $x=\frac{\pi}{6}+\frac{\pi}{6}=\frac{2\pi}{6}$, $x=\frac{2\pi}{6}+\frac{\pi}{6}=\frac{3\pi}{6}$, and $x=\frac{3\pi}{6}+\frac{\pi}{6}=\frac{4\pi}{6}$. We found that the midline is $y=1$ in **Example 2.1.1**, so we additionally add the dashed line $y=1$. We sketch the graph, starting on the midline at the first quarter mark, mimicking the behavior/shape of the graph of $y=\sin(x)$, as we move from left to right through corresponding quarter periods. Noting that the amplitude is 2, the maximum points will be 2 units above the midline and the minimum points will be 2 units below the midline.

Figure 2.1. 21

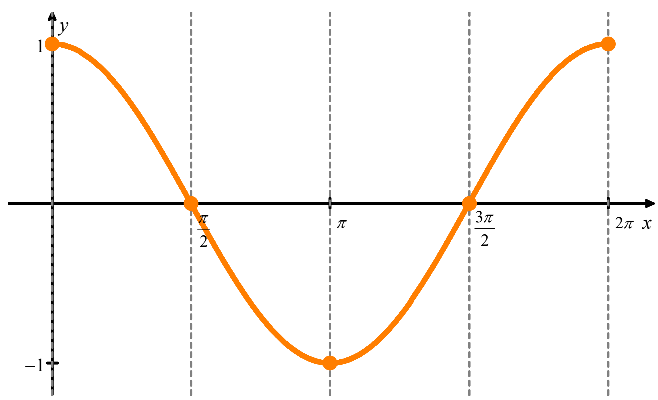


$$y = f(x) = 2\sin(3x) + 1$$

□

In **Solution 2**, while the amplitude is $|A| = 2$, it is important to confirm that $A = +2$. Had it been the case that $A = -2$, the resulting graph would have been a reflection of our current graph across its midline. And what if our function had been a transformation of $y = \cos(x)$ rather than $y = \sin(x)$? Since the period of $y = \cos(x)$ is also 2π, quarter marks would occur at the same locations as those of $y = \sin(x)$.

Figure 2.1. 22



$$y = \cos(x)$$

With its midline of $y = 0$, the graph of $y = \cos(x)$ behaves as follows:

- Quarter Period 1: The graph starts at a maximum point and ends on the midline.
- Quarter Period 2: The graph starts at the midline and ends at a minimum point.
- Quarter Period 3: The graph starts at a minimum point and ends on the midline.
- Quarter Period 4: The graph starts at the midline and ends at a maximum point.

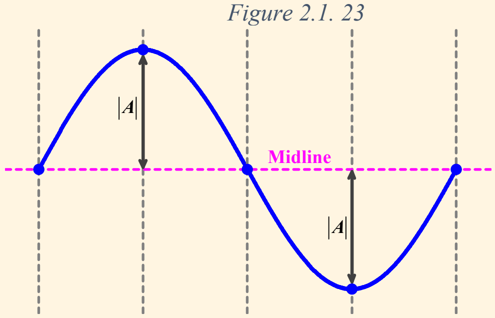
A summary of the technique from **Solution 2**, a 'shortcut' method for applying transformations, follows.

Graphing Sinusoidal Functions by Quarter Marks

To graph $S(x) = A\sin(\omega x - \phi) + B$ or $C(x) = A\cos(\omega x - \phi) + B$, $\omega > 0$, determine the period, phase shift, amplitude, and vertical shift (midline). Then,

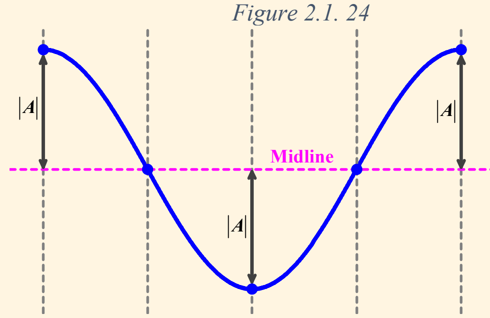
- Divide the period by 4, resulting in the width of each quarter period.
- Sketch quarter marks with dashed vertical lines. The first quarter mark is located left or right from $x = 0$ by the amount of the phase shift. The remaining four quarter marks are determined by consecutively moving right by the width of a quarter period.
- Sketch the midline with a dashed horizontal line, and choose the graph that matches your function from the following models.

Figure 2.1. 23



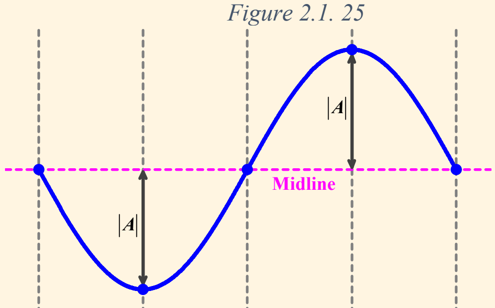
$y = A\sin(\omega x - \phi) + B,\ A > 0$

Figure 2.1. 24



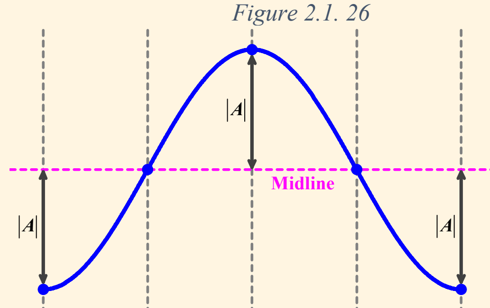
$y = A\cos(\omega x - \phi) + B,\ A > 0$

Figure 2.1. 25



$y = A\sin(\omega x - \phi) + B,\ A < 0$

Figure 2.1. 26



$y = A\cos(\omega x - \phi) + B,\ A < 0$

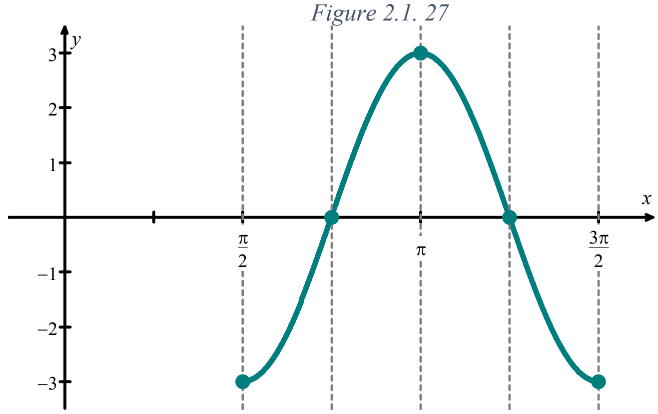
- Follow your model in marking points located at the intersections of the midline with quarter marks. Add the points that are located a distance of $|A|$ above or below the midline.
- Connect these five points and extend in the shape of a sinusoidal curve.

Example 2.1.6. Graph one cycle of the function $g(x) = -3\cos(2x - \pi)$.

Solution. We apply the technique of graphing by quarter marks. In **Example 2.1.2**, we found that the period of $g(x) = -3\cos(2x - \pi)$ is π, the phase shift is $\frac{\pi}{2}$ units to the right, the amplitude is 3 and the midline is $y = 0$.

First of all, a quarter period will have length $\frac{1}{4} \cdot \pi = \frac{\pi}{4}$. With the phase shift being $\frac{\pi}{2}$ units to the right, the first quarter mark appears at $x = \frac{\pi}{2}$ with subsequent quarter marks at $x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$, $x = \frac{3\pi}{4} + \frac{\pi}{4} = \pi$, $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, and $x = \frac{5\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{2}$. We use dashed lines to indicate quarter marks and keep in mind that the midline is $y = 0$. Since $|A| = 3$, the maximum points will be 3 units above the midline and the minimum points will be 3 units below the midline. With $A = -3$, the behavior of the graph in each quarter period will mimic that of $y = A\cos(\omega x - \phi) + B$ with $A < 0$.

Figure 2.1. 27



$y = g(x) = -3\cos(2x - \pi)$

□

2.1 Exercises

1. Why are the sine and cosine functions called periodic functions?

2. How does the graph of $y=\sin(x)$ compare with the graph of $y=\cos(x)$? Explain how you could horizontally translate the graph of $y=\sin(x)$ to obtain the graph of $y=\cos(x)$.

3. For the function $f(x)=A\cos(\omega x-\phi)+B$, what constants affect the range and how do they affect the range?

In Exercises 4 – 15, graph one cycle of the given function. State the period of the function.

4. $y=3\sin(x)$
5. $y=\sin(3x)$
6. $y=-2\cos(x)$
7. $y=\frac{1}{2}\cos(x)$
8. $y=\sin\left(-\frac{1}{2}x\right)$
9. $y=\cos(-x)+3$
10. $y=\cos\left(x-\frac{\pi}{2}\right)$
11. $y=-\sin\left(x+\frac{\pi}{3}\right)$
12. $y=\sin\left(2\left(x-\frac{\pi}{2}\right)\right)$
13. $y=-\frac{1}{3}\cos\left(x+\frac{\pi}{3}\right)$
14. $y=\cos(3x)+4$
15. $y=\sin\left(x+\frac{\pi}{4}\right)-2$

In Exercises 16 – 27, graph two full cycles of each function. State the domain, range, and period. If applicable, describe the horizontal (phase) shift and vertical shift of the function.

16. $y=2\sin(x)$
17. $y=\frac{2}{3}\cos(x)$
18. $y=-3\sin(x)$
19. $y=\sin(4x)$
20. $y=\sin(\pi x)$
21. $y=\cos(2x)$
22. $y=\cos\left(\frac{\pi}{2}x\right)$
23. $y=\cos\left(\frac{\pi}{2}x\right)+2$
24. $y=3\cos\left(\frac{6}{5}x\right)$
25. $y=\sin\left(x-\frac{\pi}{2}\right)-1$
26. $y=2\sin(3x+\pi)$
27. $y=5\sin(\pi x)-2$

In Exercises 28 – 39, state the period, phase shift, amplitude, and vertical shift of the given function. Graph at least one cycle of the function.

28. $y=3\sin(x)$
29. $y=\sin(3x)$
30. $y=-2\cos(x)$

31. $y=\cos\left(x-\frac{\pi}{2}\right)$

32. $y=-\sin\left(x+\frac{\pi}{3}\right)$

33. $y=\sin(2x-\pi)$

34. $y=-\frac{1}{3}\cos\left(\frac{1}{2}x+\frac{\pi}{3}\right)$

35. $y=\cos(3x-2\pi)+4$

36. $y=\sin\left(-x-\frac{\pi}{4}\right)-2$

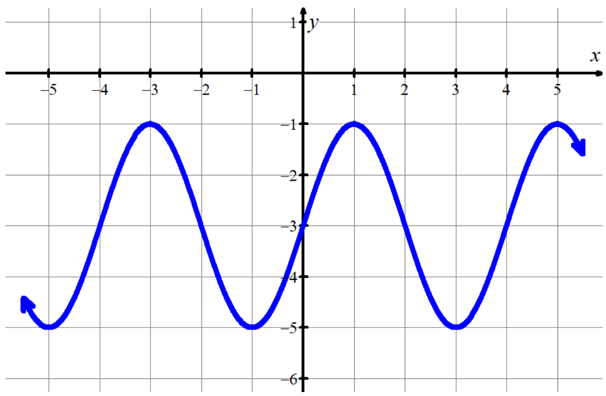
37. $y=\frac{2}{3}\cos\left(-4x+\frac{\pi}{2}\right)+1$

38. $y=-\frac{3}{2}\cos\left(2x+\frac{\pi}{3}\right)-\frac{1}{2}$

39. $y=4\sin(-2\pi x+\pi)$

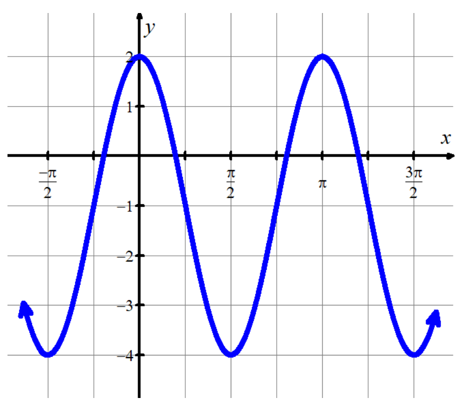
40. Write an equation of the form $S(x)=A\sin(\omega x-\phi)+B$ for the sine function whose graph is shown below.

Figure Ex2.1. 1



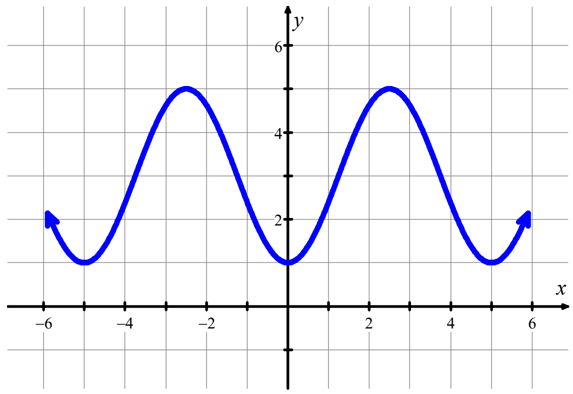
41. Write an equation of the form $C(x)=A\cos(\omega x-\phi)+B$ for the cosine function whose graph is shown below.

Figure Ex2.1. 2



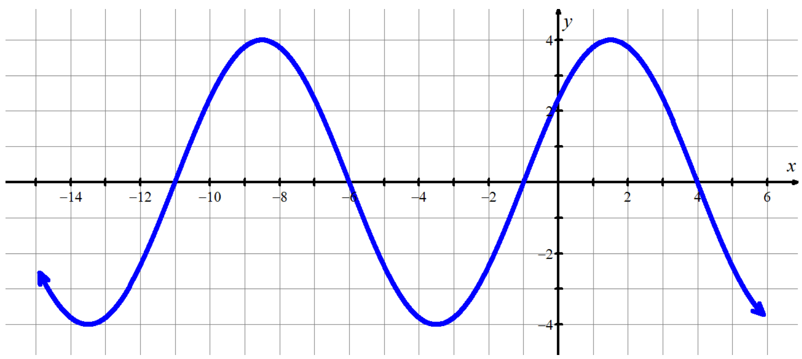
42. Write an equation of the form $C(x) = A\cos(\omega x - \phi) + B$ for the cosine function whose graph is shown below.

Figure Ex2.1. 3



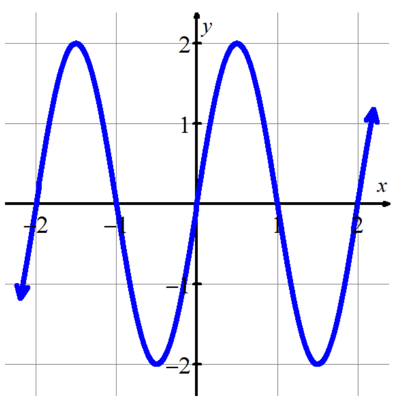
43. Write an equation of the form $S(x) = A\sin(\omega x - \phi) + B$ for the sine function whose graph is shown below.

Figure Ex2.1. 4



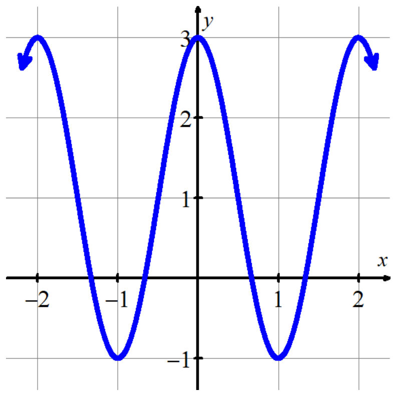
44. Write an equation of the form $S(x) = A\sin(\omega x - \phi) + B$ for the sine function whose graph is shown below.

Figure Ex2.1. 5



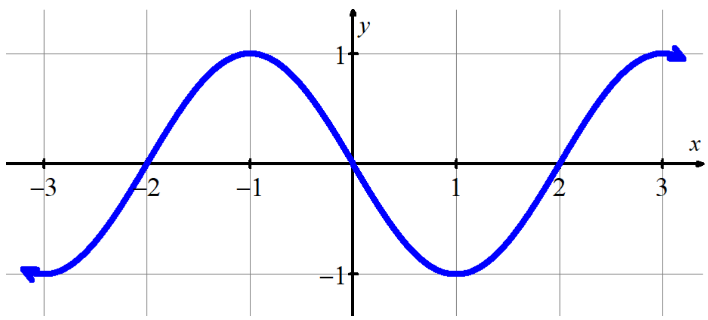
45. Write an equation of the form $C(x) = A\cos(\omega x - \phi) + B$ for the cosine function whose graph is shown below.

Figure Ex2.1. 6



46. Write an equation of the form $S(x) = A\sin(\omega x - \phi) + B$ for the sine function whose graph is shown below.

Figure Ex2.1. 7



47. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn.

a. Find the period, amplitude, and vertical shift of $h(t)$.

b. Find a formula for the height function $h(t)$.

c. How high off the ground is a person after 5 minutes?

In Exercises 48 – 49, verify the identity by using technology to graph the right and left sides.

48. $\sin^2(x)+\cos^2(x)=1$

49. $\cos(x)=\sin\left(\frac{\pi}{2}-x\right)$

In Exercises 50 – 53, graph the function with the help of technology and discuss the results with your classmates.

50. $f(x)=\cos(3x)+\sin(x)$. Is this function periodic? If so, what is the period?

51. $f(x)=\frac{\sin(x)}{x}$. What appears to be the horizontal asymptote of the graph?

52. $f(x)=x\sin(x)$. Graph $y=\pm x$ on the same set of axes and describe the behavior of f.

53. $f(x)=\sin\left(\frac{1}{x}\right)$. What's happening as $x\to 0$?

2.2 Graphs of the Other Trigonometric Functions

Learning Objectives

- **Graph tangent, cotangent, secant, and cosecant functions and their transformations.**
- **Identify vertical asymptotes, period, domain, and range of these functions.**
- **Determine whether each function is even or odd.**

Having graphed the sine and cosine functions, we move on to the remaining four trigonometric functions, starting with the tangent and cotangent. Graphs of the tangent and cotangent functions will be a bit of a challenge with their shorter period and vertical asymptotes. Then, moving on to secant and cosecant functions, we find these graphs follow nicely from graphs of the sine and cosine.

Graph of the Tangent Function

We recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and construct a table of values for the tangent function over the interval $[0, 2\pi]$. To conserve space, we have not included columns for sine and cosine values, although that is common practice.

x	$\tan(x)$	$(x, \tan(x))$
0	0	$(0,0)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4},1\right)$
$\frac{\pi}{3}$	$\sqrt{3}$	$\left(\frac{\pi}{3},\sqrt{3}\right)$
$\frac{\pi}{2}$	Not defined	
$\frac{2\pi}{3}$	$-\sqrt{3}$	$\left(\frac{2\pi}{3},-\sqrt{3}\right)$

x	$\tan(x)$	$(x, \tan(x))$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4},-1\right)$
π	0	$(\pi,0)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4},1\right)$
$\frac{4\pi}{3}$	$\sqrt{3}$	$\left(\frac{4\pi}{3},\sqrt{3}\right)$
$\frac{3\pi}{2}$	Not defined	

x	$\tan(x)$	$(x, \tan(x))$
$\frac{5\pi}{3}$	$-\sqrt{3}$	$\left(\frac{5\pi}{3},-\sqrt{3}\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4},-1\right)$
2π	0	$(2\pi,0)$

To determine the behavior of the graph of $y = \tan(x)$ when x is close to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, we will look at some values for $\tan(x)$ on both sides of $x = \frac{\pi}{2}$ and on both sides of $x = \frac{3\pi}{2}$.

- Following are some values for $\tan(x)$ when x is less than, but close to, $\frac{\pi}{2}$. With $\frac{\pi}{2} \approx 1.571$, we include approximate values of the tangent for the indicated radian measures of x. Note that these values of $\tan(x)$ are positive, getting larger and larger as x approaches $\frac{\pi}{2}$ from the left.

  The result is a vertical asymptote at $x = \frac{\pi}{2}$.

x	1.5	1.55	1.56	1.57	$\cdots$	$\frac{\pi}{2} \approx 1.571$
$\tan(x)$	14	48	93	1256	$\cdots$	undefined

  Mathematical notation: As $x \to \frac{\pi}{2}^{-}$, $\tan(x) \to \infty$.

- When x is greater than, but close to, $\frac{\pi}{2}$, as x approaches $\frac{\pi}{2}$ from the right, the values of $\tan(x)$ get smaller and smaller, approaching negative infinity.

x	1.7	1.6	1.59	1.58	$\cdots$	$\frac{\pi}{2} \approx 1.571$
$\tan(x)$	-8	-34	-52	-109	$\cdots$	undefined

  Mathematical notation: As $x \to \frac{\pi}{2}^{+}$, $\tan(x) \to -\infty$.

- Noting that $\frac{3\pi}{2} \approx 4.712$, we look at approximate values for $\tan(x)$ when x is close to $\frac{3\pi}{2}$.

x	4.6	4.65	4.7	4.71	$\cdots$	$\frac{3\pi}{2} \approx 4.712$
$\tan(x)$	9	16	81	419	$\cdots$	undefined

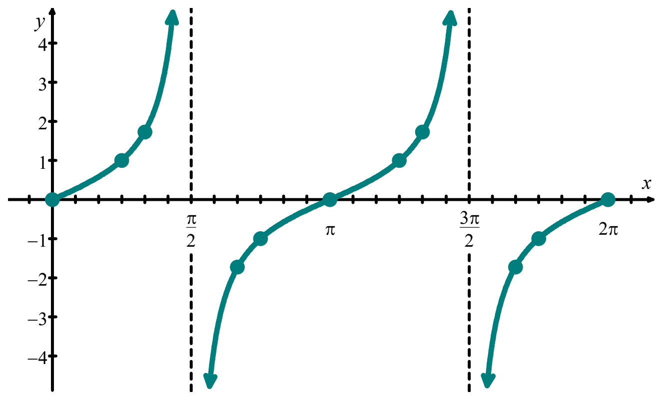
As $x \to \frac{3\pi}{2}^{-}$, $\tan(x) \to \infty$

x	4.8	4.75	4.73	4.72	$\cdots$	$\frac{3\pi}{2} \approx 4.712$
$\tan(x)$	-11	-27	-57	-131	$\cdots$	undefined

As $x \to \frac{3\pi}{2}^{+}$, $\tan(x) \to -\infty$

Thus, we have vertical asymptotes at $x=\frac{\pi}{2}$ and at $x=\frac{3\pi}{2}$. We also know something about the behavior of the graph as it approaches these vertical asymptotes from each side. Plotting all of the information we have gathered about the graph of $y=\tan(x)$ over the interval $[0,2\pi]$, we have the following:

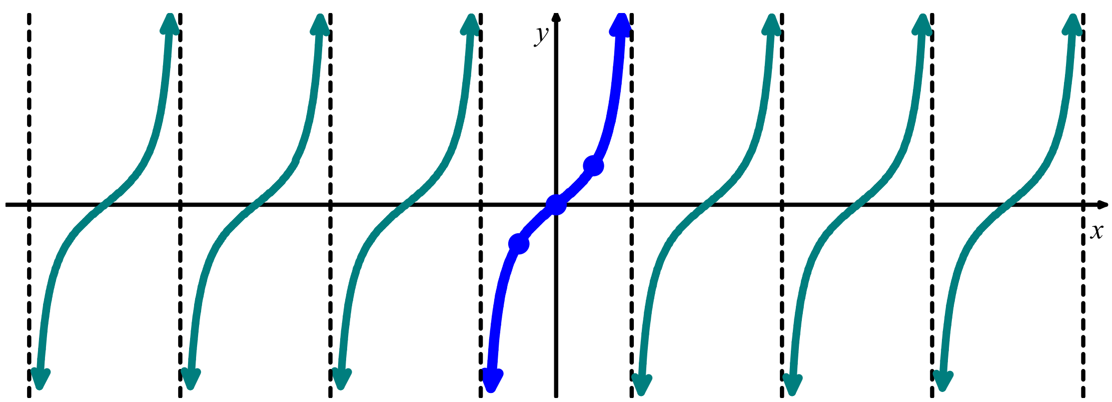
Figure 2.2. 1



The graph of $y=\tan(x)$ over $[0,2\pi]$

Below, we extend the graph of $y=\tan(x)$ by 'copy and paste'.

Figure 2.2. 2



The graph of $y=\tan(x)$ with fundamental cycle highlighted in blue

From the graph, it appears that the tangent function is periodic, with period π. This is, in fact, the case as we will prove in **Section 3.1**, following the introduction of the sum identity for tangent. We take as our fundamental cycle for $y=\tan(x)$ the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

From the graph, we see that the domain of the tangent function, $y=\tan(x)$, includes all real numbers x except for $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots$. These are the x-values for which $\cos(x)=0$ and, subsequently, are the only real numbers for which $y = \tan(x)$ is undefined. Thus, the domain of $y = \tan(x)$ is all real numbers x, excluding $x = \frac{\pi}{2} + \pi k$ for any integer k. The range of $y = \tan(x)$, as observed from the graph, includes all real numbers.

The graph of $y = \tan(x)$ suggests symmetry through the origin. Indeed, $y = \tan(x)$ is odd since $y = \sin(x)$ is odd and $y = \cos(x)$ is even, as shown below.

$$\begin{aligned}\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} \\ &= \frac{-\sin(x)}{\cos(x)} \\ &= -\tan(x)\end{aligned}$$

Graph of the Cotangent Function

It should be no surprise that the graph of the cotangent function behaves similarly to the graph of the tangent function. Noting that $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we construct a table of values for $y = \cot(x)$ over the interval $[0, 2\pi]$.

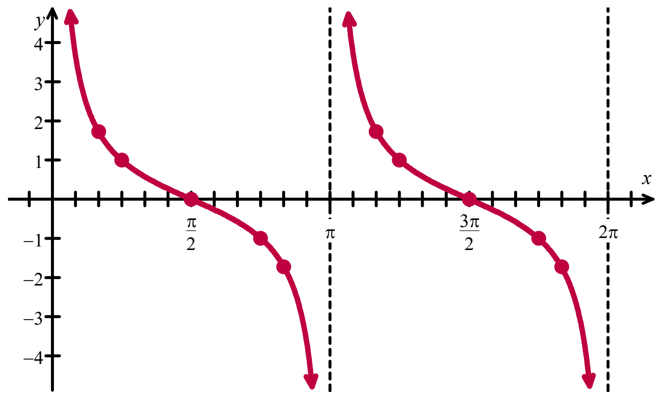
x	$\cot(x)$	$(x, \cot(x))$
0	Not defined	
$\frac{\pi}{6}$	$\sqrt{3}$	$\left(\frac{\pi}{6}, \sqrt{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$

x	$\cot(x)$	$(x, \cot(x))$
$\frac{5\pi}{6}$	$-\sqrt{3}$	$\left(\frac{5\pi}{6}, -\sqrt{3}\right)$
π	Not defined	
$\frac{7\pi}{6}$	$\sqrt{3}$	$\left(\frac{7\pi}{6}, \sqrt{3}\right)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$

x	$\cot(x)$	$(x, \cot(x))$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
$\frac{11\pi}{6}$	$-\sqrt{3}$	$\left(\frac{11\pi}{6}, -\sqrt{3}\right)$
2π	Not defined	

Asymptotes occur at the x-values where $\cot(x)$ is not defined (when $\sin(x) = 0$) resulting in the following graph.

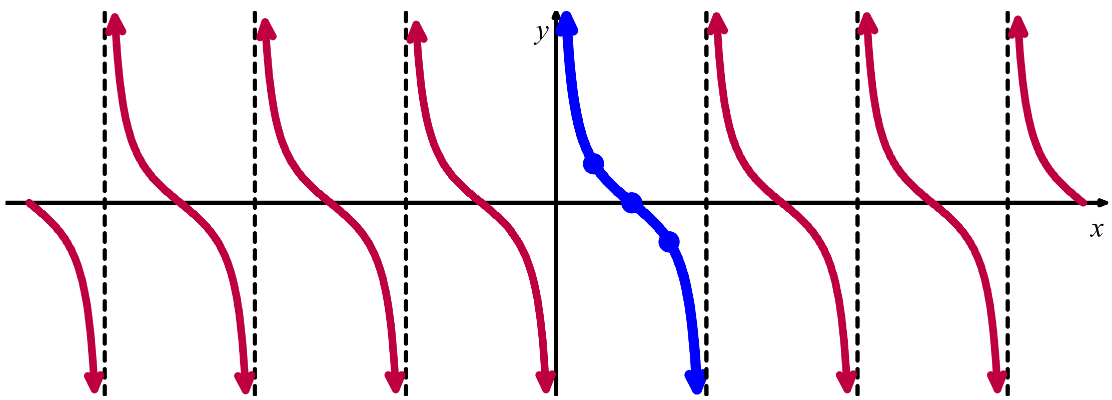
Figure 2.2. 3



The graph of $y = \cot(x)$ over $[0, 2\pi]$

It clearly appears that the period of the cotangent function is π, which is indeed the case and will be revisited in **Section 3.1**. We take as our fundamental cycle the interval $(0, \pi)$. A more complete graph of $y = \cot(x)$ follows, with the fundamental cycle highlighted.

Figure 2.2. 4



The graph of $y = \cot(x)$ with fundamental cycle highlighted in blue

We see from the graph that the apparent domain of the cotangent function is all real numbers x except for $x = 0, \pm\pi, \pm 2\pi, \ldots$. These are the values where $\sin(x) = 0$ and are the only real numbers for which $y = \cot(x)$ is not defined. Thus, the domain of $y = \cot(x)$ is all real numbers x, excluding $x = \pi k$, for any integer k. The range of $y = \cot(x)$ includes all real numbers.

As with the tangent, the graph of $y = \cot(x)$ suggests symmetry through the origin. An argument similar to that used for the tangent verifies that this is the case and that the cotangent is an odd function. Try it!

On the intervals between their vertical asymptotes, both $y=\tan(x)$ and $y=\cot(x)$ are **continuous and smooth**. In other words, they are continuous and smooth **on their domains**. Other properties of the tangent and cotangent functions are summarized below.

Properties of the Tangent and Cotangent Functions

The function $y=\tan(x)$

- has domain of all real numbers except $x=\frac{\pi}{2}+\pi k$ for integers k
- has range $(-\infty,\infty)$
- is odd
- has period π

The function $y=\cot(x)$

- has domain of all real numbers except $x=\pi k$ for integers k.
- has range $(-\infty,\infty)$
- is odd
- has period π

Graphing Transformations of the Tangent and Cotangent Functions

Graphing transformations of the tangent and cotangent functions, $J(x)=A\tan(\omega x-\phi)+B$ and $K(x)=A\cot(\omega x-\phi)+B$, $\omega>0$, is similar to graphing transformations of the sine and cosine. We can plot points or use quarter marks. Examples of each of these methods will follow.

Graphing Tangent and Cotangent Functions by Plotting Points

To graph $J(x)=A\tan(\omega x-\phi)+B$ or $K(x)=A\cot(\omega x-\phi)+B$,

1. Draw vertical asymptotes as dashed lines at locations where the function is not defined, after determining the equations of the vertical asymptotes as follows.
   - For the tangent function, solve $\omega x-\phi=\pm\frac{\pi}{2}$.
   - For the cotangent function, solve $\omega x-\phi=0$ and $\omega x-\phi=\pi$.
2. Choose values for the argument, $\omega x-\phi$, that result in tangent or cotangent values of 0 and ± 1.
   - For the tangent function, let $\omega x-\phi=-\frac{\pi}{4}, 0, \frac{\pi}{4}$.
   - For the cotangent function, let $\omega x-\phi=\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$.
3. Find the corresponding *x*- and *y*-values.
4. Plot these points, and connect them with a smooth curve that approaches the vertical asymptotes. Copy to the left and right for showing multiple periods of the graph.

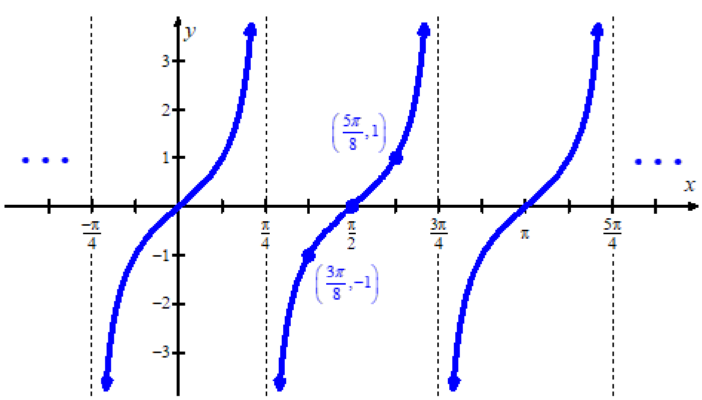
Example 2.2.1. Draw the graph of the function $y=\tan(2x-\pi)$.

Solution. To determine the vertical asymptotes, we find the x-values for which $\tan(2x-\pi)$ is not defined. These values occur when $2x-\pi$ is equal to $\pm\frac{\pi}{2}$. Additionally, we identify points on the graph whose x-values result in $y=\tan(2x-\pi)$ being zero or ± 1. These points occur when $2x-\pi$ is equal to zero or $\pm\frac{\pi}{4}$.

$2x-\pi$	x	$y=\tan(2x-\pi)$	Asymptote or Point on Graph
$-\frac{\pi}{2}$	$\frac{\pi}{4}$	Not Defined	Vertical Asymptote: $x=\frac{\pi}{4}$
$-\frac{\pi}{4}$	$\frac{3\pi}{8}$	-1	Point: $\left(\frac{3\pi}{8},-1\right)$
0	$\frac{\pi}{2}$	0	Point: $\left(\frac{\pi}{2},0\right)$
$\frac{\pi}{4}$	$\frac{5\pi}{8}$	1	Point: $\left(\frac{5\pi}{8},1\right)$
$\frac{\pi}{2}$	$\frac{3\pi}{4}$	Not Defined	Vertical Asymptote: $x=\frac{3\pi}{4}$

We draw the vertical asymptotes $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$ as dashed lines. Then, after plotting the points $\left(\frac{3\pi}{8},-1\right)$, $\left(\frac{\pi}{2},0\right)$, and $\left(\frac{5\pi}{8},1\right)$, we connect them with a smooth curve and extend the curve so that it approaches the vertical asymptotes. To show more than one period of the graph, we 'copy and paste' to the left and right.

Figure 2.2. 5



□

To graph $J(x) = A\tan(\omega x - \phi) + B$ or $K(x) = A\cot(\omega x - \phi) + B$ using quarter marks, keep in mind that some properties of the tangent and cotangent functions differ from those of the sine and cosine functions.

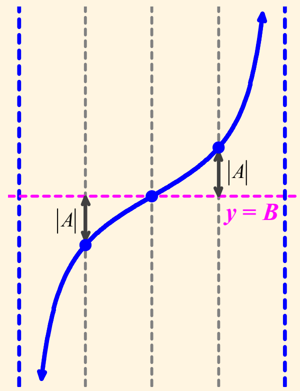
- The period of both $y = \tan(x)$ and $y = \cot(x)$ is π, so the period of transformations will be $\frac{\pi}{\omega}$.
- Since the fundamental cycle of the tangent starts at $-\frac{\pi}{2}$ instead of 0, the horizontal shift of $\frac{\phi}{\omega}$ is applied **after** determining the period. ('Phase shift' is unique to sine and cosine.)
- The vertical scaling is $|A|$, but tangent and cotangent do not have amplitude since they do not possess the wavelike characteristics of the sine and cosine.
- The vertical shift of B is not referred to as the 'midline'.

Graphing Tangent and Cotangent Functions by Quarter Marks

To graph $J(x) = A\tan(\omega x - \phi) + B$ or $K(x) = A\cot(\omega x - \phi) + B$, $\omega > 0$, determine the period, horizontal shift, vertical scaling, and vertical shift.

- Determine the location of the first quarter mark:
  - For $y = J(x)$, start at a distance of $\frac{\text{period}}{2}$ to the left of $x = 0$. Then move left or right by the amount of the phase shift.
  - For $y = K(x)$, start at $x = 0$ and move left or right by the amount of the phase shift.
- Divide the period by 4, resulting in the width of each quarter period. From the first quarter mark, move right by that width, consecutively, to identify the locations of the remaining four quarter marks. Sketch vertical asymptotes at the first and last quarter marks.
- Sketch $y = B$ with a dashed horizontal line, and choose the graph that matches your function from the models that follow these guidelines.
- Mark the point where the third quarter mark intersects the line $y = B$. Follow your model in marking points at the second and fourth quarter marks, $|A|$ units above or below $y = B$.
- Connect these three points with a curve in the shape of your model, approaching the vertical asymptotes.

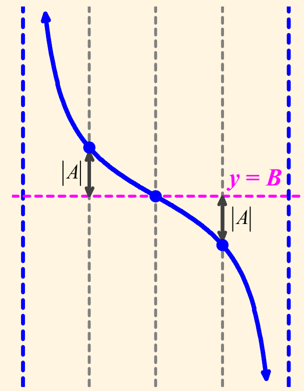
Figure 2.2. 6



$y = A\tan(\omega x - \phi) + B,\ A > 0$ or

$y = A\cot(\omega x - \phi) + B,\ A < 0$

Figure 2.2. 7



$y = A\cot(\omega x - \phi) + B,\ A > 0$ or

$y = A\tan(\omega x - \phi) + B,\ A < 0$

Example 2.2.2. Graph one cycle of the function $f(x) = 3 - \tan\left(\frac{x}{2}\right)$.

Solution. Before identifying the period, horizontal shift, vertical scaling, and vertical shift, we rewrite the function $f(x) = 3 - \tan\left(\frac{x}{2}\right)$ in the format $y = A\tan(\omega x - \phi) + B$ as follows:

$$f(x) = -\tan\left(\frac{1}{2}x - 0\right) + 3$$

With $A = -1$, $\omega = \frac{1}{2}$, $\phi = 0$ and $B = 3$, we have

Period is $\frac{\pi}{\omega} = \frac{\pi}{(1/2)} = 2\pi$.

Vertical scaling is $|A| = |-1| = 1$.

Horizontal shift is $\frac{\phi}{\omega} = \frac{0}{(1/2)} = 0$.

Vertical shift is $B = 3$.

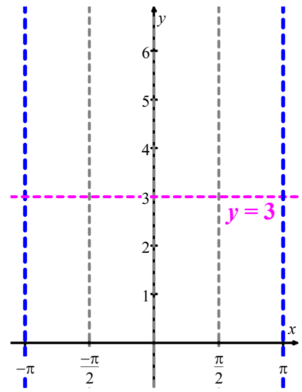
We proceed with graphing by quarter marks.

- Dividing the period of 2π by 2, we have a starting point π units to the left of $x = 0$. Since the horizontal shift is 0, we place the first quarter mark at $x = -\pi$.
- From the period of 2π, the width of each quarter period is $\frac{2\pi}{4} = \frac{\pi}{2}$, resulting in quarter marks at

  $x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$ and π.

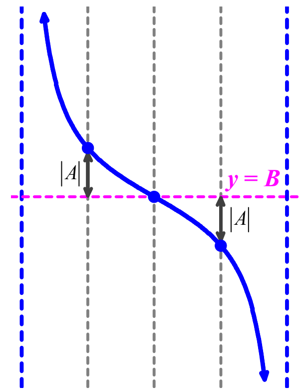
- Since the vertical shift is 3, we include the line $y=3$ in a sketch with the quarter marks, the first and last of which are vertical asymptotes. We note that our function matches the model $y = A\tan(\omega x - \phi) + B$, with $A<0$.

Figure 2.2. 8



Graph shows quarter marks, vertical asymptotes and line for vertical shift

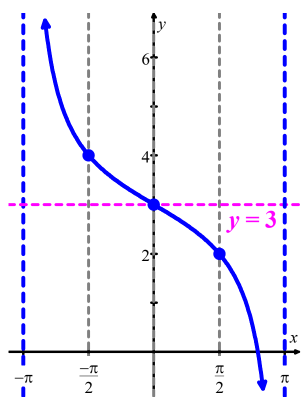
Figure 2.2. 9



$y = A\tan(\omega x - \phi) + B,\ A<0$

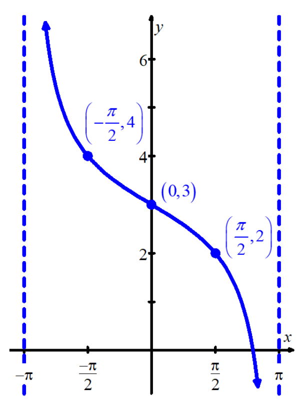
- We mark the point located at the intersection of the lines $y=3$ and $x=0$, the third quarter mark. Following the model, at the second quarter mark we plot a point $|A|=1$ unit above the line $y=3$. The third point is 1 unit below the line $y=3$ at the fourth quarter mark.
- Connecting the points and approaching the vertical asymptotes, we have the following graph.

Figure 2.2. 10



$y = f(x) = 3 - \tan\left(\frac{x}{2}\right)$

Figure 2.2. 11



(without construction lines)

□

Example 2.2.3. Graph one cycle of the function $g(x) = 3\cot\left(\frac{\pi}{4}x + \frac{\pi}{4}\right) + 1$.

Solution. We first rewrite the function $g(x)=3\cot\left(\frac{\pi}{4}x+\frac{\pi}{4}\right)+1$ in a format that is easier to work with. Noting that $y=A\cot(\omega x-\phi)+B=A\cot\left[\omega\left(x-\frac{\phi}{\omega}\right)\right]+B$, we go a step further in this example to simplify identifying the horizontal shift.

$$\begin{aligned} g(x)&=3\cot\left(\frac{\pi}{4}x+\frac{\pi}{4}\right)+1 \\ &=3\cot\left[\frac{\pi}{4}(x+1)\right]+1 \\ &=3\cot\left[\frac{\pi}{4}(x-(-1))\right]+1 \end{aligned}$$

We see that $A=3$, $\omega=\frac{\pi}{4}$, $\frac{\phi}{\omega}=-1$ and $B=1$, from which

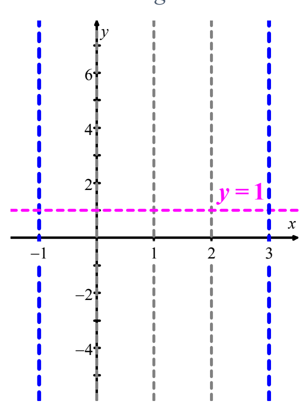
Period is $\frac{\pi}{\omega}=\frac{\pi}{(\pi/4)}=4$. Vertical scaling is $|A|=|3|=3$.

Horizontal shift is $\frac{\phi}{\omega}=-1$. Vertical shift is $B=1$.

As in the previous example, we will graph using quarter marks.

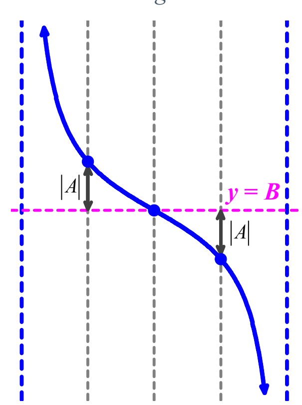
- We begin at $x=0$, the starting point for quarter marks in graphing $y=\cot(x)$. Applying the horizontal shift of -1 takes us to the left by one unit for a first quarter mark at $x=-1$.
- Dividing the period of 4 into four equal quarter periods of width 1 unit, quarter marks can be placed at $x=-1,\ 0,\ 1,\ 2,$ and 3. The first and fifth quarter marks are locations of asymptotes.
- The vertical shift is 1, so we add the line $y=1$ to our graph. With $A=3$, we select $y=A\cot(\omega x-\phi)+B$, $A>0$, as our model.

Figure 2.2. 12



Graph shows quarter marks, vertical asymptotes and line for vertical shift

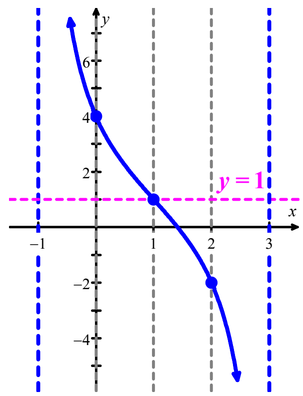
Figure 2.2. 13



$y=A\cot(\omega x-\phi)+B,\ A>0$

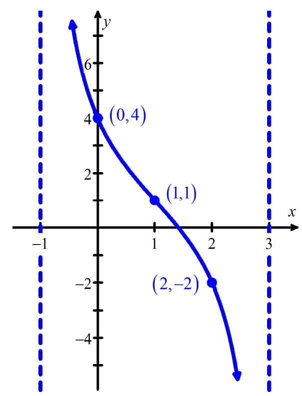
- At the third quarter mark, we plot the point $(1,1)$. At the second quarter mark, with $|A|=3$, we plot a point 3 units above the line $y=1$, at $(0,4)$. At the fourth quarter mark we drop 3 units below the line $y=1$ to plot the point $(2,-2)$.
- Applying the graphical behavior of the cotangent function, we have the following graph.

Figure 2.2. 14



$y=g(x)=3\cot\left(\frac{\pi}{4}x+\frac{\pi}{4}\right)+1$

Figure 2.2. 15



The final graph

□

Before moving on, we note the requirement that $\omega>0$. Should we be given the formula for a tangent or cotangent function in which the coefficient of x is negative, we can use the fact that these functions are odd to rewrite the formula before proceeding. For example, say we are given $f(x)=\tan(-3x)+1$. Then $f(x)=\tan(-3x)+1=\tan(-(3x))+1=-\tan(3x)+1$.

Graph of the Secant Function

We use values of the cosine function in its fundamental cycle, $[0,2\pi]$, to determine values for the secant.

The domain of the secant function excludes all odd multiples of $\frac{\pi}{2}$ since these are the values of x for which $\cos(x)=0$. In our table based on the fundamental cycle of $y=\cos(x)$, the secant is undefined at $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$. These are both *x*-values at which vertical asymptotes occur.

x	$\cos(x)$	$\sec(x)=\dfrac{1}{\cos(x)}$	$(x,\sec(x))$
0	1	1	$(0,1)$
$\dfrac{\pi}{4}$	$\dfrac{1}{\sqrt{2}}$	$\sqrt{2}$	$\left(\dfrac{\pi}{4},\sqrt{2}\right)$
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$	2	$\left(\dfrac{\pi}{3},2\right)$
$\dfrac{\pi}{2}$	0	Not defined	
$\dfrac{2\pi}{3}$	$-\dfrac{1}{2}$	-2	$\left(\dfrac{2\pi}{3},-2\right)$
$\dfrac{3\pi}{4}$	$-\dfrac{1}{\sqrt{2}}$	$-\sqrt{2}$	$\left(\dfrac{3\pi}{4},-\sqrt{2}\right)$
π	-1	-1	$(\pi,-1)$

x	$\cos(x)$	$\sec(x)=\dfrac{1}{\cos(x)}$	$(x,\sec(x))$
$\dfrac{5\pi}{4}$	$-\dfrac{1}{\sqrt{2}}$	$-\sqrt{2}$	$\left(\dfrac{5\pi}{4},-\sqrt{2}\right)$
$\dfrac{4\pi}{3}$	$-\dfrac{1}{2}$	-2	$\left(\dfrac{4\pi}{3},-2\right)$
$\dfrac{3\pi}{2}$	0	Not defined	
$\dfrac{5\pi}{3}$	$\dfrac{1}{2}$	2	$\left(\dfrac{5\pi}{3},2\right)$
$\dfrac{7\pi}{4}$	$\dfrac{1}{\sqrt{2}}$	$\sqrt{2}$	$\left(\dfrac{7\pi}{4},\sqrt{2}\right)$
2π	1	1	$(2\pi,1)$

To determine the behavior of the graph of $y=\sec(x)$ when x is close to the vertical asymptotes $x=\dfrac{\pi}{2}$ and $x=\dfrac{3\pi}{2}$, we look at some values for $\sec(x)$ on both sides of $x=\dfrac{\pi}{2}$ and on both sides of $x=\dfrac{3\pi}{2}$.

- Following are some values for $\sec(x)$ when x is less than, but close to, $\dfrac{\pi}{2}$. We note that $\dfrac{\pi}{2}\approx 1.571$ and include approximate values of the secant for the indicated radian measures of x.

x	1.5	1.55	1.56	1.57	$\cdots$	$\dfrac{\pi}{2}\approx 1.571$
$\sec(x)=\dfrac{1}{\cos(x)}$	14	48	93	1256	$\cdots$	undefined

Values of $\sec(x)$ are positive, getting larger and larger as x approaches $\dfrac{\pi}{2}$ from the left:

$$\text{As } x\to\frac{\pi}{2}^{-},\ \sec(x)\to\infty$$

- We next look at approximate values of $\sec(x)$ when x is greater than, but close to, $\dfrac{\pi}{2}$.

x	1.7	1.6	1.59	1.58	$\cdots$	$\dfrac{\pi}{2}\approx 1.571$
$\sec(x)=\dfrac{1}{\cos(x)}$	-8	-34	-52	-109	$\cdots$	undefined

As x approaches $\frac{\pi}{2}$ from the right, the values of $\sec(x)$ approach negative infinity:

$$\text{As } x \to \frac{\pi}{2}^+, \ \sec(x) \to -\infty$$

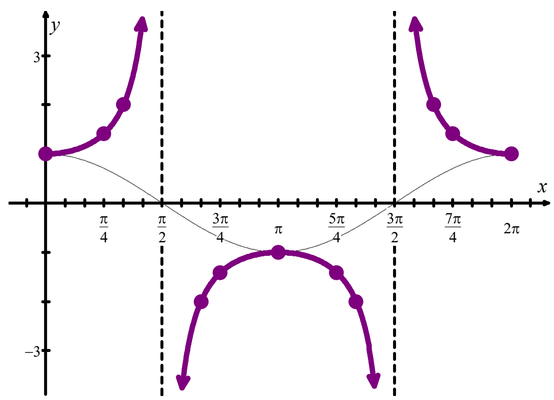
- Using a similar analysis, which we leave to the reader,

$$\text{As } x \to \frac{3\pi}{2}^-, \ \sec(x) \to -\infty$$

$$\text{As } x \to \frac{3\pi}{2}^+, \ \sec(x) \to \infty$$

Plotting points and asymptotes, with smooth curves that echo the behavior noted above, we have the following graph of $y = \sec(x)$ over the interval $[0, 2\pi]$:

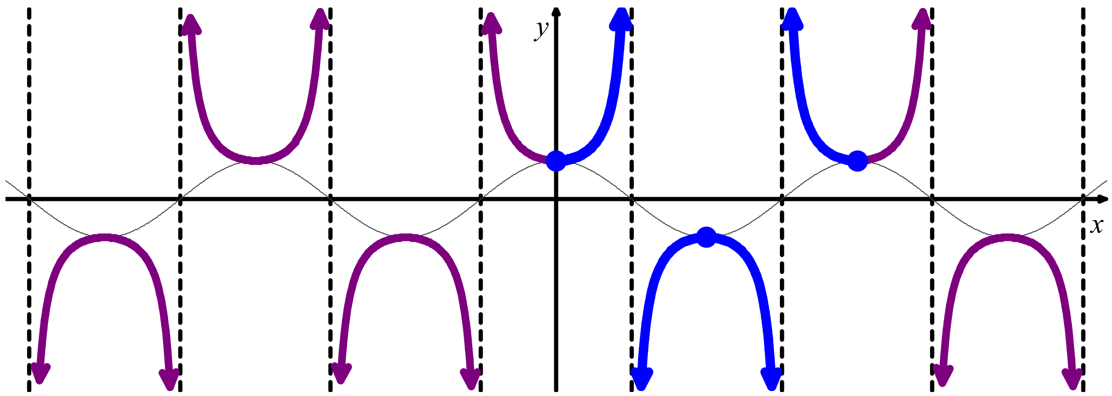
Figure 2.2. 16



The graph of $y = \sec(x)$ over $[0, 2\pi]$

Next, we use 'copy and paste' to extend this graph horizontally. Notice the light 'penciled-in' sketch of $y = \cos(x)$ on both graphs. We will find that the graph of the cosine is a handy tool in sketching the secant. At each x-value for which $\cos(x) = 0$, the graph of $y = \sec(x)$ has a vertical asymptote. The points where $\cos(x) = \pm 1$ are also points where $\sec(x) = \pm 1$. By plotting only the asymptotes and the points where $\sec(x) = \pm 1$, a rough graph of $y = \sec(x)$ can quickly be drawn by sketching the 'U' shapes of the secant function.

Figure 2.2. 17



The graph of $y = \sec(x)$ with fundamental cycle highlighted in blue

Since $\cos(x)$ is periodic with period 2π, it follows that $\sec(x)$ is also periodic with a period of 2π.[3] Due to the close relationship between the cosine and secant, the fundamental cycle of the secant function is the same as that of the cosine function. We previously noted that the domain of the secant function excludes all odd multiples of $\frac{\pi}{2}$. The range of $y = \sec(x)$, as observed graphically, includes all real numbers y such that $y \le -1$ or $y \ge 1$, or equivalently $|y| \ge 1$. By thinking of the secant function as being the reciprocal of the cosine function, a similar result can be obtained algebraically.

Graph of the Cosecant Function

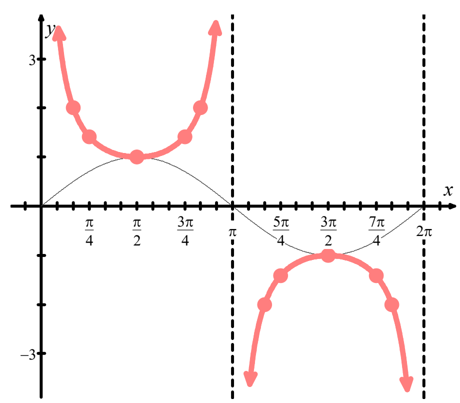
As one would expect, to graph $y = \csc(x)$ we begin with $y = \sin(x)$ and take reciprocals of the corresponding *y*-values. Here, we encounter issues at $x = 0$, $x = \pi$ and $x = 2\pi$. These are locations of vertical asymptotes. Following is a table of values for the cosecant. We leave the analysis of the graph's behavior near its asymptotes to the reader, and proceed with graphing the fundamental cycle of $y = \csc(x)$, followed by an extended graph of $y = \csc(x)$. A light 'penciled-in' sketch of $y = \sin(x)$ is included for reference.

[3] Provided $\sec(\alpha)$ and $\sec(\beta)$ are defined, $\sec(\alpha) = \sec(\beta)$ if and only if $\cos(\alpha) = \cos(\beta)$. Hence, $\sec(x)$ inherits its period from $\cos(x)$.

x	$\sin(x)$	$\csc(x)=\frac{1}{\sin(x)}$	$(x,\csc(x))$
0	0	Not defined	
$\frac{\pi}{6}$	$\frac{1}{2}$	2	$\left(\frac{\pi}{6},2\right)$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\left(\frac{\pi}{4},\sqrt{2}\right)$
$\frac{\pi}{2}$	1	1	$\left(\frac{\pi}{2},1\right)$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\left(\frac{3\pi}{4},\sqrt{2}\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	2	$\left(\frac{5\pi}{6},2\right)$
π	0	Not defined	

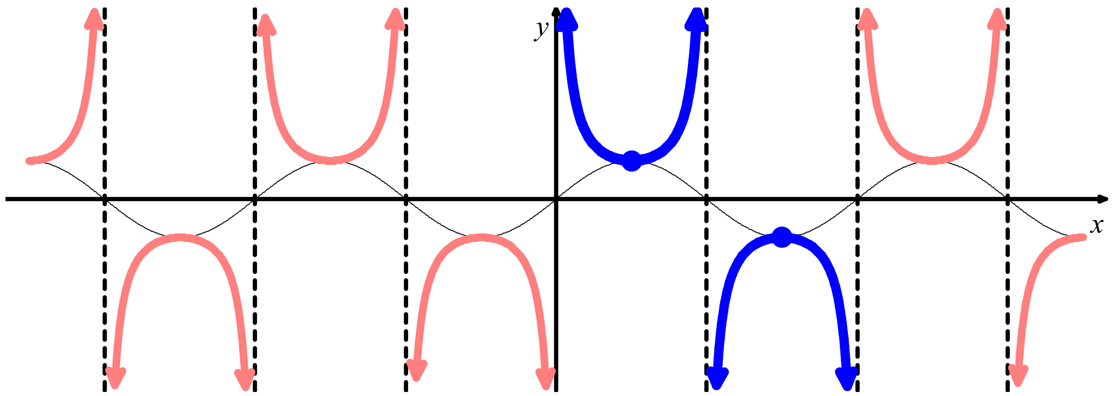
x	$\sin(x)$	$\csc(x)=\frac{1}{\sin(x)}$	$(x,\csc(x))$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	-2	$\left(\frac{7\pi}{6},-2\right)$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4},-\sqrt{2}\right)$
$\frac{3\pi}{2}$	-1	-1	$\left(\frac{3\pi}{2},-1\right)$
$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$\left(\frac{7\pi}{4},-\sqrt{2}\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	-2	$\left(\frac{11\pi}{6},-2\right)$
2π	0	Not defined	

Figure 2.2. 18



The graph of $y=\csc(x)$ over $[0,2\pi]$

Figure 2.2. 19



The graph of $y = \csc(x)$ with fundamental cycle highlighted in blue

Since $y = \sin(x)$ and $y = \cos(x)$ are horizontal shifts of each other, so too are $y = \csc(x)$ and $y = \sec(x)$. As with the tangent and cotangent functions, both $y = \sec(x)$ and $y = \csc(x)$ are continuous and smooth on their domains. Other properties of the secant and cosecant functions are summarized below. Note that all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

Properties of the Secant and Cosecant Functions

The function $y = \sec(x)$

- has domain of all real numbers except $x = \frac{\pi}{2} + \pi k$ for integers k
- has range $(-\infty, -1] \cup [1, \infty)$
- is even
- has period 2π

The function $y = \csc(x)$

- has domain of all real numbers except $x = \pi k$ for integers k
- has range $(-\infty, -1] \cup [1, \infty)$
- is odd
- has period 2π

Graphs of Transformations of the Secant and Cosecant Functions

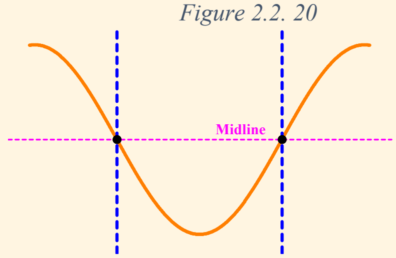
To graph the transformation of a secant or cosecant function, $F(x) = A\sec(\omega x - \phi) + B$ or $G(x) = A\csc(\omega x - \phi) + B$, $\omega > 0$, we use a corresponding cosine or sine function, respectively, as a guide. We may graph the cosine or sine function by plotting points or using quarter marks. Either way, the following describes the technique of using graphs of a cosine or sine function to graph a secant or cosecant function, respectively.

Graphing Secant and Cosecant Functions by Guide Functions

To graph $F(x) = A\sec(\omega x - \phi) + B$, $\omega > 0$, begin by graphing $y = A\cos(\omega x - \phi) + B$ as a guide. To graph $G(x) = A\csc(\omega x - \phi) + B$, $\omega > 0$, begin by graphing $y = A\sin(\omega x - \phi) + B$ as a guide.

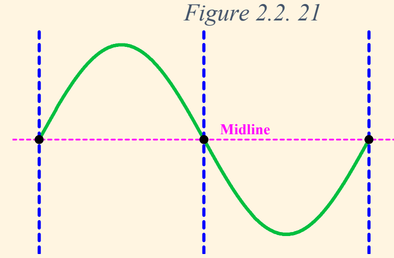
- Sketch vertical asymptotes at points where $\cos(\omega x - \phi)$ or $\sin(\omega x - \phi)$ is zero, or where the graph of the guide function crosses the midline.

Figure 2.2. 20



$y = A\cos(\omega x - \phi) + B$

Figure 2.2. 21



$y = A\sin(\omega x - \phi) + B$

- Plot a point at any location where the guide graph has a maximum or minimum value.

Figure 2.2. 22

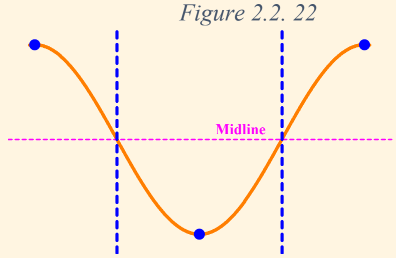
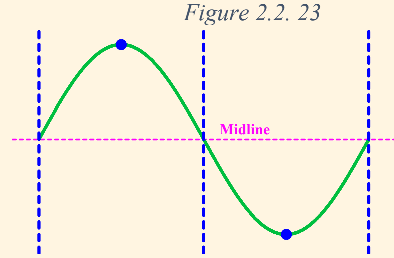


Figure 2.2. 23



- Maximum values of the guide function are low points above the midline on the secant/cosecant curve. Minimum values of the guide function are high points below the midline on the secant/cosecant curve. Passing through the points plotted in the previous step, sketch U-shaped curves that approach the corresponding vertical asymptotes.

Figure 2.2. 24

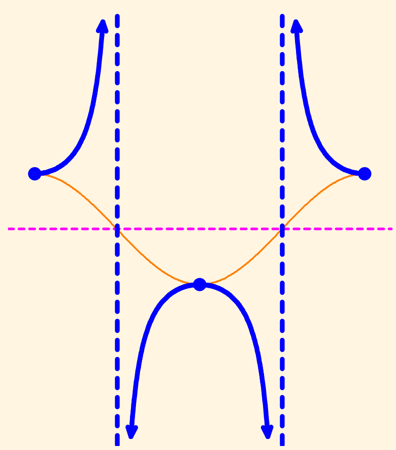
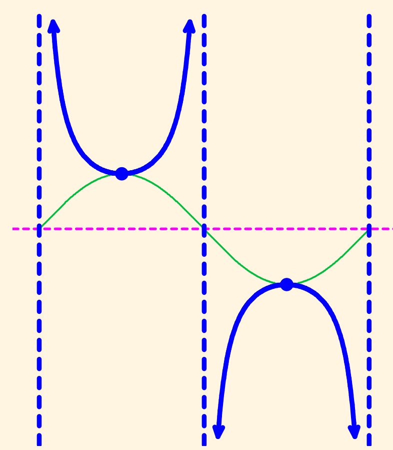


Figure 2.2. 25



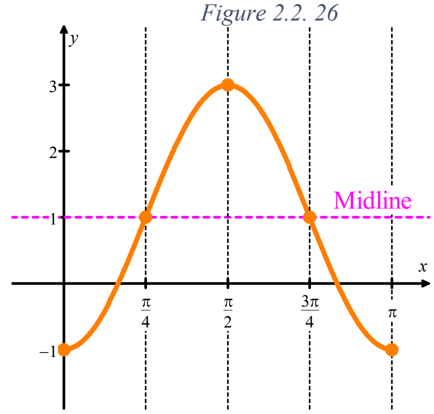
In the next two examples, we use graphs of the cosine and sine to sketch transformations of the secant and cosecant functions, respectively.

Example 2.2.4. Graph one cycle of the function $f(x)=1-2\sec(2x)$.

Solution. Before proceeding, we rewrite $f(x)=1-2\sec(2x)$ in the format $f(x)=-2\sec(2x-0)+1$. This gives us the guide function $y=-2\cos(2x-0)+1$, which we will graph using quarter marks.

The period of $y=-2\cos(2x-0)+1$ is $\frac{2\pi}{2}=\pi$, the amplitude is $|-2|=2$, and the midline is $y=1$. The phase shift is 0, so with the width of each quarter period being $\frac{\pi}{4}$, we plot quarter marks at $x=0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π. Using the model $y=A\cos(\omega x-\phi)+B$, $A<0$, we get the following graph.

Figure 2.2. 26

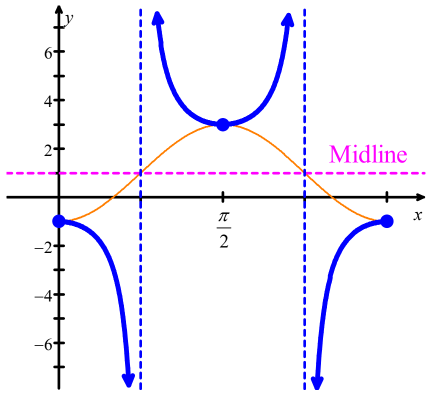


$y=-2\cos(2x-0)+1$

Now, we use this graph of $y=-2\cos(2x-0)+1$ as a guide for sketching $f(x)=-2\sec(2x-0)+1$ by

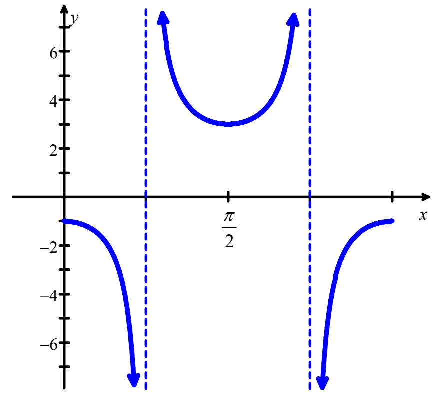
1. Drawing vertical asymptotes at $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$, the x-values at which the guide function intersects the midline. (Note that for these two x-values, $\cos(2x)=0$, resulting in $\sec(2x)$ being undefined.)
2. Plotting points at $(0,-1)$, $\left(\frac{\pi}{2},3\right)$, and $(\pi,-1)$, locations where the guide function has maximum and minimum values.
3. Sketching U-shaped curves through points, approaching asymptotes.

Figure 2.2. 27



$y = f(x) = 1 - 2\sec(2x)$ with guide function $y = -2\cos(2x - 0) + 1$

Figure 2.2. 28



One cycle of $y = f(x) = 1 - 2\sec(2x)$

□

For an explanation of why the above technique works, consider $F(x) = A\sec(\omega x - \phi) + B$ with $A > 0$, noting that $f(x) = \dfrac{A}{\cos(\omega x - \phi)} + B$.

- Since $f(x) = \dfrac{A + B\cos(\omega x - \phi)}{\cos(\omega x - \phi)}$, $f(x)$ is undefined, or has vertical asymptotes, when the denominator $\cos(\omega x - \phi) = 0$. This occurs when $y = A\cos(\omega x - \phi) + B = B$ or when the graph of $y = A\cos(\omega x - \phi) + B$ crosses its midline.
- If the fraction $\dfrac{A}{\cos(\omega x - \phi)}$ is positive, its minimum value occurs when $\cos(\omega x - \phi) = 1$, so the minimum value of $f(x)$, above the midline, is where $y = A\cos(\omega x - \phi) + B = A + B$ attains its maximum value.
- If the fraction $\dfrac{A}{\cos(\omega x - \phi)}$ is negative, its maximum value occurs when $\cos(\omega x - \phi) = -1$, so the maximum value of $f(x)$, below the midline, is where $y = A\cos(\omega x - \phi) + B = -A + B$ attains its minimum value.

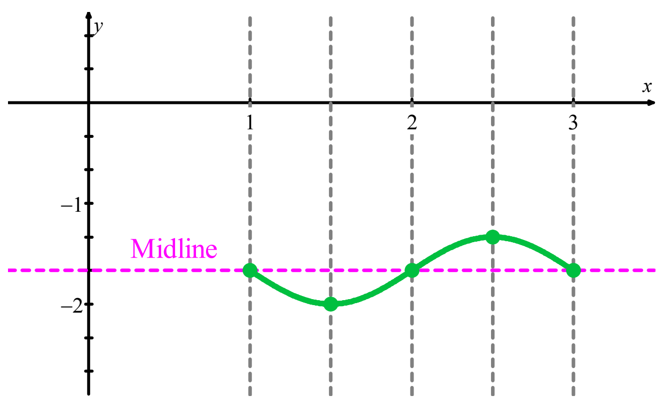
Example 2.2.5. Graph one cycle of the function $g(x) = \dfrac{\csc(\pi - \pi x) - 5}{3}$.

Solution. We begin by rewriting the function g in the format $y == A\csc(\omega x - \phi) + B$.

$$\begin{aligned} g(x) &= \frac{\csc(\pi - \pi x) - 5}{3} \\ &= \frac{1}{3}\csc(\pi - \pi x) - \frac{5}{3} \\ &= \frac{1}{3}\csc(-\pi x + \pi) - \frac{5}{3} \\ &= \frac{1}{3}\csc\left[-(\pi x - \pi)\right] - \frac{5}{3} \\ &= -\frac{1}{3}\csc(\pi x - \pi) - \frac{5}{3} \quad \text{from odd property of cosecant} \end{aligned}$$

We proceed to graph $g(x) = -\frac{1}{3}\csc(\pi x - \pi) - \frac{5}{3}$ by first graphing $y = -\frac{1}{3}\sin(\pi x - \pi) - \frac{5}{3}$. We find the period of $y = -\frac{1}{3}\sin(\pi x - \pi) - \frac{5}{3}$ is $\frac{2\pi}{\pi} = 2$, the amplitude is $\left|-\frac{1}{3}\right| = \frac{1}{3}$, and the midline is $y = -\frac{5}{3}$. Since $\pi x - \pi = \pi(x-1)$, the phase shift is 1. The width of each quarter period is $\frac{2}{4} = \frac{1}{2}$. Quarter marks occur at $x = 1, \frac{3}{2}, 2, \frac{5}{2},$ and 3. Using the model $y = A\sin(\omega x - \phi) + B$, $A < 0$, we get the following graph.

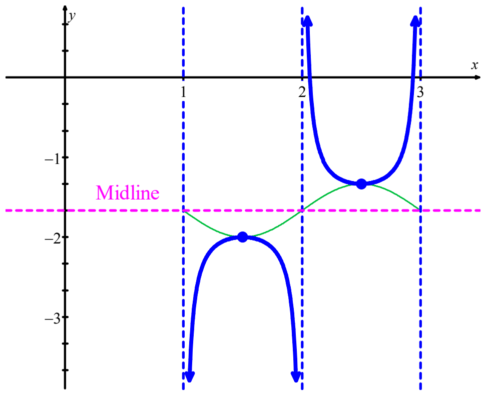
Figure 2.2. 29



$$y = -\frac{1}{3}\sin(\pi x - \pi) - \frac{5}{3}$$

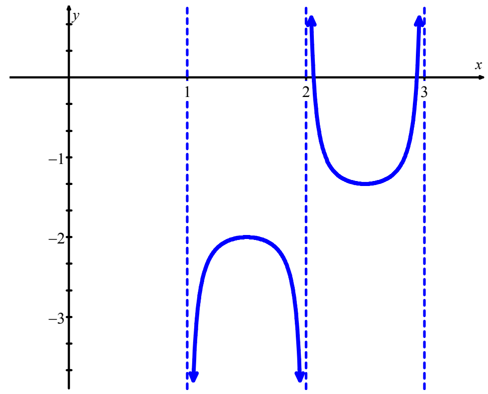
Using the transformed sine graph as a guide in sketching one cycle of $g(x) = -\frac{1}{3}\csc(\pi x - \pi) - \frac{5}{3}$, we plot vertical asymptotes at points of intersection with the midline. We add points at maximum and minimum values. Finally, U-shaped curves pass through points and approach asymptotes.

Figure 2.2. 30



$y = g(x) = \dfrac{\csc(\pi - \pi x) - 5}{3}$ with guide function $y = -\dfrac{1}{3}\sin(\pi x - \pi) - \dfrac{5}{3}$

Figure 2.2. 31



One cycle of $y = g(x) = \dfrac{\csc(\pi - \pi x) - 5}{3}$

□

In real world applications, sine and cosine functions are often used in place of cosecant and secant functions, respectively.

2.2 Exercises

In Exercises 1 – 12, graph one cycle of the given function. State the period of the function.

1. $y=\tan\left(x-\frac{\pi}{3}\right)$
2. $y=2\tan\left(\frac{1}{4}x\right)-3$
3. $y=\frac{1}{3}\tan(-2x-\pi)+1$
4. $y=\cot\left(x+\frac{\pi}{6}\right)$
5. $y=-11\cot\left(\frac{1}{5}x\right)$
6. $y=\frac{1}{3}\cot\left(2x+\frac{3\pi}{2}\right)+1$
7. $y=\sec\left(x-\frac{\pi}{2}\right)$
8. $y=-\csc\left(x+\frac{\pi}{3}\right)$
9. $y=-\frac{1}{3}\sec\left(\frac{1}{2}x+\frac{\pi}{3}\right)$
10. $y=\csc(2x-\pi)$
11. $y=\sec(3x-2\pi)+4$
12. $y=\csc\left(-x-\frac{\pi}{4}\right)-2$

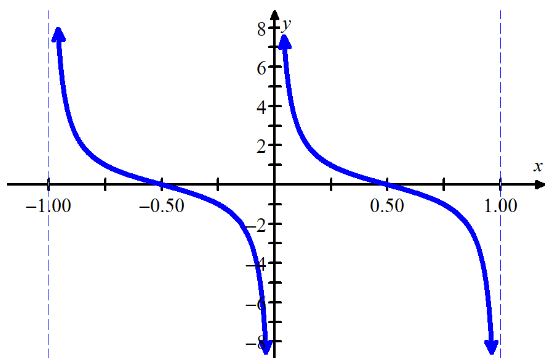
In Exercises 13 – 33, graph two full cycles of each function. State the period and asymptotes. If applicable, describe the horizontal shift and vertical shift of the function.

13. $y=\tan(x)$
14. $y=\cot(x)$
15. $y=2\tan(4x-3\pi)$
16. $y=\tan\left(\frac{\pi}{2}x\right)$
17. $y=\cot\left(x-\frac{\pi}{2}\right)$
18. $y=4\cot(x)$
19. $y=\tan\left(x+\frac{\pi}{4}\right)$
20. $y=\pi\tan(\pi x-\pi)-\pi$
21. $y=-3\cot(2x)$
22. $y=\sec(x)$
23. $y=\csc(x)$
24. $y=2\sec\left(\frac{\pi}{4}(x+1)\right)$
25. $y=6\csc\left(\frac{\pi}{3}x+\pi\right)$
26. $y=2\csc(x)$
27. $y=-\frac{1}{4}\csc(x)$
28. $y=4\sec(3x)$
29. $y=7\sec(5x)$
30. $y=\frac{3}{2}\csc(\pi x)$
31. $y=2\csc\left(x+\frac{\pi}{4}\right)-1$
32. $y=-\sec\left(x-\frac{\pi}{3}\right)-2$
33. $y=\frac{7}{5}\csc\left(x-\frac{\pi}{4}\right)$

In Exercises 34 – 39, find an equation for the graph of each function.

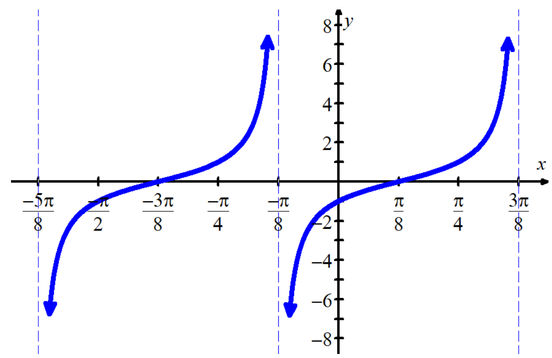
34.

Figure Ex2.2. 1



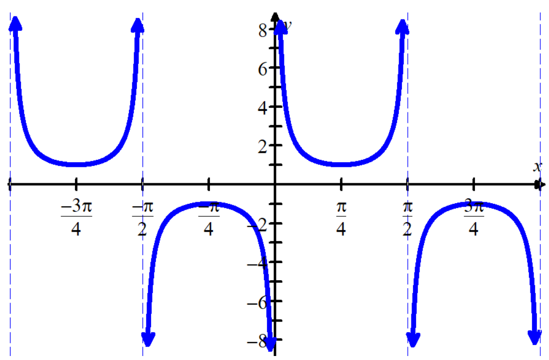
35.

Figure Ex2.2. 2



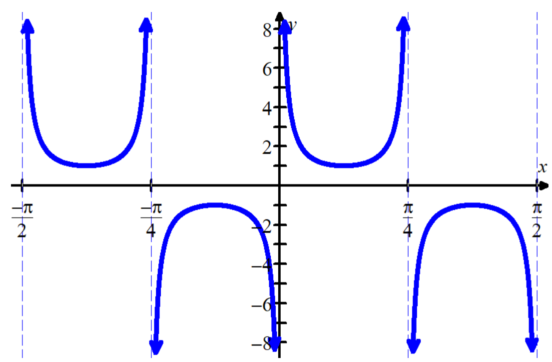
36.

Figure Ex2.2. 3



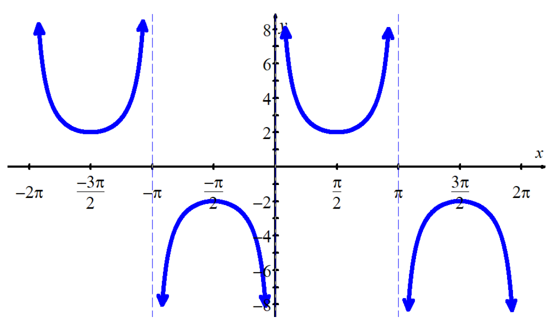
37.

Figure Ex2.2. 4



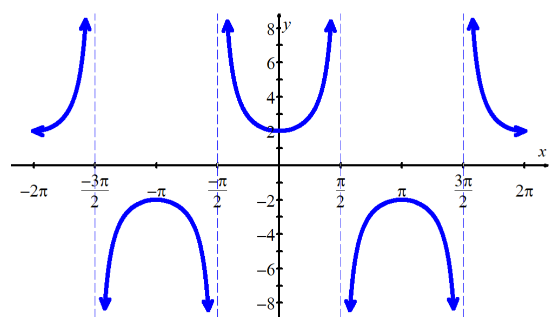
38.

Figure Ex2.2. 5



39.

Figure Ex2.2. 6



40. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the boat and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. The

boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x)=1.5\sec(x)$.

a. What is a reasonable domain for $d(x)$?

b. Graph $d(x)$ on the domain.

c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.

d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the nearest hundredth.

e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the nearest hundredth.

f. What is the minimum distance between the fisherman and the boat? When does this occur?

41. A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x)=250{,}000\csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g(x)$ on the interval $[0,30]$.

b. Evaluate $g(5)$ and interpret the result.

c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?

d. Find and discuss the meaning of any vertical asymptotes.

42. The function $f(x)=20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

a. Graph the function $f(x)$ on the interval $[0,5]$.

b. Find and interpret the vertical stretching factor, the period and any asymptotes.

c. Evaluate $f(1)$ and $f(2.5)$ and discuss the function's values at those inputs.

43. Verify the identity $\tan(x+\pi)=\tan(x)$ by using technology to graph the right and left sides.

44. Graph the function $f(x)=x-\tan(x)$ with the help of technology. Graph $y=x$ on the same set of axes and describe the behavior of f.

2.3 Applications of Radian Measure

Learning Objectives

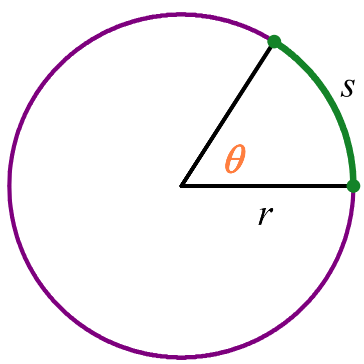
- **Determine arc length.**
- **Determine area of a sector of a circle.**
- **Solve problems involving linear and angular speed.**

This section focuses on applications involving radian measure. We begin with a topic first introduced in **Section 1.1**, the correlation between arc length and radian measure of an angle.

Arc Length

We found in **Section 1.1** that, for a circle with radius r, the radian measure of a central angle is $\theta = \frac{s}{r}$, where s is the length of the arc subtending the angle θ. Solving for s, we have a formula for arc length: $s = r\theta$.

Figure 2.3. 1



Length of a Circular Arc: In a circle of radius r, the length of an arc that subtends a central angle of measure θ radians is $s = r\theta$.

Note that in the above formula, θ is the radian measure of an angle. As in the following example, when an angle is given in degree measure, we must first convert to radians.

Example 2.3.1. Find the length of an arc on a circle of radius 10 units that subtends a central angle of 215°.

Solution. To determine arc length, we must first convert the angle measure to radians.

$$215^\circ = 215\left(\frac{\pi}{180}\right) \text{ radians}$$
$$= \frac{43\pi}{36} \text{ radians}$$

We next use $r = 10$ units and $\theta = \dfrac{43\pi}{36}$ radians to determine the arc length:

$$\text{arc length} = r\theta$$
$$= (10 \text{ units})\left(\frac{43\pi}{36}\right)$$
$$\approx 37.52 \text{ units}$$

□

Example 2.3.2. Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part 1 to determine the radian measure for Mercury's movement in one day.

Solution.

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r$$
$$= 2\pi(36 \text{ million miles})$$
$$= 72\pi \text{ million miles}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the approximate distance traveled in one day:

$$(0.0114)(72\pi \text{ million miles}) \approx 2.58 \text{ million miles}$$

2. We use the arc length, which is the distance traveled, to determine the radian measure θ for Mercury's movement in one day.

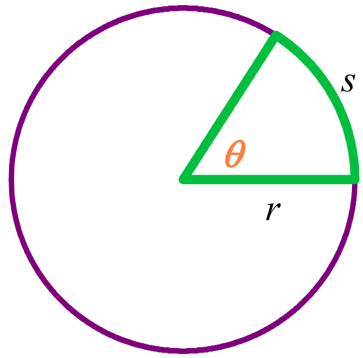
$$\theta = \frac{\text{arc length}}{\text{radius}}$$
$$\approx \frac{2.58 \text{ million miles}}{36 \text{ million miles}}$$
$$\approx 0.072 \text{ radians}$$

□

Area of a Sector

We next determine the **area of a sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc.

Figure 2.3. 2



Consider the ratio of the area of the sector to the area of the circle. This ratio is equivalent to the ratio of the measure of the central angle of the sector (θ radians) to the measure of the central angle of the circle (2π radians). For sector area A and central angle θ, we have the following:

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{\theta}{2\pi}\left(\pi r^2\right)$$

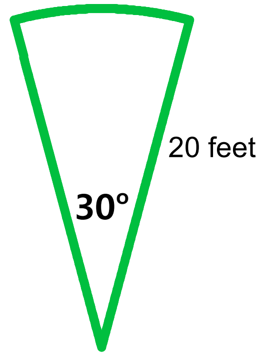
$$A = \frac{\theta r^2}{2}$$

> **Area of a Sector of a Circle:** In a circle with radius r, the area of a sector having central angle of measure θ radians is $A = \frac{1}{2}\theta r^2$.

Be careful! As in the arc length formula, θ must be in radian measure when calculating the area of a sector of a circle.

Example 2.3.3. An automatic sprinkler sprays a distance of 20 feet while rotating 30 degrees. What is the area of the sector of grass the sprinkler waters?

Figure 2.3. 3



Solution. We begin by converting the angle into radians.

$$30^\circ = 30\left(\frac{\pi}{180}\right) \text{ radians}$$
$$= \frac{\pi}{6} \text{ radians}$$

The area of the sector can then be determined using $\theta = \frac{\pi}{6}$ radians and $r = 20$ feet.

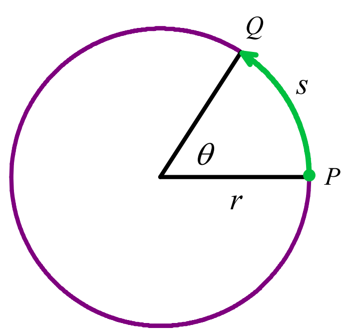
$$\text{Area} = \frac{1}{2}\theta r^2$$
$$= \frac{1}{2}\left(\frac{\pi}{6}\right)(20 \text{ feet})^2$$
$$\approx 104.72 \text{ ft}^2$$

□

Linear and Angular Speed

We end **Chapter 2** with applications involving circular motion. Suppose an object is moving as pictured below, along a circular path of radius r from the point P to the point Q in an amount of time t.

Figure 2.3. 4



In the previous figure, s represents the distance traveled. The **speed** of this object,[4] denoted v, is the ratio of distance traveled to time traveled, or $v = \dfrac{\text{distance}}{\text{time}}$. With t representing time, we have

$$\begin{aligned} v &= \frac{s}{t} \\ &= \frac{r\theta}{t} \quad \textbf{since } s = r\theta \\ &= r \cdot \frac{\theta}{t} \end{aligned}$$

The quantity $\dfrac{\theta}{t}$ is called the **angular speed** of the object.[5] It is denoted by ω, read as 'omega'. The quantity ω is the rate of change of the angle θ with respect to time and thus has units $\dfrac{\text{radians}}{\text{time}}$.

In the case of an object traveling along a circular arc, if the path of that object is uncurled to form a line segment, then the speed of the object on the line segment would be the same as the speed on the circle. For this reason, the quantity v is often called the **linear speed** of the object in order to distinguish it from the angular speed, ω.

Putting together the ideas of the previous paragraphs, we get the following:

Speed for Circular Motion

For an object moving on a circular path of radius r in time t,

- The **linear speed** of the object is $v = \dfrac{s}{t}$, where s is the arc length along the object's path.
- The **angular speed** of the object is $\omega = \dfrac{\theta}{t}$, where θ is the radian measure of the central angle passed through by the object.
- The **linear speed** may also be defined as $v = r\omega$, where ω is the constant angular speed of the object.

It is worth commenting on the units here. The units of v are $\dfrac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\dfrac{\text{radians}}{\text{time}}$. Thus, the left side of the equation $v = r\omega$ has units $\dfrac{\text{length}}{\text{time}}$, and the right

[4] We are assuming here that speed is constant, so use the designation 'speed' instead of 'average speed'.
[5] We are assuming here that angular speed is constant, so use the designation 'angular speed' instead of 'average angular speed'.

side has units $\text{length}\cdot\frac{\text{radians}}{\text{time}}=\frac{\text{length}\cdot\text{radians}}{\text{time}}$. Since radian measure is just a number, $\frac{\text{length}\cdot\text{radians}}{\text{time}}$ is a multiple of $\frac{\text{length}}{\text{time}}$, and thus the units are consistent.

Example 2.3.4. Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle that completes one revolution in (approximately) 24 hours. The path traced out by a point during this 24-hour period is the latitude of the point. Salt Lake Community College is 40.7608° North Latitude and the radius of the circle of revolution at this latitude is approximately 2999 miles. Find the linear speed, in miles per hour, of Salt Lake Community College as the world turns.

Solution. To use the formula $v = r\omega$, we first need to compute the angular speed ω. The Earth makes one revolution in 24 hours, and one revolution is 2π radians, so

$$\begin{aligned}\omega &= \frac{\theta}{t}\\ &= \frac{2\pi \text{ radians}}{24 \text{ hours}}\\ &= \frac{\pi}{12} \text{ radians/hour}\end{aligned}$$

The linear speed is

$$\begin{aligned}v &= r\omega\\ &= (2999)\left(\frac{\pi}{12}\right)\\ &\approx 785 \text{ miles/hour}\end{aligned}$$

□

Example 2.3.5. An old vinyl record is played on a turntable rotating at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution. We find angular speed by dividing the total angular rotation by the time. We then convert to radians per second.

$$\begin{aligned}\text{angular speed} &= \frac{45 \text{ rotations}}{1 \text{ minute}}\\ &= \frac{45 \text{ rotations}}{1 \text{ minute}}\cdot\frac{2\pi \text{ radians}}{1 \text{ rotation}}\cdot\frac{1 \text{ minute}}{60 \text{ seconds}}\\ &= \frac{3\pi}{2} \text{ radians/second}\end{aligned}$$

□

Example 2.3.6. A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution. Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road. The equation $v = r\omega$ allows us to find linear speed given angular speed. We begin by converting angular speed from revolutions (rotations) per minute to radians per minute.

$$\frac{180\text{ rotations}}{1\text{ minute}} \cdot \frac{2\pi\text{ radians}}{1\text{ rotation}} = 360\pi\ \frac{\text{radians}}{\text{minute}}$$

The diameter of the wheels is 28 inches, for a radius of 14 inches, and we have the following:

$$\begin{aligned} v &= r\omega \\ &= (14)(360\pi) \\ &= 5040\pi\ \frac{\text{inches}}{\text{minute}} \end{aligned}$$

Since radians are a dimensionless measure, it is not necessary to include them. Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour:

$$v = 5040\pi\ \frac{\text{inches}}{\text{minute}} \cdot \frac{1\text{ foot}}{12\text{ inches}} \cdot \frac{1\text{ mile}}{5280\text{ feet}} \cdot \frac{60\text{ minutes}}{1\text{ hour}} \approx 14.99\text{ miles per hour}$$

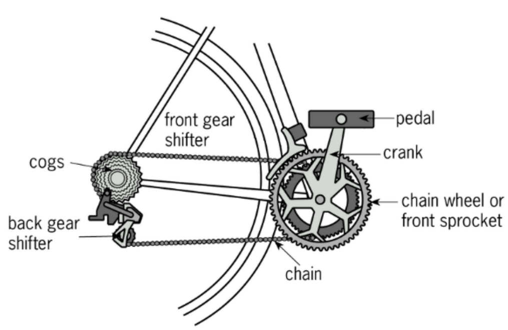
Thus, the speed of the bicycle is approximately 15 miles per hour.

□

Example 2.3.7. A bicycle with 27-inch diameter wheels has its gears set so that the chain is attached to a 4-inch diameter front sprocket and a 2-inch diameter back cog. If the cyclist is peddling at one revolution per second, how fast is the bicycle moving in miles per hour?

Solution. Here, the two gears are joined by a chain and the key is to realize that every point on the chain is moving at the same rate.

Figure 2.3. 5



Each turn of the front sprocket will turn the back cog $\frac{4}{2}=2$ times. Thus, each rotation of the pedal will result in two revolutions of the wheel. With the cyclist peddling at one revolution per second, and the circumference of the wheel being $2\pi r = 27\pi$ inches, the bicycle travels $2(27\pi)=54\pi$ inches per second. We convert to miles per hour as follows:

$$v = 54\pi\frac{\text{inches}}{\text{second}}\cdot\frac{1\text{ foot}}{12\text{ inches}}\cdot\frac{1\text{ mile}}{5280\text{ feet}}\cdot\frac{60\text{ seconds}}{1\text{ minute}}\cdot\frac{60\text{ minutes}}{1\text{ hour}}$$

After simplifying, we find that the bicycle is moving at approximately 9.6 miles per hour.

□

2.3 Exercises

In Exercises 1 – 6, round your answers to two decimal places.

1. Find the length of an arc on a circle of radius 12 inches that subtends a central angle of $\frac{\pi}{4}$ radians.

2. Find the length of an arc on a circle of radius 5.02 miles that subtends a central angle of $\frac{\pi}{3}$.

3. Find the length of an arc on a circle of diameter 14 meters that subtends a central angle of $\frac{5\pi}{6}$.

4. Find the length of an arc on a circle of radius 10 centimeters that subtends a central angle of 50°.

5. Find the length of an arc on a circle of radius 5 inches that subtends a central angle of 220°.

6. Find the length of an arc on a circle of diameter 12 meters that subtends a central angle of 63°.

In Exercises 7 – 12, compute the areas of the circular sectors with the given central angles and radii. Round your answers to two decimal places.

7. $\theta = \frac{\pi}{6}$, $r = 12$

8. $\theta = \frac{5\pi}{4}$, $r = 100$

9. $\theta = 330^\circ$, $r = 9.3$

10. $\theta = \pi$, $r = 1$

11. $\theta = 240^\circ$, $r = 5$

12. $\theta = 1^\circ$, $r = 117$

13. A yo-yo which is 2.25 inches in diameter spins at a rate of 4500 revolutions per minute. How fast is the edge of the yo-yo spinning in miles per hour? Round your answer to two decimal places.

14. How many revolutions per minute would the yo-yo in **Exercise 13** have to complete if the edge of the yo-yo is to be spinning at a rate of 42 miles per hour? Round your answer to two decimal places.

15. In the yo-yo trick ‘Around the World’, the performer throws the yo-yo so that it sweeps out a vertical circle whose radius is the yo-yo string. If the yo-yo string is 28 inches long and the yo-yo takes 3 seconds to complete one revolution of the circle, compute the speed of the yo-yo in miles per hour. Round your answer to two decimal places.

16. A computer hard drive contains a circular disk with diameter 2.5 inches and spins at a rate of 7200 RPM (revolutions per minute). Find the linear speed of a point on the edge of the disk in miles per hour. Round your answer to two decimal places.

17. The Giant Wheel at Cedar Point Amusement Park is a circle with diameter 128 feet. It sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming the riders are at the edge of the circle, how fast are they traveling in miles per hour? Round your answer to two decimal places.

18. A truck with 32-inch diameter wheels is traveling at 60 miles per hour. Find the angular speed of the wheels in radians per minute. How many revolutions per minute do the wheels make? Round your answer to two decimal places.

19. A CD has diameter 120 millimeters. When playing audio, the angular speed varies to keep linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed. Round your answer to two decimal places.

20. Imagine a rope tied around the Earth at the equator. Show that you need to add only 2π feet of length to the rope in order to lift it one foot above the ground around the entire equator. (You do NOT need to know the radius of the Earth to show this.)

21. A bicycle with 19-inch diameter wheels has its gears set so that the chain is attached to a 6-inch radius front sprocket and a 2-inch radius back cog. The cyclist pedals at 180 revolutions per minute. How fast is the bicycle moving in miles per hour? Round your answer to two decimal places.

CHAPTER 3
TRIGONOMETRIC IDENTITIES

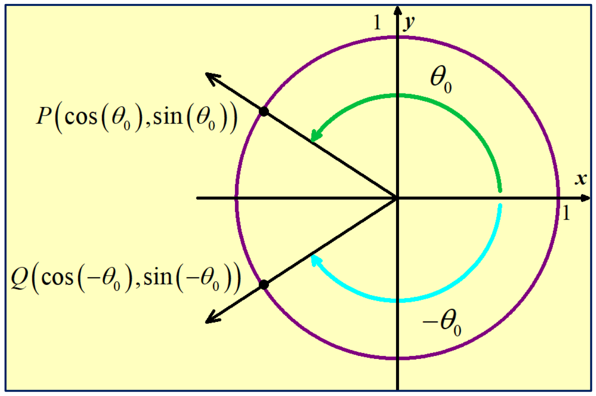


Figure 3.0. 1

Chapter Outline

Introduction

In Chapter 3, we return to the exciting world of identities. Recall that identities are equations relating variables that are true for any valid input in the relationship. We experienced several identities related to the Pythagorean Theorem in Chapter 1. Here, we will be identifying and exploring many new identities through the use of symmetries and patterns we viewed in the graphs of functions from Chapter 2.

Hopefully, these will not come across as just a laundry-list of formulas to memorize. These identities tend to be easier to understand, recall, and manipulate if you understand why they work, rather than just accepting them as facts. Pay careful attention to what the identities mean.

Section 3.1 covers the even/odd identities related to a trigonometric function's symmetry. As well, it addresses the sum and difference identities, which connect sine, cosine, and tangent with multivariate inputs. Section 3.2 explores the double- and half-angle identities that have application in simplifying trigonometric expressions and finding exact values of trigonometric functions of non-standard angles.

Throughout Chapter 3, attention will be paid to finding exact values of trigonometric functions, writing trigonometric expressions in varying formats, and verifying trigonometric identities. These skills will be important in the future study of Calculus.

3.1 Using Trigonometric Identities

Learning Objectives

- **State the even/odd identities.**
- **Use even/odd identities in simplifying trigonometric expressions.**
- **Use even/odd identities in verifying trigonometric identities.**
- **State the sum and difference identities for sine, cosine, and tangent.**
- **Use sum and difference identities to find values of trigonometric functions.**
- **Use sum and difference identities in verifying trigonometric identities.**
- **State and apply the co-function identities.**

In **Section 1.5**, we saw the utility of Pythagorean identities, along with the quotient and reciprocal identities. Not only did these identities help us compute values of trigonometric functions, they were also useful in simplifying expressions. In this section, we formally introduce the even/odd identities[1], while recalling their graphical significance from **Chapter 2**. After establishing the even/odd identities, we move on to sum and difference identities and co-function identities, further increasing our ability to find trigonometric function values and verify trigonometric identities.

The Even/Odd Identities

Theorem 3.1. The Even/Odd Identities: For all angles θ for which the following are defined,

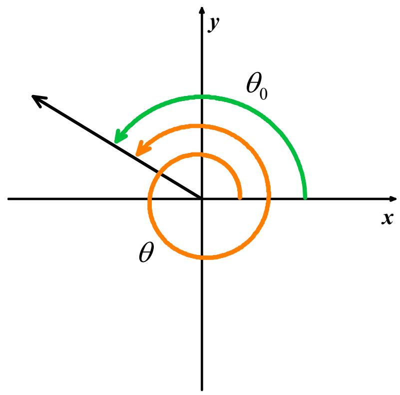
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

In light of the quotient and reciprocal identities, it suffices to show that $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$. The remaining four trigonometric functions can be expressed in terms of $\sin(\theta)$ and $\cos(\theta)$, so the proofs of their even/odd identities are left as exercises.

[1] As mentioned at the end of **Section 1.4**, properties of the trigonometric functions, when thought of as functions of angles in radian measure, hold equally well if we view these functions as functions of real numbers. Not surprisingly, the even/odd properties of the trigonometric functions are so named because they identify cosine and secant as even functions, while the remaining four trigonometric functions are odd.

Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ such that $0 \le \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive x-axis to the terminal side of θ as shown in the following illustration.) Since θ and θ_0 are coterminal, $\sin(\theta) = \sin(\theta_0)$ and $\cos(\theta) = \cos(\theta_0)$.

Figure 3.1. 1



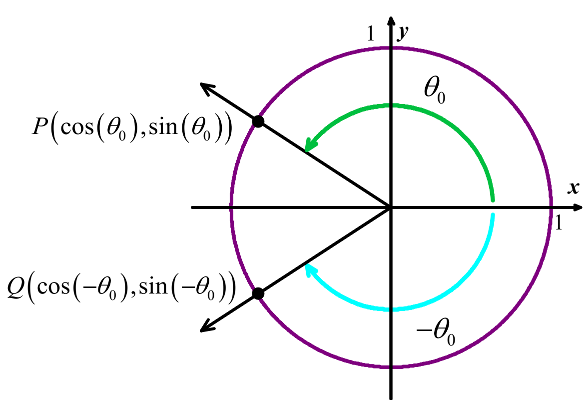
We now consider the angles $-\theta$ and $-\theta_0$. Since θ is coterminal with θ_0, there is some integer k for which $\theta = \theta_0 + 2\pi k$. It follows that

$$\begin{aligned} -\theta &= -\theta_0 - 2\pi k \\ &= -\theta_0 + 2\pi(-k) \end{aligned}$$

Since k is an integer, so is $-k$, which means $-\theta$ is coterminal with $-\theta_0$. Hence, $\sin(-\theta) = \sin(-\theta_0)$ and $\cos(-\theta) = \cos(-\theta_0)$.

We let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, that lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$.

Figure 3.1. 2



Since θ_0 and $-\theta_0$ sweep out congruent sectors of the Unit Circle, it follows that the points P and Q are symmetric about the *x*-axis. Thus, $\sin(-\theta_0) = -\sin(\theta_0)$ and $\cos(-\theta_0) = \cos(\theta_0)$. Since the sines and cosines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, we get $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$, as required.

Using Even/Odd Identities in Simplifying Expressions

The even/odd identities are readily demonstrated using any of the standard angles noted in **Section 1.3**. Their true utility, however, lies not in computation but in simplifying expressions involving the trigonometric functions.

Example 3.1.1. Use identities to fully simplify[2] the expression $(1+\sin(x))(1+\sin(-x))$.

Solution. We begin with the odd identity $\sin(-\theta) = -\sin(\theta)$, substituting x for θ.

$$\begin{aligned}(1+\sin(x))(1+\sin(-x)) &= (1+\sin(x))(1-\sin(x)) \\ &= 1-\sin^2(x) && \text{difference of squares} \\ &= \cos^2(x) && \text{Pythagorean identity}\end{aligned}$$

□

Using Even/Odd Identities in Verifying Trigonometric Identities

Looking back at **Section 1.5**, where we began verifying identities, we can now add the even/odd identities to the Pythagorean, quotient, and reciprocal identities as tools in verifying other trigonometric identities.

Example 3.1.2. Verify the identity $\dfrac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos(\theta)-\sin(\theta)$.

Solution. We begin with the left, more complicated, side.

[2] 'Fully simplify' is sometimes subject to interpretation.

$$\begin{aligned}\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}&=\frac{(\sin(-\theta))^2-(\cos(-\theta))^2}{\sin(-\theta)-\cos(-\theta)}\\&=\frac{(-\sin(\theta))^2-(\cos(\theta))^2}{-\sin(\theta)-\cos(\theta)} &&\text{even / odd identities}\\&=\frac{(\sin(\theta))^2-(\cos(\theta))^2}{-\sin(\theta)-\cos(\theta)}\\&=\frac{(\sin(\theta)-\cos(\theta))(\sin(\theta)+\cos(\theta))}{-(\sin(\theta)+\cos(\theta))} &&\text{difference of squares}\\&=\frac{\sin(\theta)-\cos(\theta)}{-1}\\&=\cos(\theta)-\sin(\theta)\end{aligned}$$ □

The Sum and Difference Identities for Cosine

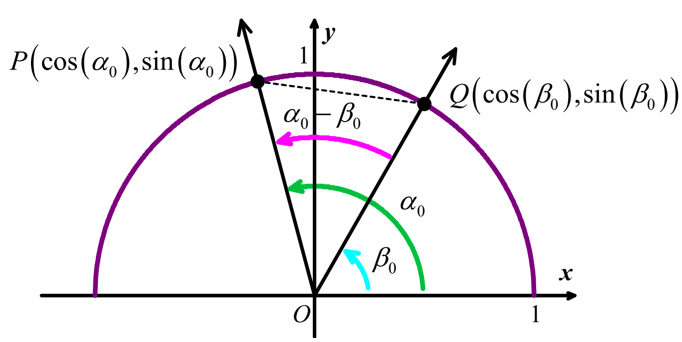
We begin with a theorem introducing the sum and difference identities for cosine, followed by a proof of that theorem. Then, after the introduction of the co-function identities, sum and difference identities for sine and tangent will logically follow.

Theorem 3.2. Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$

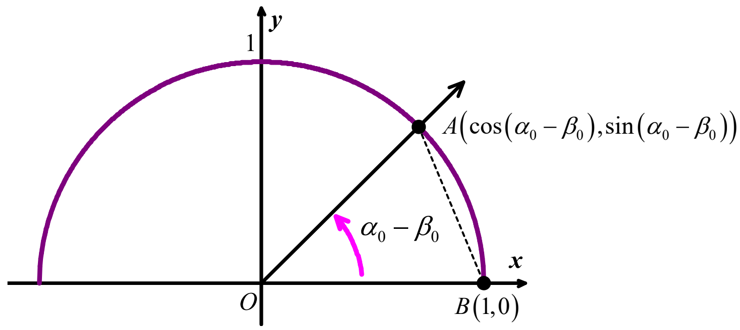
We first prove the result for differences. As in the proof of even/odd identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, with $0\le\alpha_0<2\pi$ and $0\le\beta_0<2\pi$. Since α and α_0 are coterminal, as are β and β_0, it follows that $\alpha-\beta$ is coterminal with $\alpha_0-\beta_0$. Consider the following case where $\alpha_0\ge\beta_0$:

Figure 3.1. 3



Since the angle POQ (above) is congruent to the angle AOB (below), the distance between P and Q is equal to the distance between A and B.[3]

Figure 3.1. 4



With distance QP equal to distance BA, we use the distance formula to find

$$\sqrt{[\cos(\alpha_0)-\cos(\beta_0)]^2+[\sin(\alpha_0)-\sin(\beta_0)]^2}=\sqrt{[\cos(\alpha_0-\beta_0)-1]^2+[\sin(\alpha_0-\beta_0)-0]^2}$$

Or, after squaring both sides,

$$[\cos(\alpha_0)-\cos(\beta_0)]^2+[\sin(\alpha_0)-\sin(\beta_0)]^2=[\cos(\alpha_0-\beta_0)-1]^2+[\sin(\alpha_0-\beta_0)-0]^2$$

We expand the left side and apply a Pythagorean identity to get

$$\begin{aligned}&[\cos(\alpha_0)-\cos(\beta_0)]^2+[\sin(\alpha_0)-\sin(\beta_0)]^2\\&=\cos^2(\alpha_0)-2\cos(\alpha_0)\cos(\beta_0)+\cos^2(\beta_0)+\sin^2(\alpha_0)-2\sin(\alpha_0)\sin(\beta_0)+\sin^2(\beta_0)\\&=[\sin^2(\alpha_0)+\cos^2(\alpha_0)]+[\sin^2(\beta_0)+\cos^2(\beta_0)]-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\\&=2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

Turning our attention to the right side, and again applying a Pythagorean identity, we have

$$\begin{aligned}[\cos(\alpha_0-\beta_0)-1]^2+[\sin(\alpha_0-\beta_0)-0]^2&=\cos^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)+1+\sin^2(\alpha_0-\beta_0)\\&=[\sin^2(\alpha_0-\beta_0)+\cos^2(\alpha_0-\beta_0)]+1-2\cos(\alpha_0-\beta_0)\\&=2-2\cos(\alpha_0-\beta_0)\end{aligned}$$

We put both sides together, $2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)=2-2\cos(\alpha_0-\beta_0)$, then simplify as follows:

$$\begin{aligned}2-2\cos(\alpha_0-\beta_0)&=2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) && \text{swap sides}\\-2\cos(\alpha_0-\beta_0)&=-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) && \text{subtract 2 from each side}\\\cos(\alpha_0-\beta_0)&=\cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0) && \text{divide through by } -2\end{aligned}$$

[3] In the illustrations, the triangles POQ and AOB are congruent. However, $\alpha_0-\beta_0$ could be 0 or it could be π, neither of which makes a triangle. Or, $\alpha_0-\beta_0$ could be larger than π, which makes a triangle, just not the one we've drawn. You should think about these three cases.

Since α and α_0, β and β_0, $\alpha-\beta$ and $\alpha_0-\beta_0$, are all coterminal pairs of angles, we have

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$$

This verifies the difference identity for cosine when $\alpha_0 \geq \beta_0$. In the case where $\alpha_0 \leq \beta_0$, we can apply the above argument to the angle $\beta_0-\alpha_0$ to obtain the identity

$$\cos(\beta_0-\alpha_0)=\cos(\beta_0)\cos(\alpha_0)+\sin(\beta_0)\sin(\alpha_0)$$

Applying the even identity of cosine, we get

$$\begin{aligned}\cos(\beta_0-\alpha_0)&=\cos(-(-\beta_0+\alpha_0))\\&=\cos(\alpha_0-\beta_0)\end{aligned}$$

It follows that $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$.

To verify the sum identity for cosine, we use the difference identity, along with even/odd identities:

$$\begin{aligned}\cos(\alpha+\beta)&=\cos(\alpha-(-\beta))\\&=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta) && \text{difference identity for cosine}\\&=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(-\beta) && \text{even identity of cosine}\\&=\cos(\alpha)\cos(\beta)+\sin(\alpha)(-\sin(\beta)) && \text{odd identity of sine}\\&=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\end{aligned}$$

We put these newfound identities to good use in the following examples.

Example 3.1.3. Find the exact value of $\cos(15^\circ)$.

Solution. In order to use a sum or difference identity to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles for which we know the values of their sines and cosines. One way to do this is to write $15^\circ = 45^\circ - 30^\circ$. Then we use the difference identity for cosine to get

$$\begin{aligned}\cos(15^\circ)&=\cos(45^\circ-30^\circ)\\&=\cos(45^\circ)\cos(30^\circ)+\sin(45^\circ)\sin(30^\circ)\\&=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\&=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\&=\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

□

The Co-function Identities

We begin with an example.

Example 3.1.4. Verify the identity $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$, where θ is any angle in radians.

Solution. This is a straightforward application of the difference identity for cosine.

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta)\\ &= (0)\cos(\theta)+(1)\sin(\theta)\\ &= \sin(\theta)\end{aligned}$$

□

The identity verified in **Example 3.1.4**, namely $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$, is the first of the co-function identities. Next, by replacing θ with $\left(\frac{\pi}{2}-\theta\right)$ in $\sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right)$, we get

$$\begin{aligned}\sin\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right)\\ &= \cos(\theta)\end{aligned}$$

This says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for sine and cosine, the remaining trigonometric functions follow suit. Their proofs are left as exercises.

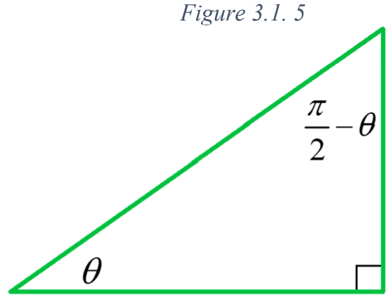
Theorem 3.3. Co-function Identities: For all angles θ, measured in radians,[4] for which the following are defined,

- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc(\theta)$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$
- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec(\theta)$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$

Note that in the case of an acute angle θ, the angles θ and $\frac{\pi}{2}-\theta$ are the two acute angles in a right triangle, as demonstrated below.

[4] If θ is measured in degrees, replace $\frac{\pi}{2}$ with 90 degrees in each identity.

Figure 3.1. 5



Example 3.1.5. Use the result from **Example 3.1.3**, $\cos\left(15^\circ\right)=\dfrac{\sqrt{6}+\sqrt{2}}{4}$, to find the exact value of $\sin\left(75^\circ\right)$.

Solution. From the co-function identities, we have $\sin\left(90^\circ-\theta\right)=\cos(\theta)$, where θ is in degrees.

With $\sin\left(75^\circ\right)=\sin\left(90^\circ-15^\circ\right)$, it follows that $\sin\left(75^\circ\right)=\cos\left(15^\circ\right)$. Then, since $\cos\left(15^\circ\right)=\dfrac{\sqrt{6}+\sqrt{2}}{4}$, we have $\sin\left(75^\circ\right)=\dfrac{\sqrt{6}+\sqrt{2}}{4}$.

□

With the co-function identities in place, we are now in the position to derive the sum and difference identities for sine.

The Sum and Difference Identities for Sine

We begin with the sum identity.

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) && \text{co-function identity}\\
&= \cos\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right)\\
&= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta) && \text{difference identity for cosine}\\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) && \text{co-function identities}
\end{aligned}$$

We can derive the difference identity for sine by rewriting $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$ and using the sum identity and the even/odd identities. Again, we leave the details to the reader.

Theorem 3.4. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

Example 3.1.6. Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$.

Solution. As in **Example 3.1.3**, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of standard angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is

$$\begin{aligned}\frac{19\pi}{12}&=\frac{16\pi}{12}+\frac{3\pi}{12}\\&=\frac{4\pi}{3}+\frac{\pi}{4}\end{aligned}$$

Proceeding, we have

$$\begin{aligned}\sin\left(\frac{19\pi}{12}\right)&=\sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\&=\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \quad \text{sum identity for sine}\\&=\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\&=-\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\&=\frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

□

Example 3.1.7. If α is a Quadrant II angle with $\sin(\alpha)=\frac{5}{13}$ and β is a Quadrant III angle with $\tan(\beta)=2$, find $\sin(\alpha-\beta)$.

Solution. We are given $\sin(\alpha)=\frac{5}{13}$, but to use the difference identity for sine we will also need to find $\cos(\alpha)$, $\sin(\beta)$, and $\cos(\beta)$.[5]

- We use the Pythagorean identity $\sin^2(\alpha)+\cos^2(\alpha)=1$, along with $\sin(\alpha)=\frac{5}{13}$, to find $\cos(\alpha)$:

---

[5] In this solution, we use Pythagorean identities to determine cos(α), sin(β) and cos(β). These values may also be found graphically, as in **Example 1.6.3**.

$$\left(\frac{5}{13}\right)^2+\cos^2(\alpha)=1$$
$$\cos^2(\alpha)=\frac{144}{169}$$
$$\cos(\alpha)=\pm\frac{12}{13}$$
$$\cos(\alpha)=-\frac{12}{13} \quad \text{since } \alpha \text{ is a Quadrant II angle}$$

- We next use a different Pythagorean identity, $\tan^2(\beta)+1=\sec^2(\beta)$, along with $\tan(\beta)=2$, to find $\cos(\beta)$:

$$(2)^2+1=\sec^2(\beta)$$
$$\sec(\beta)=\pm\sqrt{5}$$
$$\sec(\beta)=-\sqrt{5} \qquad \text{since } \beta \text{ is a Quadrant III angle}$$
$$\frac{1}{\cos(\beta)}=-\sqrt{5} \qquad \text{reciprocal identity for secant}$$
$$\cos(\beta)=-\frac{1}{\sqrt{5}}$$

- Having $\cos(\beta)=-\frac{1}{\sqrt{5}}$ and $\tan(\beta)=2$, we use the quotient identity for tangent, $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, to determine $\sin(\beta)$:

$$\sin(\beta)=\tan(\beta)\cos(\beta) \quad \text{from quotient identity for tangent}$$
$$\sin(\beta)=(2)\left(-\frac{1}{\sqrt{5}}\right)$$
$$\sin(\beta)=-\frac{2}{\sqrt{5}}$$

Now that we have the necessary values, we proceed to determine the value of $\sin(\alpha-\beta)$:

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) \quad \text{difference identity for sine}$$
$$=\left(\frac{5}{13}\right)\left(-\frac{1}{\sqrt{5}}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2}{\sqrt{5}}\right)$$
$$=-\frac{29}{13\sqrt{5}}$$
$$=-\frac{29\sqrt{5}}{65}$$

□

The Sum and Difference Identities for Tangent

In the next example, we use the sum identities for sine and cosine to determine a sum identity for tangent.

Example 3.1.8. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution. We start with the quotient identity for tangent, along with sum identities for sine and cosine.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\end{aligned}$$

Our next goal is to get our terms in a format that we can rewrite as tangents, i.e., $\frac{\sin(\alpha)}{\cos(\alpha)}=\tan(\alpha)$ and $\frac{\sin(\beta)}{\cos(\beta)}=\tan(\beta)$. With this goal in mind, we multiply the numerator and denominator by $\frac{1}{\cos(\alpha)\cos(\beta)}$:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\cdot\frac{\frac{1}{\cos(\alpha)\cos(\beta)}}{\frac{1}{\cos(\alpha)\cos(\beta)}} = \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}$$

After simplifying, we have

$$\tan(\alpha+\beta) = \frac{\frac{\sin(\alpha)}{\cos(\alpha)}+\frac{\sin(\beta)}{\cos(\beta)}}{1-\frac{\sin(\alpha)}{\cos(\alpha)}\cdot\frac{\sin(\beta)}{\cos(\beta)}} = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

Thus, our result is the formula $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$. Naturally, this result is limited to those cases where all of the tangents are defined.

□

The formula developed in **Example 3.1.8** for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$. The reader is encouraged to fill in the details. Below, we summarize the sum and difference identities for sine, cosine, and tangent.

Theorem 3.5. Sum and Difference Identities: For all angles α and β for which the following are defined,

- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

In the statement of **Theorem 3.5**, we have combined the cases for the sum '+' and difference '−' of angles into one formula. The convention is that if you want the formula for the sum '+' of two angles, use the top sign in the formula; for the difference '−' use the bottom sign. For example,

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

Example 3.1.9. Find the exact value of $\tan(75^\circ)$.

Solution. We use the sum identity for tangent, first writing $\tan(75^\circ)$ as $\tan(45^\circ + 30^\circ)$.

$$\begin{aligned}
\tan(75^\circ) &= \tan(45^\circ + 30^\circ) \\
&= \frac{\tan(45^\circ) + \tan(30^\circ)}{1 - \tan(45^\circ)\tan(30^\circ)} \quad \text{sum identity for tangent} \\
&= \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)\left(\frac{1}{\sqrt{3}}\right)} \\
&= \left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) \Bigg/ \left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) \quad \text{or} \quad \frac{\sqrt{3}+1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}-1} \\
&= \frac{\sqrt{3}+1}{\sqrt{3}-1}
\end{aligned}$$

□

As promised in **Section 2.2**, we finish this section by proving algebraically that the period of the tangent function is π. Recall that a function f is periodic if there is a real number p so that $f(x+p) = f(x)$ for all real numbers x in the domain of f. The smallest positive number p, if it exists, is called the period of f. To prove that the period of $y = \tan(x)$ is π, we appeal to the sum identity for tangent:

$$\begin{aligned}\tan(x+\pi) &= \frac{\tan(x)+\tan(\pi)}{1-\tan(x)\tan(\pi)} \\ &= \frac{\tan(x)+0}{1-(\tan(x))(0)} \\ &= \tan(x)\end{aligned}$$

This tells us that the function $y=\tan(x)$ is periodic and that its period is at most π. To show it is exactly π, suppose p is a positive real number so that $\tan(x+p)=\tan(x)$ for all real numbers x. For $x=0$, we have

$$\begin{aligned}\tan(p) &= \tan(0+p) \\ &= \tan(0) \qquad \text{from } \tan(x+p)=\tan(x) \\ &= 0\end{aligned}$$

This means p is a multiple of π. The smallest positive multiple of π is π itself, so we have established that the period of the tangent function is π. We leave it to the reader to establish that the period of the cotangent function is also π.[6]

[6] Certainly, mimicking the proof for the period of tan(*x*) is an option. For another approach, consider transforming tan(*x*) to cot(*x*) using identities.

3.1 Exercises

1. We know $g(x)=\cos(x)$ is an even function while $f(x)=\sin(x)$ and $h(x)=\tan(x)$ are odd functions. What about $G(x)=\cos^2(x)$, $F(x)=\sin^2(x)$ and $H(x)=\tan^2(x)$? Are they even, odd, or neither? Why?

2. Examine the graph of $f(x)=\sec(x)$ on the interval $[-\pi,\pi]$. How can we tell whether the function $f(x)=\sec(x)$ is even or odd from its graph?

In Exercises 3 – 8, use identities to fully simplify the expression.

3. $\sin(-x)\cos(-x)\csc(-x)$

4. $\csc(x)+\cos(x)\cot(-x)$

5. $\dfrac{\cot(t)+\tan(t)}{\sec(-t)}$

6. $3\sin^3(t)\csc(t)+\cos^2(t)+2\cos(-t)\cos(t)$

7. $-\tan(-x)\cot(-x)$

8. $\dfrac{-\sin(-x)\cos(x)\sec(x)\csc(x)\tan(x)}{\cot(x)}$

In Exercises 9 – 23, use the sum and difference identities to find the exact value. You may have need of the quotient, reciprocal, or even/odd identities as well.

9. $\cos(75^\circ)$

10. $\sec(165^\circ)$

11. $\sin(105^\circ)$

12. $\csc(195^\circ)$

13. $\cot(255^\circ)$

14. $\tan(375^\circ)$

15. $\cos\left(\dfrac{13\pi}{12}\right)$

16. $\sin\left(\dfrac{11\pi}{12}\right)$

17. $\tan\left(\dfrac{13\pi}{12}\right)$

18. $\cos\left(\dfrac{7\pi}{12}\right)$

19. $\tan\left(\dfrac{17\pi}{12}\right)$

20. $\sin\left(\dfrac{\pi}{12}\right)$

21. $\cot\left(\dfrac{11\pi}{12}\right)$

22. $\csc\left(\dfrac{5\pi}{12}\right)$

23. $\sec\left(-\dfrac{\pi}{12}\right)$

24. If α is a Quadrant IV angle with $\cos(\alpha)=\dfrac{\sqrt{5}}{5}$ and $\sin(\beta)=\dfrac{\sqrt{10}}{10}$ where $\dfrac{\pi}{2}<\beta<\pi$, find

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha+\beta)$ (c) $\sin(\alpha-\beta)$ (d) $\cos(\alpha-\beta)$

25. If $\tan(\alpha) = -2$ where $\frac{3\pi}{2} < \alpha < 2\pi$, and β is a Quadrant II angle with $\tan(\beta) = -\frac{1}{3}$, find $\tan(\alpha + \beta)$.

26. If $\csc(\alpha) = 3$ where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find

(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha + \beta)$ (c) $\sin(\alpha - \beta)$ (d) $\cos(\alpha - \beta)$

27. If $\sin(\alpha) = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find

(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha - \beta)$ (c) $\tan(\alpha - \beta)$

28. If $\sec(\alpha) = -\frac{5}{3}$ where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$ where $\pi < \beta < \frac{3\pi}{2}$, find

(a) $\csc(\alpha - \beta)$ (b) $\sec(\alpha + \beta)$ (c) $\cot(\alpha + \beta)$

In Exercises 29 – 32, verify the identity or show that it is not an identity. Assume all quantities are defined.

29. $\frac{1}{1+\cos(x)} - \frac{1}{1-\cos(-x)} = -2\cot(x)\csc(x)$

30. $\frac{\tan(x)}{\sec(x)}\sin(-x) = \cos^2(x)$

31. $\frac{\sec(-x)}{\tan(x)+\cot(x)} = -\sin(-x)$

32. $\frac{1+\sin(x)}{\cos(x)} = \frac{\cos(x)}{1+\sin(-x)}$

In Exercises 33 – 51, verify the identity. Assume all quantities are defined.

33. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$

34. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$

35. $\tan(-t^2 + 1) = -\tan(t^2 - 1)$

36. $\csc(-\theta - 5) = -\csc(\theta + 5)$

37. $\sec(-6t) = \sec(6t)$

38. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

39. $\cos(\theta - \pi) = -\cos(\theta)$

40. $\sin(\pi - \theta) = \sin(\theta)$

41. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

42. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

43. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

44. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

45. $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\dfrac{1+\cot(\alpha)\tan(\beta)}{1-\cot(\alpha)\tan(\beta)}$

46. $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{1-\tan(\alpha)\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

47. $\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin(\alpha)\sin(\beta)$

48. $\dfrac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)}=\dfrac{\sin(\alpha)\cos(\alpha)+\sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha)-\sin(\beta)\cos(\beta)}$

49. $\dfrac{\sin(t+h)-\sin(t)}{h}=\cos(t)\left(\dfrac{\sin(h)}{h}\right)+\sin(t)\left(\dfrac{\cos(h)-1}{h}\right)$

50. $\dfrac{\cos(t+h)-\cos(t)}{h}=\cos(t)\left(\dfrac{\cos(h)-1}{h}\right)-\sin(t)\left(\dfrac{\sin(h)}{h}\right)$

51. $\dfrac{\tan(t+h)-\tan(t)}{h}=\left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

52. Verify the even/odd identities for tangent, cosecant, secant, and cotangent.

53. Verify the co-function identities for tangent, cosecant, secant, and cotangent.

54. Verify the difference identities for sine and tangent.

3.2 Multiple Angle Identities

Learning Objectives

- **State the double-angle identities for sine, cosine, and tangent.**
- **Find trigonometric values of double-angles.**
- **Verify identities that include double-angles.**
- **State and apply the power reduction formulas for sine and cosine.**
- **State and apply the half-angle formulas for sine, cosine, and tangent.**
- **State and apply the product-to-sum formulas for sine and cosine.**
- **State and apply the sum-to-product formulas for sine and cosine.**

The identities in this section follow easily from the sum identities that were introduced in **Section 3.1**. These new identities will allow us to evaluate exact values of trigonometric functions that we could not find prior to this section. We will also use identities to break down or rewrite trigonometric expressions in ways that will be useful in Calculus.

Double-Angle Identities

Recall the sum identities from **Section 3.1**:

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$
$$\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

Double-angle identities relate trigonometric functions of double angles 2θ with trigonometric functions of an angle θ. To derive these, we set $\alpha=\theta$ and $\beta=\theta$ in the above sum identities. Following **Theorem 3.6**, we will look at some of these derivations.

Theorem 3.6. Double-Angle Identities: For all angles θ for which the following are defined,

- $\sin(2\theta)=2\sin(\theta)\cos(\theta)$
- $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$

  Other forms: $\cos(2\theta)=2\cos^2(\theta)-1$ or $\cos(2\theta)=1-2\sin^2(\theta)$
- $\tan(2\theta)=\dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

The double-angle identity for cosine, $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$, is derived as follows:

$$\begin{aligned}\cos(2\theta)&=\cos(\theta+\theta)\\&=\cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta) \quad \text{sum identity for cosine}\\&=\cos^2(\theta)-\sin^2(\theta)\end{aligned}$$

To arrive at the alternate form $\cos(2\theta)=2\cos^2(\theta)-1$, we apply the Pythagorean identity.

$$\begin{aligned}\cos(2\theta)&=\cos^2(\theta)-\sin^2(\theta)\\&=\cos^2(\theta)-\left[1-\cos^2(\theta)\right] \quad \text{from } \sin^2(\theta)+\cos^2(\theta)=1\\&=\cos^2(\theta)-1+\cos^2(\theta)\\&=2\cos^2(\theta)-1\end{aligned}$$

Likewise, the Pythagorean identity may be used to derive the form $\cos(2\theta)=1-2\sin^2(\theta)$ from $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$. We leave the proofs of the remaining double-angle identities to the reader.

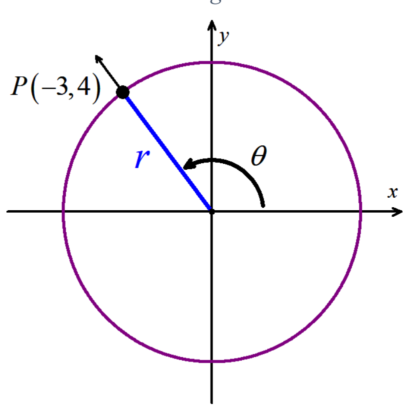
Trigonometric Values of Double-Angles

Now that we have established the double-angle identities, we put them to good use in determining trigonometric values of double-angles.

Example 3.2.1. Suppose point $P(-3,4)$ lies on the terminal side of angle θ, when θ is plotted in standard position. Find $\sin(2\theta)$ and $\cos(2\theta)$. Determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.

Solution. Point $P(-3,4)$ lies on the circle of radius r, where $r=\sqrt{x^2+y^2}$.

Figure 3.2. 1



With $x=-3$ and $y=4$, we find $r=\sqrt{(-3)^2+(4)^2}=\sqrt{25}=5$. Then $\sin(\theta)=\frac{y}{r}=\frac{4}{5}$ and $\cos(\theta)=\frac{x}{r}=-\frac{3}{5}$. It follows that

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta) && \text{from double-angle identity}\\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

and

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) && \text{from double-angle identity}\\ &= \left(-\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2\\ &= -\frac{7}{25}\end{aligned}$$

Since both the sine and cosine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III.

□

Example 3.2.2. If $\sin(\theta)=x$ for $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

Solution. If your first reaction to '$\sin(\theta)=x$' is that x should be the cosine of θ, then you have indeed learned something. However, context is everything. Here, x is just a variable. It does not necessarily represent the x-coordinate of a point on the Unit Circle. Here, x represents the quantity $\sin(\theta)$, and what we wish to know is how to express $\sin(2\theta)$ in terms of x. We will see more of this kind of thing in **Chapter 4** and, as usual, this is something we need for Calculus.

We start with the double-angle identity for sine:

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ &= 2x\cos(\theta) && \text{from the problem statement that } \sin(\theta)=x\end{aligned}$$

We need to write $\cos(\theta)$ in terms of x to finish the problem, and once again return to the Pythagorean identity to help us out.

$$\begin{aligned}\sin^2(\theta)+\cos^2(\theta) &= 1 && \text{Pythagorean identity}\\ x^2+\cos^2(\theta) &= 1 && \text{from the problem statement}\\ \cos(\theta) &= \pm\sqrt{1-x^2}\\ \cos(\theta) &= \sqrt{1-x^2} && \cos(\theta)\geq 0 \text{ since } -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\end{aligned}$$

Finally, back to finding an expression for $\sin(2\theta)$, we have an answer of $\sin(2\theta)=2x\sqrt{1-x^2}$.

□

Verifying Identities that Include Double-Angles

We return to verifying trigonometric identities.

Example 3.2.3. Verify the identity $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

Solution. Starting with the more complicated right side of the equation,

$$\begin{aligned}
\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} && \text{Pythagorean identity} \\
&= \frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\left(\dfrac{1}{\cos^2(\theta)}\right)} && \text{quotient and reciprocal identities} \\
&= 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\left(\frac{\cos^2(\theta)}{1}\right) \\
&= 2\sin(\theta)\cos(\theta) \\
&= \sin(2\theta) && \text{double-angle identity for sine}
\end{aligned}$$

□

Example 3.2.4. Verify the identity $\tan(2\theta) = \dfrac{2}{\cot(\theta)-\tan(\theta)}$.

Solution. In this case, we begin with the left side of the equation (explanation to follow).

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)} && \text{double-angle identity for tangent} \\
&= \frac{2\tan(\theta)}{\left(1-\tan^2(\theta)\right)} \cdot \frac{\left(\dfrac{1}{\tan(\theta)}\right)}{\left(\dfrac{1}{\tan(\theta)}\right)} && \text{goal: numerator of 2} \\
&= \frac{2}{\dfrac{1}{\tan(\theta)} - \tan(\theta)} \\
&= \frac{2}{\cot(\theta)-\tan(\theta)} && \text{reciprocal identity for cotangent}
\end{aligned}$$

□

Example 3.2.4 is a case where the more complicated side of the initial equation appears to be on the right, but we chose to start with the left side in order to change from the double angle 2θ to the angle θ, which was used on the right side. Beginning with the right side would require some thinking ahead. Try it!

When using identities to simplify a trigonometric expression, solve a trigonometric equation, or verify a trigonometric identity, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated, although generally one of the paths will result in a simpler solution. In verifying identities, the strategies established in **Section 1.5** will help, but there is no substitute for practice.

One more note before moving on to power reduction and half-angle formulas. While double-angle identities could be established for cosecant, secant, and cotangent, the identities already established in this section may be used in their place. Recall that cosecant, secant, and cotangent are reciprocals of sine, cosine, and tangent, respectively. Thus, for example, $\sec(2\theta)=\dfrac{1}{\cos(2\theta)}$ and so any of the three double-angle identities for cosine may be used in determining $\sec(2\theta)$.

Power Reduction Formulas

While the double-angle identities allow us to write $\cos(2\theta)$ as powers of sine and/or cosine, in Calculus we occasionally do the reverse; that is, reduce the power of sine or cosine. Solving the identity $\cos(2\theta)=1-2\sin^2(\theta)$ for $\sin^2(\theta)$ and the identity $\cos(2\theta)=2\cos^2(\theta)-1$ for $\cos^2(\theta)$ result in the following aptly-named 'power reduction' formulas.

Theorem 3.7. Power Reduction Formulas: For all angles θ,

- $\sin^2(\theta)=\dfrac{1-\cos(2\theta)}{2}$
- $\cos^2(\theta)=\dfrac{1+\cos(2\theta)}{2}$

Example 3.2.5. Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and/or difference of cosines to the first power.

Solution. We begin with a straightforward application of **Theorem 3.7**:

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right) && \text{power reduction formulas}\\ &= \frac{1}{4}\left(1-\cos^2(2\theta)\right)\\ &= \frac{1}{4}-\frac{1}{4}\cos^2(2\theta)\\ &= \frac{1}{4}-\frac{1}{4}\left(\frac{1+\cos(2(2\theta))}{2}\right) && \text{replace } \theta \text{ with } 2\theta \text{ in power reduction formula}\\ &= \frac{1}{4}-\frac{1}{8}-\frac{1}{8}\cos(4\theta)\\ &= \frac{1}{8}-\frac{1}{8}\cos(4\theta)\end{aligned}$$

□

Half-Angle Formulas

Another application of the power reduction formulas is the half-angle formulas. To start, we apply the power reduction formula to $\sin^2\left(\frac{\theta}{2}\right)$:

$$\begin{aligned}\sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} && \text{replace } \theta \text{ with } \frac{\theta}{2} \text{ in power reduction formula}\\ &= \frac{1-\cos(\theta)}{2}\end{aligned}$$

We obtain a formula for $\sin\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we obtain a half-angle formula for cosine. The half-angle formula for tangent results from using a quotient identity. Following is a summary of these formulas.

Theorem 3.8. Half-Angle Formulas: For all angles θ for which the following are defined,

- $\sin\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\cos\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

The choice of '+' or '–' depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 3.2.6. Use a half-angle formula to find the exact value of $\cos(15^\circ)$.

Solution. To use the half-angle formula, we note that $15^\circ=\frac{30^\circ}{2}$.

$$\cos\left(15^\circ\right)=\cos\left(\frac{30^\circ}{2}\right)$$
$$=\pm\sqrt{\frac{1+\cos\left(30^\circ\right)}{2}} \quad \text{half-angle formula for cosine}$$

Because we know that $\cos\left(15^\circ\right)$ is positive (15º is in Quadrant I),

$$\cos\left(15^\circ\right)=+\sqrt{\frac{1+\sqrt{3}/2}{2}}$$
$$=\sqrt{\frac{1+\sqrt{3}/2}{2}\cdot\frac{\sqrt{2}}{\sqrt{2}}}$$
$$=\frac{\sqrt{2+\sqrt{3}}}{2}$$

□

Back in **Example 3.1.3**, we found $\cos\left(15^\circ\right)$ by using the difference identity for cosine. In that case, we determined $\cos\left(15^\circ\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$. The reader is encouraged to prove that these two expressions are equal.

Example 3.2.7. Suppose $-\pi<\theta<0$ and $\cos(\theta)=-\frac{3}{5}$. Find $\sin\left(\frac{\theta}{2}\right)$.

Solution. If $-\pi<\theta<0$, then $-\frac{\pi}{2}<\frac{\theta}{2}<0$, which means $\sin\left(\frac{\theta}{2}\right)<0$.

$$\sin\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos(\theta)}{2}} \quad \text{half-angle formula for sine}$$
$$=-\sqrt{\frac{1-(-3/5)}{2}}$$
$$=-\sqrt{\frac{1+3/5}{2}}$$

We simplify as follows:

$$-\sqrt{\frac{1+3/5}{2}\cdot\frac{\sqrt{5}}{\sqrt{5}}}=-\sqrt{\frac{5+3}{10}}=-\frac{2}{\sqrt{5}}$$

Thus, $\sin\left(\frac{\theta}{2}\right)=-\frac{2}{\sqrt{5}}$, or $-\frac{2\sqrt{5}}{5}$ with a rationalized denominator.

□

Example 3.2.8. Prove the identity $\tan\left(\frac{\theta}{2}\right)=\frac{\sin(\theta)}{1+\cos(\theta)}$.

Solution. We start with the more complicated side, which is the right side.

$$\frac{\sin(\theta)}{1+\cos(\theta)} = \frac{\sin\left(2\left(\frac{\theta}{2}\right)\right)}{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{1+\left(2\cos^2\left(\frac{\theta}{2}\right)-1\right)} \quad \text{double angle identities}$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

□

The next example uses identities already established in this section to rewrite a trigonometric expression with argument 3θ as the sum of trigonometric expressions with argument θ.

Example 3.2.9. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

Solution. The double angle identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ expresses $\cos(2\theta)$ in terms of $\cos(\theta)$.

We are asked to find such an identity for $\cos(3\theta)$. First, we consider 3θ as $(2\theta + \theta)$ and apply the sum identity for cosine.

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta+\theta) \\
&= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) && \text{sum identity for cosine} \\
&= \left(2\cos^2(\theta)-1\right)\cos(\theta) - \left(2\sin(\theta)\cos(\theta)\right)\sin(\theta) && \text{double angle identities} \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\left(1-\cos^2(\theta)\right)\cos(\theta) && \text{Pythagorean identity} \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\
&= 4\cos^3(\theta) - 3\cos(\theta)
\end{aligned}$$

Thus, $\cos(3\theta)$ can be expressed as the polynomial $4\cos^3(\theta) - 3\cos(\theta)$.

□

Having just shown how we could rewrite $\cos(3\theta)$ as the sum of powers of $\cos(\theta)$, it might occur to you that similar operations could be applied to $\cos(4\theta)$ or $\cos(5\theta)$ to rewrite the expressions as sums of powers of $\cos(\theta)$. Try it!

The following formulas are particularly useful in spring-mass-dashpot systems and in electrical circuits.

Product-to-Sum Formulas

Our next batch of identities, the product-to-sum formulas[7], are easily verified by expanding each of the right sides in accordance with the sum and difference identities. The details are left as exercises.

Theorem 3.9. Product-to-Sum Formulas: For all angles α and β,

- $\sin(\alpha)\sin(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\cos(\alpha)\cos(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta)=\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Example 3.2.10. Write $\cos(2\theta)\cos(6\theta)$ as a sum.

Solution. Identifying $\alpha=2\theta$ and $\beta=6\theta$, we use the product-to-sum formula for $\cos(\alpha)\cos(\beta)$:

$$\begin{aligned}\cos(2\theta)\cos(6\theta)&=\frac{1}{2}\left[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)\right]\\&=\frac{1}{2}\left[\cos(-4\theta)+\cos(8\theta)\right]\\&=\frac{1}{2}\cos(4\theta)+\frac{1}{2}\cos(8\theta) &&\text{even property of cosine}\end{aligned}$$

□

Sum-to-Product Formulas

Related to the product-to-sum formulas are the sum-to-product formulas. These are easily verified using the product-to-sum formulas and, as such, their proofs are left as exercises.

[7] These are also known as the Prosthaphaeresis Formulas and have a rich history. Conduct some research on them as your schedule allows.

Theorem 3.10. Sum-to-Product Formulas: For all angles α and β,

- $\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right)$
- $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$
- $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

Example 3.2.11. Write $\sin(\theta) - \sin(3\theta)$ as a product.

Solution. The sum-to-product formula for $\sin(\alpha) - \sin(\beta)$, with $\alpha = \theta$ and $\beta = 3\theta$, yields the following:

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta) && \text{odd property of sine} \end{aligned}$$

□

The identities established in **Chapter 3** will prove useful throughout the remainder of this textbook. In **Chapter 4**, these identities will be particularly helpful in solving trigonometric equations.

3.2 Exercises

In Exercises 1 – 10, use the given information about θ to determine the exact value.

1. If $\sin(\theta) = -\frac{7}{25}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin(2\theta)$.

2. If $\cos(\theta) = \frac{28}{53}$ and $0 < \theta < \frac{\pi}{2}$, find $\cos(2\theta)$.

3. If $\tan(\theta) = \frac{12}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos(2\theta)$.

4. If $\csc(\theta) = 4$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin(2\theta)$.

5. If $\cos(\theta) = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$, find $\sin(2\theta)$.

6. If $\sin(\theta) = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos(2\theta)$.

7. If $\cos(\theta) = \frac{12}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos(2\theta)$.

8. If $\sin(\theta) = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin(2\theta)$.

9. If $\sec(\theta) = \sqrt{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\tan(2\theta)$.

10. If $\tan(\theta) = -2$ and $\frac{\pi}{2} < \theta < \pi$, find $\tan(2\theta)$.

In Exercises 11 – 25, use half-angle formulas to find the exact value. You may have need of the quotient, reciprocal, or even/odd identities as well.

11. $\cos(75^\circ)$

12. $\sin(105^\circ)$

13. $\cos(67.5^\circ)$

14. $\sin(157.5^\circ)$

15. $\tan(112.5^\circ)$

16. $\cos\left(\frac{7\pi}{12}\right)$

17. $\sin\left(\frac{\pi}{12}\right)$

18. $\cos\left(\frac{\pi}{8}\right)$

19. $\sin\left(\frac{5\pi}{8}\right)$

20. $\tan\left(\frac{7\pi}{8}\right)$

21. $\cos\left(-\frac{11\pi}{12}\right)$

22. $\sin\left(\frac{11\pi}{12}\right)$

23. $\tan\left(\frac{5\pi}{12}\right)$

24. $\tan\left(-\frac{3\pi}{12}\right)$

25. $\tan\left(-\frac{3\pi}{8}\right)$

In Exercises 26 – 39, use the given information about θ to find the exact value.

26. If $\sin(\theta) = -\frac{7}{25}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\left(\frac{\theta}{2}\right)$.

27. If $\cos(\theta) = \frac{28}{53}$ and $0 < \theta < \frac{\pi}{2}$, find $\cos\left(\frac{\theta}{2}\right)$.

28. If $\tan(\theta) = \frac{12}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\left(\frac{\theta}{2}\right)$.

29. If $\csc(\theta) = 4$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin\left(\frac{\theta}{2}\right)$.

30. If $\cos(\theta) = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$, find $\tan\left(\frac{\theta}{2}\right)$.

31. If $\sin(\theta) = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\sin\left(\frac{\theta}{2}\right)$.

32. If $\cos(\theta) = \frac{12}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\tan\left(\frac{\theta}{2}\right)$.

33. If $\sin(\theta) = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$, find $\cos\left(\frac{\theta}{2}\right)$.

34. If $\sec(\theta) = \sqrt{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\left(\frac{\theta}{2}\right)$.

35. If $\tan(\theta) = -2$ and $\frac{\pi}{2} < \theta < \pi$, find $\cos\left(\frac{\theta}{2}\right)$.

36. If $\tan(\theta) = -\frac{4}{3}$ and θ is in Quadrant IV, find $\cos\left(\frac{\theta}{2}\right)$.

37. If $\sin(\theta) = -\frac{12}{13}$ and θ is in Quadrant III, find $\sin\left(\frac{\theta}{2}\right)$.

38. If $\csc(\theta) = 7$ and θ is in Quadrant II, find $\cos\left(\frac{\theta}{2}\right)$.

39. If $\sec(\theta) = -4$ and θ is in Quadrant II, find $\tan\left(\frac{\theta}{2}\right)$.

In Exercises 40 – 45, write the given product as a sum. You may need to use an even/odd identity.

40. $\cos(3\theta)\cos(5\theta)$

41. $\sin(2\theta)\sin(7\theta)$

42. $\sin(9\theta)\cos(\theta)$

43. $\cos(2\theta)\cos(6\theta)$

44. $\sin(3\theta)\sin(2\theta)$

45. $\cos(\theta)\sin(3\theta)$

In Exercises 46 – 51, write the given sum as a product. You may need to use an even/odd or co-function identity.

46. $\cos(3\theta)+\cos(5\theta)$

47. $\sin(2\theta)-\sin(7\theta)$

48. $\sin(5\theta)-\cos(6\theta)$

49. $\sin(9\theta)-\sin(-\theta)$

50. $\sin(\theta)+\cos(\theta)$

51. $\cos(\theta)-\sin(\theta)$

52. Use the double-angle identity $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ to verify the double-angle identity $\cos(2\theta) = 2\cos^2(\theta) - 1$.

53. Use the double-angle identities for $\cos(2\theta)$ and $\sin(2\theta)$ to verify the double-angle identity for $\tan(2\theta)$.

54. Without using your calculator, show that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$.

In Exercises 55 – 72, verify the identity. Assume all quantities are defined.

55. $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$

56. $\cos(2\theta) = \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$

57. $\tan(2\theta) = \frac{2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)-1}$

58. $\left[\sin(\theta)+\cos(\theta)\right]^2 = 1+\sin(2\theta)$

59. $\left[\cos(\theta)-\sin(\theta)\right]^2 = 1-\sin(2\theta)$

60. $\tan(2\theta) = \frac{1}{1-\tan(\theta)} - \frac{1}{1+\tan(\theta)}$

61. $\csc(2\theta)=\dfrac{\tan(\theta)+\cot(\theta)}{2}$

62. $\sec(2\theta)=\dfrac{\cos(\theta)}{\cos(\theta)+\sin(\theta)}+\dfrac{\sin(\theta)}{\cos(\theta)-\sin(\theta)}$

63. $\dfrac{1}{\cos(\theta)-\sin(\theta)}+\dfrac{1}{\cos(\theta)+\sin(\theta)}=\dfrac{2\cos(\theta)}{\cos(2\theta)}$

64. $\dfrac{1}{\cos(\theta)-\sin(\theta)}-\dfrac{1}{\cos(\theta)+\sin(\theta)}=\dfrac{2\sin(\theta)}{\cos(2\theta)}$

65. $8\sin^4(\theta)=\cos(4\theta)-4\cos(2\theta)+3$

66. $8\cos^4(\theta)=\cos(4\theta)+4\cos(2\theta)+3$

67. $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$

68. $\cos(4\theta)=8\cos^4(\theta)-8\cos^2(\theta)+1$

69. $\sin(4\theta)=4\sin(\theta)\cos^3(\theta)-4\sin^3(\theta)\cos(\theta)$

70. $32\sin^2(\theta)\cos^4(\theta)=2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)$

71. $32\sin^4(\theta)\cos^2(\theta)=2-\cos(2\theta)-2\cos(4\theta)+\cos(6\theta)$

72. $\cos(8\theta)=128\cos^8(\theta)-256\cos^6(\theta)+160\cos^4(\theta)-32\cos^2(\theta)+1$ (HINT: Use result for **68**.)

73. Suppose θ is a Quadrant I angle with $\sin(\theta)=x$. Verify the following formulas.

(a) $\cos(\theta)=\sqrt{1-x^2}$ (b) $\sin(2\theta)=2x\sqrt{1-x^2}$ (c) $\cos(2\theta)=1-2x^2$

74. Discuss with your classmates how each of the formulas, if any, in **Exercise 73** change if we assume θ is a Quadrant II, III, or IV angle.

75. Suppose θ is a Quadrant I angle with $\tan(\theta)=x$. Verify the following formulas.

(a) $\cos(\theta)=\dfrac{1}{\sqrt{x^2+1}}$ (b) $\sin(\theta)=\dfrac{x}{\sqrt{x^2+1}}$

(c) $\sin(2\theta)=\dfrac{2x}{x^2+1}$ (d) $\cos(2\theta)=\dfrac{1-x^2}{x^2+1}$

76. Discuss with your classmates how each of the formulas, if any, in **Exercise 75** change if we assume θ is a Quadrant II, III, or IV angle.

77. If $\sin(\theta) = \frac{x}{2}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of x.

78. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

79. Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$. Compute $\sin\left(\frac{\theta}{2}\right)$ in both cases.

80. In **Exercise 67**, we had you verify an identity that expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

81. Verify the Product-to-Sum Identities.

82. Verify the Sum-to-Product Identities.

CHAPTER 4
TRIGONOMETRIC EQUATIONS

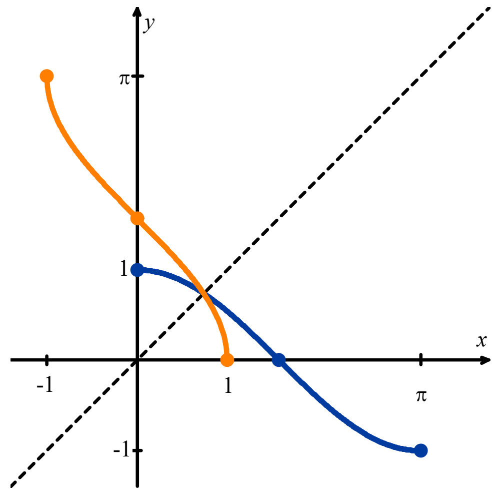


Figure 4.0. 1

Chapter Outline

Introduction

Sections 4.1 and 4.2 are all about inverse trigonometric functions — their appearances, their patterns, and using them to solve problems. We establish that trigonometric functions *can be undone* and what it means to undo them. Section 4.1 covers inverse sine and cosine functions, while Section 4.2 looks at the remaining inverse functions: tangent, cotangent, secant, and cosecant. This leads directly into Sections 4.3 and 4.4, which are all about solving equations containing trigonometric functions.

Section 4.3 will discuss solving equations containing only one trigonometric function (possibly used multiple times in the equation). Section 4.4 will discuss solving equations containing multiple trigonometric functions using identities, factoring, and other solving techniques.

By now, you likely have a good deal of experience solving equations. You have experience with equations containing operations such as polynomials, radicals (roots), rational expressions (fractions), exponentials, and logarithms. Solving some equations containing trigonometric functions will involve only one or two steps (as with linear equations), and some will involve multiple steps with new twists.

Keep in mind that, as with any function, solving for a variable buried within a function will involve using inverse functions. Given the cyclical nature of trigonometric functions, simply using inverse functions will only locate one of possibly many viable solutions. Much as with solving a quadratic equation, you will have to use your knowledge of domain and range to expand a primary solution into a secondary solution and even into a span of infinitely many solutions.

4.1 Inverse Sine and Cosine Functions

Learning Objectives

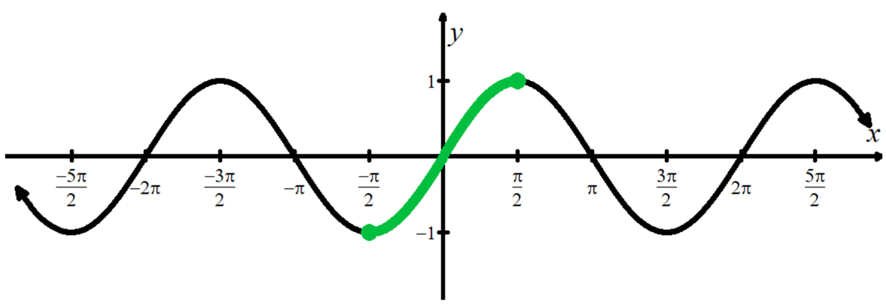
- **Define the inverse sine and cosine functions. State and apply their properties.**
- **Find exact values of inverse sine and cosine functions, and of their composition with other trigonometric functions.**
- **Rewrite composite functions of trigonometric and inverse sine or cosine functions as algebraic expressions.**

We begin **Chapter 4** by finding inverses of the sine and cosine functions. Our immediate problem is that, owing to their periodic nature, neither of these functions is one-to-one. To remedy this, we restrict the domain of each to obtain a one-to-one function.

The Inverse Sine Function

We first consider the function $f(x)=\sin(x)$. Choosing the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ results in a one-to-one function and allows us to keep the range as $[-1,1]$.

Figure 4.1. 1



Domain of $f(x)=\sin(x)$ restricted to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

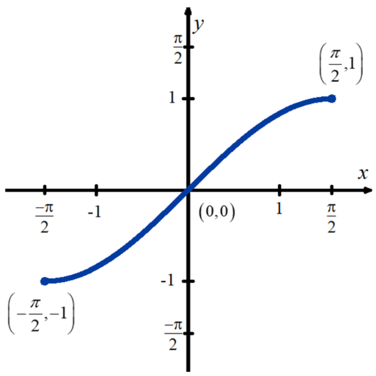
Recall that the inverse of a function f is denoted by f^{-1}. The sine function is denoted by the three letters 'sin', so the inverse of the sine function is denoted by $\sin^{-1}$. The notation for the inverse of $f(x)=\sin(x)$ is $f^{-1}(x)=\sin^{-1}(x)$, read 'inverse sine of x'.[1] Another notation for the inverse of the

[1] Due to our convention of writing $(\sin(x))^2$ as $\sin^2(x)$, $(\sin(x))^3$ as $\sin^3(x)$, and so on, it is easy to confuse $\sin^{-1}(x)$ with $(\sin(x))^{-1}$, which is equivalent to $\csc(x)$, not the inverse of $\sin(x)$. Pay attention to context!

sine function is arcsin, read 'arc-sine'. To understand the 'arc' in arcsine, recall that an inverse function, by definition, reverses the process of the original function. Thus, $f^{-1}(x)=\arcsin(x)$ represents an angle, and an angle placed in the standard position on the Unit Circle corresponds to an oriented arc on the Unit Circle.

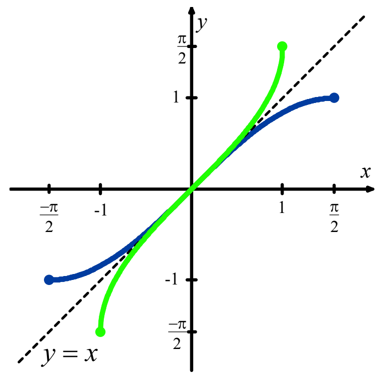
Below are graphs of $f(x)=\sin(x)$ (with restricted domain) and $f^{-1}(x)=\arcsin(x)$, where the latter is the reflection of the former across the line $y=x$. This is, of course, equivalent to switching the *x*- and *y*-coordinates.

Figure 4.1. 2



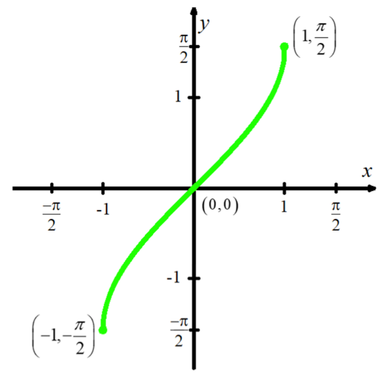
$f(x)=\sin(x),\ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$

Figure 4.1. 3



Reflection about the line $y=x$

Figure 4.1. 4



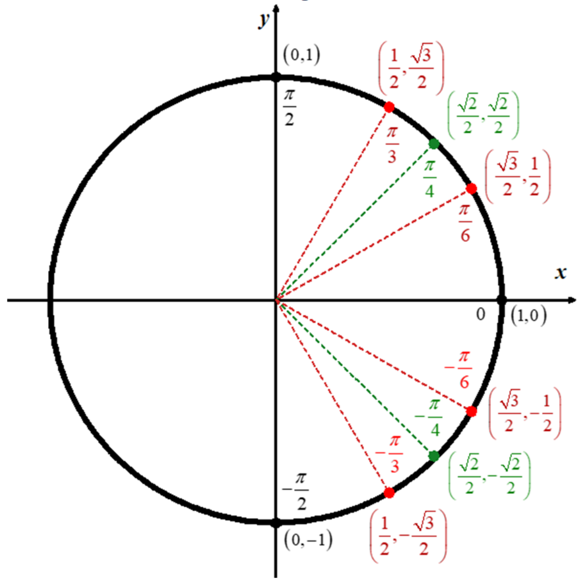
$f^{-1}(x)=\arcsin(x),\ -1\le x\le 1$

The table below includes values of the sine function for standard angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

These sine values are shown as the *y*-coordinates of points on the Unit Circle, as follows.

Figure 4.1. 5



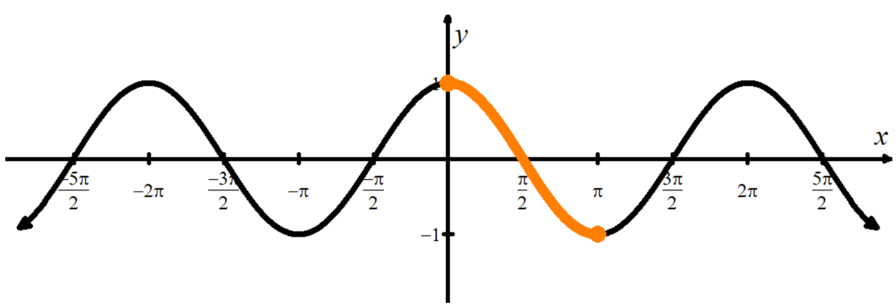
Based on these angles and associated sine values, we have the following values of the arcsine function.

y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin(y)$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

The Inverse Cosine Function

We next consider the function $g(x)=\cos(x)$. Choosing the interval $[0,\pi]$ results in a one-to-one function and allows us to keep the range $[-1,1]$.[2]

Figure 4.1. 6

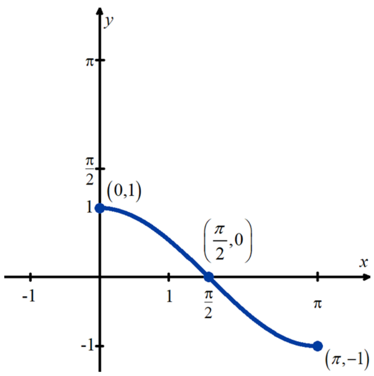


Domain of $g(x)=\cos(x)$ restricted to $[0,\pi]$

[2] We choose [0, π], as opposed to [π, 2π] or others, to have a continuous interval that includes the acute angle measures.

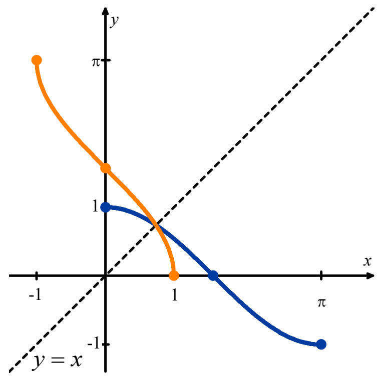
The inverse of $g(x)=\cos(x)$ is denoted as either $g^{-1}(x)=\cos^{-1}(x)$, read 'inverse cosine of *x*', or $g^{-1}(x)=\arccos(x)$, read 'arc-cosine of *x*'. Below are graphs of $g(x)=\cos(x)$ (with restricted domain) and $g^{-1}(x)=\arccos(x)$, where the latter is the reflection of the former across the line $y=x$. This is equivalent to switching the *x*- and *y*-coordinates.

Figure 4.1. 7



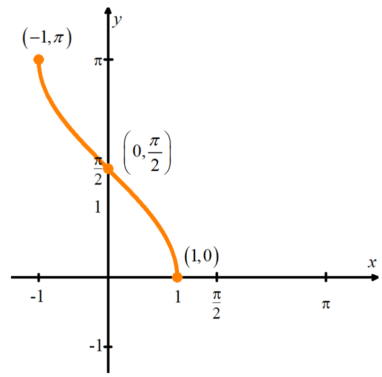
$g(x)=\cos(x)$, $0\le x\le\pi$

Figure 4.1. 8



Reflection about the line $y=x$

Figure 4.1. 9



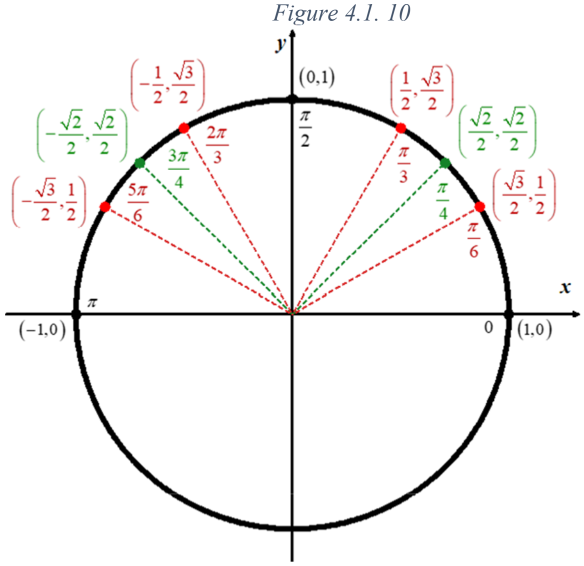
$g^{-1}(x)=\arccos(x)$, $-1\le x\le 1$

The following table shows values of the cosine function for standard angles between 0 and π.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

These cosine values are the *x*-coordinates of points on the Unit Circle, as shown below.

Figure 4.1. 10



From these angles and their associated cosine values, we get values for the arccosine function.

y	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\arccos(y)$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

Finding Values of Inverse Sine and Inverse Cosine Functions

We move on to finding values of inverse sine (arcsine) and inverse cosine (arccosine) functions. Domains and ranges are critical in determining values for each of these functions. Keep the following in mind, as well as graphs of the arcsine and arccosine, when evaluating function values.

Definition 4.1. The Arcsine and Arccosine Functions:

- For $-1 \le x \le 1$, $\arcsin(x)$ is the angle θ such that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin(\theta) = x$.

  The function $y = \arcsin(x)$ has domain $[-1,1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- For $-1 \le x \le 1$, $\arccos(x)$ is the angle θ such that $0 \le \theta \le \pi$ and $\cos(\theta) = x$.

  The function $y = \arccos(x)$ has domain $[-1,1]$ and range $[0, \pi]$.

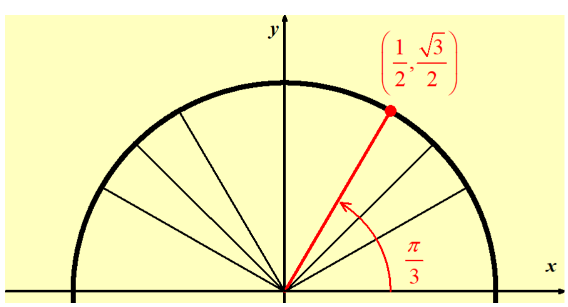
Example 4.1.1. Find the exact values of the following.

1. $\arccos\left(\frac{1}{2}\right)$ 2. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ 3. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ 4. $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution.

1. To find $\arccos\left(\frac{1}{2}\right)$, we need to find the angle θ, in radians, with $0 \le \theta \le \pi$ and $\cos(\theta) = \frac{1}{2}$.

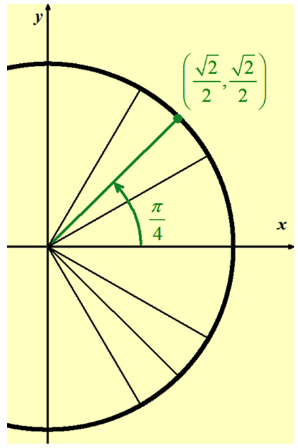
Figure 4.1. 11



We know $\theta = \frac{\pi}{3}$ meets these criteria, so $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

2. The value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ is the angle θ, in radians, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin(\theta) = \frac{\sqrt{2}}{2}$.

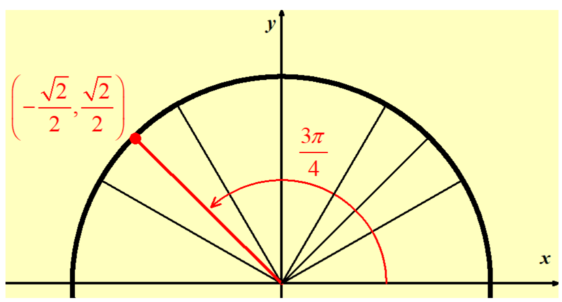
Figure 4.1. 12



Since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$, we have $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

3. We begin by observing that $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ is equivalent to $\arccos\left(-\frac{\sqrt{2}}{2}\right)$.

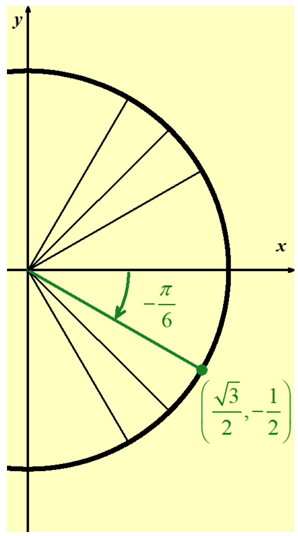
Figure 4.1. 13



The angle $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ lies in the interval $[0, \pi]$ with $\cos(\theta) = -\frac{\sqrt{2}}{2}$. Our answer is $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

4. To find $\sin^{-1}\left(-\frac{1}{2}\right)$, we seek the angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(\theta) = -\frac{1}{2}$.

Figure 4.1. 14



The answer is $\theta=-\frac{\pi}{6}$, so that $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

□

Properties of the Inverse Sine and Inverse Cosine Functions

Recall that for a function f and its inverse f^{-1}, $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$. Since $y=\sin(x)$, $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$, and $y=\arcsin(x)$, $-1\leq x\leq 1$, are inverses of each other, as are $y=\cos(x)$, $0\leq x\leq\pi$, and $y=\arccos(x)$, $-1\leq x\leq 1$, the following properties are a direct consequence.

Inverse Properties of the Arcsine and Arccosine Functions

- $\arcsin(\sin(x))=x$ provided $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$
- $\arccos(\cos(x))=x$ provided $0\leq x\leq\pi$
- $\sin(\arcsin(x))=x$ provided $-1\leq x\leq 1$
- $\cos(\arccos(x))=x$ provided $-1\leq x\leq 1$

We use these properties, when applicable, in evaluating compositions of inverse trigonometric functions with trigonometric functions.

Example 4.1.2. Find the exact values of the following.

1. $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$
2. $\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right)$

Solution.

1. Since $0 \le \frac{\pi}{6} \le \pi$, we could simply invoke the property $\arccos(\cos(x)) = x$, provided $0 \le x \le \pi$, to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.

   Knowing the exact value of $\cos\left(\frac{\pi}{6}\right)$, we can also solve this problem as follows.
$$\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \theta$$
   where θ is the angle with $0 \le \theta \le \pi$ and $\cos(\theta) = \frac{\sqrt{3}}{2}$. Thus, $\theta = \frac{\pi}{6}$ so that
$$\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

2. Since $\frac{11\pi}{6}$ does not fall between 0 and π, the inverse property does not apply. But we do know the exact values of $\cos\left(\frac{11\pi}{6}\right)$ and $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. Hence,
$$\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

□

Most of the common errors encountered in dealing with the inverse trigonometric functions come from the need to restrict the domains of the original functions so that they are one-to-one. One instance of this phenomenon is the fact that $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$ as opposed to $\frac{11\pi}{6}$, demonstrated in the previous example. We move on to the next example, where we reverse the order and evaluate compositions of trigonometric functions with inverse trigonometric functions.

Example 4.1.3. Find the exact values of the following.

1. $\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

2. $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

Solution.

1. One way to simplify $\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$ is to use the property that $\cos(\arccos(x)) = x$, provided $-1 \le x \le 1$. Since $-\frac{3}{5}$ is between -1 and 1, we find $\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$, and we are done.

   For a deeper understanding, we solve this problem a second way, using the meaning of the inverse cosine function. Let $\theta = \cos^{-1}\left(-\frac{3}{5}\right)$. Then $0 \le \theta \le \pi$ and $\cos(\theta) = -\frac{3}{5}$. Thus,

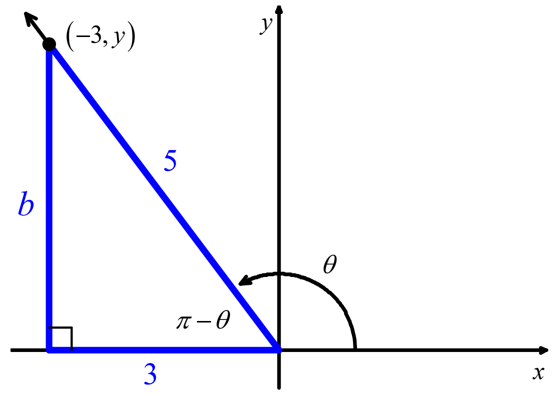
$$\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) = \cos(\theta) = -\frac{3}{5}$$

2. To evaluate $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$, as in the previous example, we let $\theta = \arccos\left(-\frac{3}{5}\right)$ so that $\cos(\theta) = -\frac{3}{5}$ for some θ, where $0 \le \theta \le \pi$. Since $\cos(\theta) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < \theta < \pi$, so that θ represents an angle in Quadrant II. We move on to evaluating $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(\theta)$.

   A geometric approach[3] to evaluating $\sin(\theta)$ is to sketch the angle θ, along with its corresponding reference angle $\pi - \theta$. We then introduce a 'reference triangle' in Quadrant II. Since $\cos(\theta) = -\frac{3}{5}$, the reference triangle will have $\cos(\pi - \theta) = \frac{3}{5}$. We label the adjacent side with length 3 and the hypotenuse with length 5.

[3] For an approach that uses the Pythagorean identity, see the next example.

Figure 4.1. 15



The Pythagorean Theorem can be used to find the length b of the opposite side.

$$b^2 + 3^2 = 5^2$$
$$b^2 = 16$$
$$b = 4 \quad \text{As a length, } b \text{ is positive.}$$

In the reference triangle, $\sin(\pi-\theta) = \frac{b}{5} = \frac{4}{5}$. Since sine is positive in Quadrant II,

$\sin(\theta) = +\sin(\pi-\theta) = \frac{4}{5}$, and we have

$$\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(\theta)$$
$$= \frac{4}{5}$$

□

The last two examples in this section have arguments containing a variable. In finding an equivalent algebraic expression for that variable, we also determine the domain on which the equivalence is valid.

Example 4.1.4. Rewrite $\tan(\arccos(x))$ as an algebraic expression of x and state the values of x for which the equivalence holds.

Solution. We begin by letting $\theta = \arccos(x)$. Then $\cos(\theta) = x$ where $0 \le \theta \le \pi$ and

$$\tan(\arccos(x)) = \tan(\theta)$$

One approach to finding $\tan(\theta)$ is to use the quotient identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. We have $\cos(\theta) = x$, and can use a Pythagorean identity to determine $\sin(\theta)$.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\sin^2(\theta) + x^2 = 1$$
$$\sin(\theta) = \pm\sqrt{1-x^2}$$

Since $0 \le \theta \le \pi$, we find $\sin(\theta) \ge 0$, and thus $\sin(\theta) = \sqrt{1-x^2}$. Then $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{\sqrt{1-x^2}}{x}$.

To determine the values of x for which the equivalence is valid, we consider our substitution $\theta = \arccos(x)$. Since the domain of $\theta = \arccos(x)$ is $[-1,1]$, we must restrict $-1 \le x \le 1$. Additionally, $\dfrac{\sqrt{1-x^2}}{x}$ is not defined for $x = 0$, so we discard this value from the interval $[-1,1]$. Hence, $\tan(\arccos(x)) = \dfrac{\sqrt{1-x^2}}{x}$ for x in $[-1,0) \cup (0,1]$.

□

Example 4.1.5. Rewrite $\cos(2\arcsin(x))$ as an algebraic expression of x and state the values of x for which the equivalence holds.

Solution. We begin by letting $\theta = \arcsin(x)$ so that θ lies in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ with $\sin(\theta) = x$.

Our goal is to express $\cos(2\arcsin(x)) = \cos(2\theta)$ in terms of x. We have three choices for rewriting $\cos(2\theta)$: $\cos^2(\theta) - \sin^2(\theta)$, $2\cos^2(\theta) - 1$, or $1 - 2\sin^2(\theta)$. Since we know that $\sin(\theta) = x$, it is easiest to use the third form.

$$\begin{aligned} \cos(2\arcsin(x)) &= \cos(2\theta) \\ &= 1 - 2\sin^2(\theta) \\ &= 1 - 2x^2 \end{aligned}$$

Since $\arcsin(x)$ is defined only for $-1 \le x \le 1$, $\cos(2\arcsin(x)) = 1 - 2x^2$ holds only for $-1 \le x \le 1$, or the interval $[-1,1]$.

□

In the previous example, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid only for $-1 \le x \le 1$, since $\arcsin(x)$ is defined for $-1 \le x \le 1$. This is similar to the fact that the equivalence $\left(\sqrt{x}\right)^2 = x$ is valid only for $x \ge 0$, since $\sqrt{x}$ is defined for $x \ge 0$. Keep in mind that it pays to be careful when determining the intervals where equivalences are valid.

4.1 Exercises

In Exercises 1 – 21, find the exact value or state that it is undefined. Give your answer in radians.

1. $\arcsin(-1)$
2. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
3. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin(0)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\sin^{-1}(2)$
8. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
9. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
10. $\arcsin(1)$
11. $\arccos\left(-\sqrt{2}\right)$
12. $\cos^{-1}(-1)$
13. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
14. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
15. $\cos^{-1}\left(\sqrt{3}\right)$
16. $\cos^{-1}\left(-\frac{1}{2}\right)$
17. $\arccos(0)$
18. $\arccos\left(\frac{1}{2}\right)$
19. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
20. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
21. $\arccos(1)$

In Exercises 22 – 47, find the exact value or state that it is undefined.

22. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
23. $\sin\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$
24. $\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$
25. $\sin(\arcsin(-0.42))$
26. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$
27. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
28. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$
29. $\cos\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$
30. $\cos(\arccos(-0.998))$
31. $\cos(\arccos(\pi))$
32. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$
33. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$
34. $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$
35. $\sin^{-1}\left(\sin\left(\frac{11\pi}{6}\right)\right)$
36. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$
37. $\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)$
38. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$
39. $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$
40. $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$
41. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$
42. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

43. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

44. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

45. $\sin\left(\sin^{-1}\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$

46. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

47. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

In Exercises 48 – 57, rewrite the quantities as equivalent algebraic expressions of x and state the values of x for which the equivalence holds.

48. $\sin(\arccos(x))$

49. $\tan(\sin^{-1}(x))$

50. $\sin(2\cos^{-1}(x))$

51. $\sin(\arccos(2x))$

52. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$

53. $\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$

54. $\sin(2\arcsin(7x))$

55. $\sin\left(2\sin^{-1}\left(\frac{x\sqrt{3}}{3}\right)\right)$

56. $\cos(2\arcsin(4x))$

57. $\sin(\sin^{-1}(x)+\cos^{-1}(x))$

58. For $\sin(\theta)=\frac{x}{2}$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, verify the identity $\theta+\sin(2\theta)=\sin^{-1}\left(\frac{x}{2}\right)+\frac{x\sqrt{4-x^2}}{2}$.

59. Show that $\arcsin(x)+\arccos(x)=\frac{\pi}{2}$ for $-1\le x\le 1$.

60. Discuss with your classmates why $\sin^{-1}\left(\frac{1}{2}\right)\neq 30^\circ$.

61. Why do the functions $f(x)=\sin^{-1}(x)$ and $g(x)=\cos^{-1}(x)$ have different ranges?

62. Since the functions $y=\cos(x)$ and $y=\cos^{-1}(x)$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

4.2 The Other Inverse Trigonometric Functions

Learning Objectives

- **Define the inverse tangent, cotangent, secant, and cosecant functions. State and apply their properties.**
- **Find exact values of the inverse tangent, cotangent, secant, and cosecant functions, and of their composition with other trigonometric functions.**
- **Rewrite composite functions of trigonometric and inverse tangent, cotangent, secant, and cosecant functions as algebraic expressions.**

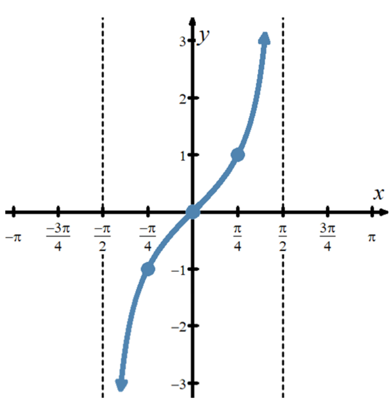
As with the sine and cosine functions, the remaining four trigonometric functions are not one-to-one. This is a necessary requirement for a function to have an inverse, so we will restrict the domain of each function to make it one-to-one before identifying its inverse.

The Inverse Tangent Function

The function $f(x)=\tan(x)$, with its domain restricted to the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, is a one-to-one function.

The graph of its inverse is the reflection of the graph of $f(x)=\tan(x)$, with restricted domain, about the line $y=x$. The inverse, $f^{-1}(x)=\tan^{-1}(x)$, read 'inverse tangent of x', may also be denoted as $\arctan(x)$, read 'arc-tangent of x'.

Figure 4.2. 1

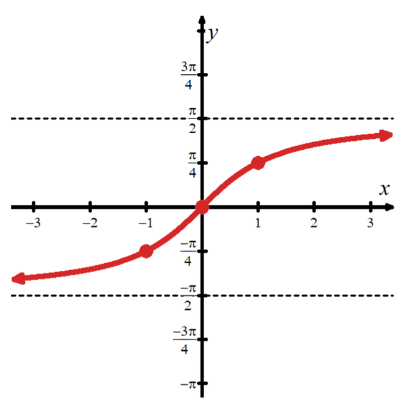


$f(x)=\tan(x),\ -\frac{\pi}{2}<x<\frac{\pi}{2}$

reflect across $y=x$

switch x- and y-coordinates

Figure 4.2. 2



$f^{-1}(x)=\arctan(x),\ -\infty<x<\infty$

Note that the marked points $\left(-\frac{\pi}{4},-1\right)$, $(0,0)$, and $\left(\frac{\pi}{4},1\right)$ on the graph of $f(x)=\tan(x)$, after swapping *x*- and *y*-coordinates to reflect the graph of $f(x)$ about the line $y=x$, become the marked points $\left(-1,-\frac{\pi}{4}\right)$, $(0,0)$, and $\left(1,\frac{\pi}{4}\right)$, respectively, on the graph of $f^{-1}(x)=\arctan(x)$.[4] Also note that the vertical asymptotes $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$ from the graph of $f(x)$ become the horizontal asymptotes $y=-\frac{\pi}{2}$ and $y=\frac{\pi}{2}$, respectively, on the graph of $f^{-1}(x)$.

The following table shows values of the tangent function for angles having standard reference angles and lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Note that $\theta\to\left(-\frac{\pi}{2}\right)^{+}$ means θ approaches $-\frac{\pi}{2}$ from the right side of $-\frac{\pi}{2}$, and $\theta\to\left(\frac{\pi}{2}\right)^{-}$ means θ approaches $\frac{\pi}{2}$ from the left side of $\frac{\pi}{2}$.

θ	$\to\left(-\frac{\pi}{2}\right)^{+}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\to\left(\frac{\pi}{2}\right)^{-}$
$\tan(\theta)$	$\to-\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\to\infty$

From these angles and their associated tangent values, we get the following values of arctangent.

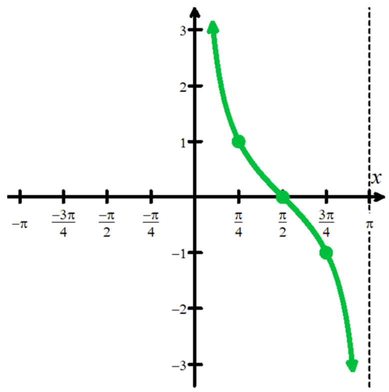
x	$\to-\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\to\infty$
$\arctan(x)$	$\to-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\to\frac{\pi}{2}$

The Inverse Cotangent Function

The function $g(x)=\cot(x)$, with its domain restricted to the interval $(0,\pi)$, is a one-to-one function. The graph of its inverse is the reflection of the graph of $g(x)=\cot(x)$, with restricted domain, about the line $y=x$. The inverse, $g^{-1}(x)=\cot^{-1}(x)$, read 'inverse cotangent of x', may also be denoted as $\operatorname{arccot}(x)$, read 'arc-cotangent of x'.

[4] To avoid 'cluttering', we have not shown the tangent and arctangent functions graphed together with the line *y*=*x*.

Figure 4.2. 3

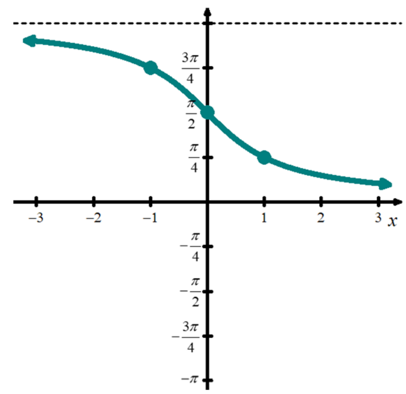


$g(x) = \cot(x)$, $0 < x < \pi$

reflect across $y = x$

switch *x*- and *y*-coordinates

Figure 4.2. 4



$g^{-1}(x) = \operatorname{arccot}(x)$, $-\infty < x < \infty$

Note that the marked points $\left(\frac{\pi}{4},1\right)$, $\left(\frac{\pi}{2},0\right)$, and $\left(\frac{3\pi}{4},-1\right)$ on the graph of $g(x) = \cot(x)$, after swapping *x*- and *y*-coordinates to reflect the graph of $g(x)$ about the line $y = x$, become the marked points $\left(1,\frac{\pi}{4}\right)$, $\left(0,\frac{\pi}{2}\right)$, and $\left(-1,\frac{3\pi}{4}\right)$, respectively, on the graph of $g^{-1}(x) = \operatorname{arccot}(x)$.[5] Also note that the vertical asymptotes $x = 0$ and $x = \pi$ from the graph of $g(x)$ become the horizontal asymptotes $y = 0$ and $y = \pi$, respectively, on the graph of $g^{-1}(x)$.

The following table shows values of the cotangent function for angles having standard reference angles and lying between 0 and π.

θ	$\to 0^+$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\to \pi^-$
$\cot(\theta)$	$\to \infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\to -\infty$

From these angles and their associated cotangent values, we get the following values of arccotangent.

x	$\to \infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\to -\infty$
$\operatorname{arccot}(x)$	$\to 0$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\to \pi$

[5] Again, to avoid 'cluttering', we have not shown the cotangent and arccotangent functions graphed together with the line $y = x$.

Finding Values of Inverse Tangent and Inverse Cotangent Functions

Following is a summary of domains and ranges for inverse tangent (arctangent) and inverse cotangent (arccotangent) functions, along with definitions that will prove helpful in determining function values.

Definition 4.2. The Arctangent and Arccotangent Functions:

- For any real value x, $\arctan(x)$ is the angle θ such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = x$.

  The function $y = \arctan(x)$ has domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- For any real value x, $\operatorname{arccot}(x)$ is the angle θ such that $0 < \theta < \pi$ and $\cot(\theta) = x$.

  The function $y = \operatorname{arccot}(x)$ has domain $(-\infty, \infty)$ and range $(0, \pi)$.

Example 4.2.1. Find the exact values of the following.

1. $\arctan\left(\sqrt{3}\right)$
2. $\operatorname{arccot}\left(-\sqrt{3}\right)$

Solution.

1. To find $\arctan\left(\sqrt{3}\right)$, we need to find the angle θ such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = \sqrt{3}$. We know $\theta = \frac{\pi}{3}$ meets these criteria, so $\arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$.
2. We let $\operatorname{arccot}\left(-\sqrt{3}\right) = \theta$. Then, we need to find the angle θ, $0 < \theta < \pi$, with $\cot(\theta) = -\sqrt{3}$. We know $\theta = \frac{5\pi}{6}$ meets these criteria, so $\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$.

□

Properties of the Inverse Tangent and Cotangent Functions

Noting that $y = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $y = \arctan(x)$, $-\infty < x < \infty$, are inverse functions, as well as $y = \cot(x)$, $0 < x < \pi$, and $y = \operatorname{arccot}(x)$, $-\infty < x < \infty$, we have the following properties.

Inverse Properties of the Arctangent and Arccotangent Functions

- $\tan(\arctan(x)) = x$ for all real numbers x
- $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
- $\arctan(\tan(x)) = x$ provided $-\frac{\pi}{2} < x < \frac{\pi}{2}$
- $\operatorname{arccot}(\cot(x)) = x$ provided $0 < x < \pi$

Example 4.2.2. Find the exact values of the following.

1. $\cot\left(\operatorname{arccot}(-5)\right)$

2. $\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)$

Solution.

1. Since $\cot\left(\operatorname{arccot}(x)\right)=x$ for all real values of x, we have $\cot\left(\operatorname{arccot}(-5)\right)=-5$.

   For a deeper understanding, we solve this problem a second way, using the definition of the inverse cotangent function. If we let $\theta=\operatorname{arccot}(-5)$, then $0<\theta<\pi$ and $\cot(\theta)=-5$. Hence,

$$\begin{aligned}\cot\left(\operatorname{arccot}(-5)\right)&=\cot(\theta)\\&=-5\end{aligned}$$

2. We start by letting $\theta=\tan^{-1}\left(-\frac{3}{4}\right)$, from which $\tan(\theta)=-\frac{3}{4}$ with $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Since $\tan(\theta)<0$, we know in fact that $-\frac{\pi}{2}<\theta<0$ and have $\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)=\sin(\theta)$. We proceed by using $\tan(\theta)=-\frac{3}{4}$ to determine $\sin(\theta)$.

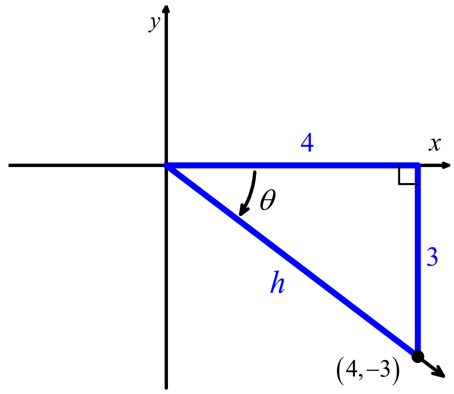
   - One way to find the value of $\sin(\theta)$ is to use the Pythagorean identity $1+\cot^2(\theta)=\csc^2(\theta)$ since this relates the reciprocals of $\tan(\theta)$ and $\sin(\theta)$, and is valid for $-\frac{\pi}{2}<\theta<0$.

$$\begin{aligned}1+\cot^2(\theta)&=\csc^2(\theta)\\1+\left(-\frac{4}{3}\right)^2&=\csc^2(\theta)\quad\text{from }\tan(\theta)=-\frac{3}{4}\\\frac{25}{9}&=\csc^2(\theta)\\\csc(\theta)&=\pm\frac{5}{3}\end{aligned}$$

     With $-\frac{\pi}{2}<\theta<0$, we must have $\csc(\theta)=-\frac{5}{3}$ so that $\sin(\theta)=-\frac{3}{5}$.

   - Another approach is to use geometry to determine $\sin(\theta)$. With $\tan(\theta)=-\frac{3}{4}$ and $-\frac{\pi}{2}<\theta<0$, we place a reference triangle in Quadrant IV. We label the opposite side with length 3 and the adjacent side with length 4.

Figure 4.2. 5



The Pythagorean Theorem can be used to find the length h of the hypotenuse.

$$\begin{aligned} 3^2+4^2 &= h^2 \\ 25 &= h^2 \\ h &= 5 \quad \text{As a length, } h \text{ is positive.} \end{aligned}$$

Since sine is negative in Quadrant IV, $\sin(\theta) = -\frac{3}{5}$.

Using either of the above methods, we get $\sin(\theta) = -\frac{3}{5}$, so that

$$\begin{aligned} \sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right) &= \sin(\theta) \\ &= -\frac{3}{5} \end{aligned}$$

□

Before moving on to inverse secant and cosecant functions, we have one last example of compositions involving inverse tangent and cotangent functions; this time with arguments containing a variable.

Example 4.2.3. Rewrite the following as algebraic expressions of x and state the values of x for which they hold.

1. $\tan(2\arctan(x))$
2. $\cos(\cot^{-1}(2x))$

Solution.

1. Let $\theta = \arctan(x)$. Then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = x$, so that $\tan(2\arctan(x)) = \tan(2\theta)$.

   Using the double-angle identity for tangent, we get

$$\begin{aligned}\tan\left(2\arctan(x)\right) &= \tan(2\theta)\\ &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}\\ &= \frac{2x}{1-x^2}\end{aligned}$$

We note that this fraction is not defined if $1-x^2=0$, which occurs when $x=\pm1$. Hence, the equivalence $\tan\left(2\arctan(x)\right)=\frac{2x}{1-x^2}$ holds for all x in $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

2. To write $\cos\left(\cot^{-1}(2x)\right)$ as an algebraic expression of x, we first let $\theta=\cot^{-1}(2x)$ so that $\cos\left(\cot^{-1}(2x)\right)=\cos(\theta)$. Then $0<\theta<\pi$ and $\cot(\theta)=2x$.

   We next use the identity $\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$, rewriting it as $\cos(\theta)=\cot(\theta)\sin(\theta)$ to get

$$\begin{aligned}\cos\left(\cot^{-1}(2x)\right) &= \cos(\theta)\\ &= \cot(\theta)\sin(\theta)\\ &= 2x\sin(\theta)\end{aligned}$$

   We still need $\sin(\theta)$ in terms of x. Since cosecant is the reciprocal of sine, and we know $\cot(\theta)=2x$, we use the Pythagorean identity $1+\cot^2(\theta)=\csc^2(\theta)$ to rewrite $\sin(\theta)$ in terms of x.

$$\begin{aligned}1+\cot^2(\theta) &= \csc^2(\theta)\\ 1+(2x)^2 &= \csc^2(\theta)\\ \csc(\theta) &= \pm\sqrt{4x^2+1}\end{aligned}$$

   Since θ is between 0 and π, $\csc(\theta)>0$. Thus, $\csc(\theta)=\sqrt{4x^2+1}$ and $\sin(\theta)=\frac{1}{\sqrt{4x^2+1}}$.

   Finally,

$$\begin{aligned}\cos\left(\cot^{-1}(2x)\right) &= 2x\sin(\theta)\\ &= \frac{2x}{\sqrt{4x^2+1}}\end{aligned}$$

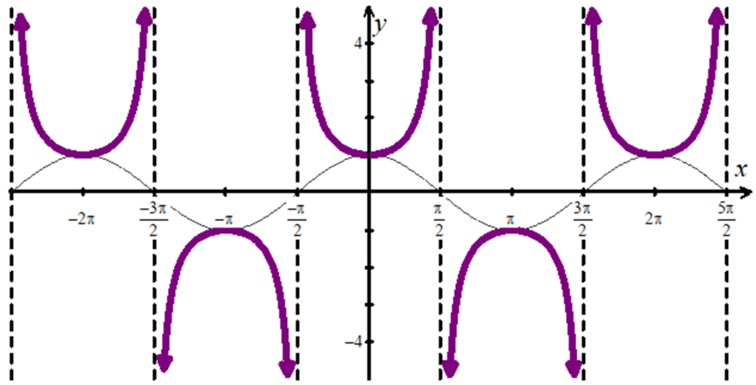
   This fraction is defined for all values of x, so $\cos\left(\cot^{-1}(2x)\right)=\frac{2x}{\sqrt{4x^2+1}}$ for all real numbers x.

□

The Inverse Secant Function

A portion of the graph of the secant function follows.

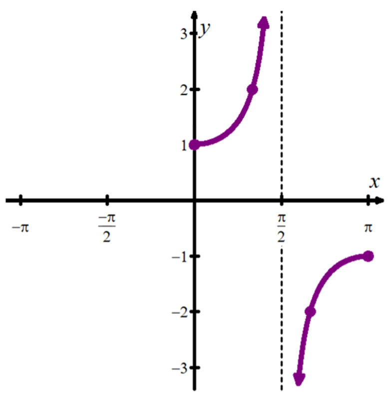
Figure 4.2. 6



$y = \sec(x)$

For the secant, no single continuous piece of its graph covers its entire range of $(-\infty,-1] \cup [1,\infty)$. Thus, to achieve a one-to-one function, we settle for a piecewise approach in which we choose one piece to cover the top of the range, $[1,\infty)$, and another piece to cover the bottom, $(-\infty,-1]$. We restrict the domain to $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$, corresponding to that of cosine, and reflect the graph of $f(x) = \sec(x)$ about the line $y = x$ to obtain the graph of $f^{-1}(x) = \sec^{-1}(x)$, read 'inverse secant of x', or $f^{-1}(x) = \operatorname{arcsec}(x)$, read 'arc-secant of x'.

Figure 4.2. 7

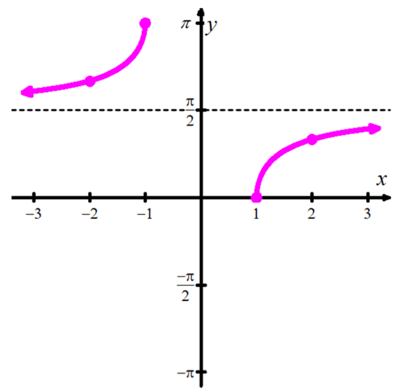


$f(x) = \sec(x)$ on $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$

reflect across $y = x$

switch *x*- and *y*-coordinates

Figure 4.2. 8



$f^{-1}(x) = \operatorname{arcsec}(x)$, $|x| \geq 1$

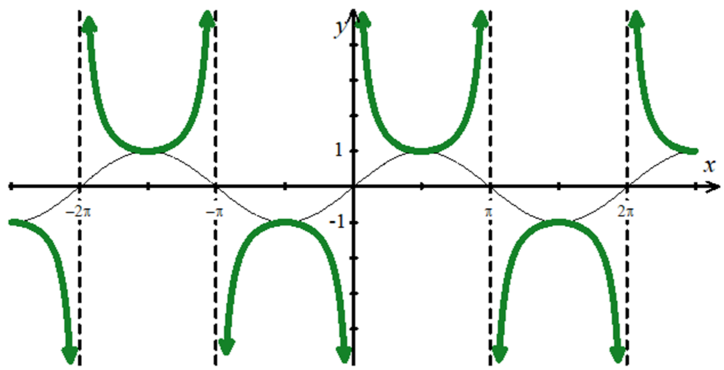
Note that the marked points $(0,1)$, $\left(\frac{\pi}{3},2\right)$, $\left(\frac{2\pi}{3},-2\right)$, and $(\pi,-1)$ on the graph of $f(x) = \sec(x)$, after swapping *x*- and *y*-coordinates to reflect the graph of $f(x)$ about the line $y = x$, become the marked points $(1,0)$, $\left(2,\frac{\pi}{3}\right)$, $\left(-2,\frac{2\pi}{3}\right)$, and $(-1,\pi)$, respectively, on the graph of $f^{-1}(x)=\text{arcsec}(x)$.[6] What observations can you make regarding the asymptotes?

For both inverse secant and inverse cosecant functions, we forego posting a table of values, as we have done for the other four inverse functions, since we often use sine and cosine to determine values of these functions. For extra practice, you might try creating such a table yourself.

The Inverse Cosecant Function

Following is a portion of the graph of the cosecant function.

Figure 4.2. 9

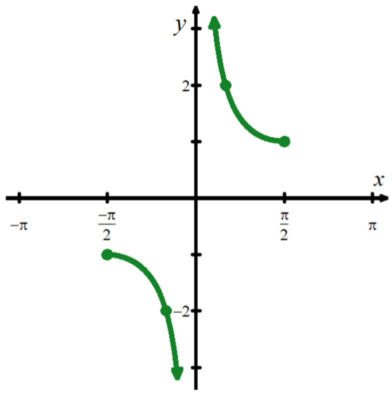


$y=\csc(x)$

To obtain a one-to-one function for determining the inverse, we restrict $g(x)=\csc(x)$ to $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, corresponding to restrictions for the sine, and reflect about the line $y=x$ to obtain $g^{-1}(x)=\csc^{-1}(x)$, read 'inverse cosecant of x', and also referred to as $\text{arccsc}(x)$, read 'arc-cosecant of x'.

[6] To avoid 'cluttering', we have not shown the secant and arcsecant functions graphed together with the line $y=x$.

Figure 4.2. 10

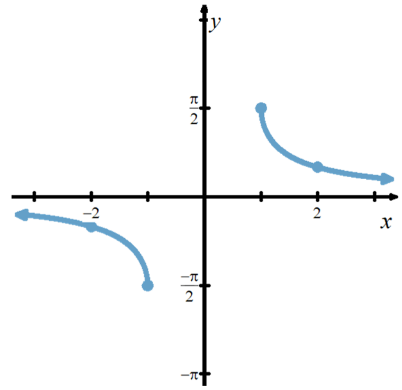


$g(x) = \csc(x)$ on $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

reflect across $y = x$

switch x- and y-coordinates

Figure 4.2. 11



$g^{-1}(x) = \text{arccsc}(x)$, $|x| \geq 1$

Note the marked points on each graph, along with asymptotes. What observations can you make about the locations of points and asymptotes in moving from one graph to the next? Also note that the domain of both the arcsecant and arccosecant functions is $(-\infty, -1] \cup [1, \infty)$, and is often written as $\{x : |x| \geq 1\}$.

Finding Values of Inverse Secant and Inverse Cosecant Functions

In finding values of inverse secant (arcsecant) and inverse cosecant (arccosecant) functions, domains and ranges are critical, as are the following definitions.

Definition 4.3. The Arcsecant and Arccosecant Functions:

- For $x \leq -1$ or $x \geq 1$, $\text{arcsec}(x)$ is the angle θ such that $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$ and $\sec(\theta) = x$.

  The function $y = \text{arcsec}(x)$ has domain $(-\infty, -1] \cup [1, \infty)$ and range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

- For $x \leq -1$ or $x \geq 1$, $\text{arccsc}(x)$ is the angle θ such that $-\frac{\pi}{2} \leq \theta < 0$ or $0 < \theta \leq \frac{\pi}{2}$ and $\csc(\theta) = x$.

  The function $y = \text{arccsc}(x)$ has domain $(-\infty, -1] \cup [1, \infty)$ and range $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

Example 4.2.4. Find the exact values of the following.

1. $\text{arcsec}(2)$
2. $\csc^{-1}(-2)$

Solution.

1. To find $\operatorname{arcsec}(2)$, we look for an angle θ such that $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$ and $\sec(\theta) = 2$. It may help us out to convert to cosine, as follows:

$$\sec(\theta) = 2$$
$$\frac{1}{\cos(\theta)} = 2 \quad \text{reciprocal identity}$$
$$\cos(\theta) = \frac{1}{2}$$

Since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, and $0 \le \frac{\pi}{3} < \frac{\pi}{2}$, we find $\theta = \frac{\pi}{3}$. Then, $\operatorname{arcsec}(2) = \theta = \frac{\pi}{3}$.

2. We let $\csc^{-1}(-2) = \theta$, so that $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$ and $\csc(\theta) = -2$. Again, we use a reciprocal identity, this time to convert to sine:

$$\csc(\theta) = -2$$
$$\frac{1}{\sin(\theta)} = -2$$
$$\sin(\theta) = -\frac{1}{2}$$

The value of θ that works here is $-\frac{\pi}{6}$ since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{2} \le -\frac{\pi}{6} < 0$. Thus, we have $\csc^{-1}(-2) = -\frac{\pi}{6}$.

□

Properties of the Inverse Secant and Inverse Cosecant Functions

Since $y = \sec(x)$, $0 \le x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \le \pi$, and $y = \operatorname{arcsec}(x)$, $|x| \ge 1$, are inverse functions, as well as $y = \csc(x)$, $-\frac{\pi}{2} \le x < 0$ or $0 < x \le \frac{\pi}{2}$, and $y = \operatorname{arccsc}(x)$, $|x| \ge 1$, we have the following properties.

Inverse Properties of the Arcsecant and Arccosecant Functions

- $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \ge 1$
- $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \ge 1$
- $\operatorname{arcsec}(\sec(x)) = x$ provided $0 \le x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \le \pi$
- $\operatorname{arccsc}(\csc(x)) = x$ provided $-\frac{\pi}{2} \le x < 0$ or $0 < x \le \frac{\pi}{2}$

Example 4.2.5. Find the exact values of the following.

1. $\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right)$

2. $\cot\left(\operatorname{arccsc}(-3)\right)$

Solution.

1. Since $\frac{5\pi}{4}$ does not fall between 0 and $\frac{\pi}{2}$, or between $\frac{\pi}{2}$ and π, we cannot use the inverse property stated above. Instead, we start by evaluating the expression:

$$\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right)=\sec^{-1}\left(-\sqrt{2}\right)$$

If we let $\theta=\sec^{-1}\left(-\sqrt{2}\right)$ then $\sec(\theta)=-\sqrt{2}$ and $0\le\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$. Since $\cos\left(\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}$ and $\frac{\pi}{2}<\frac{3\pi}{4}<\pi$, we have $\theta=\frac{3\pi}{4}$. Thus,

$$\begin{aligned}\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right)&=\sec^{-1}\left(-\sqrt{2}\right)\\&=\theta\\&=\frac{3\pi}{4}\end{aligned}$$

2. To simplify $\cot\left(\operatorname{arccsc}(-3)\right)$, we let $\theta=\operatorname{arccsc}(-3)$. Then $\csc(\theta)=-3$ and, being negative, θ lies in the interval $\left[-\frac{\pi}{2},0\right)$. We next find the value of $\cot(\theta)$ by applying a Pythagorean identity.

$$\begin{aligned}1+\cot^2(\theta)&=\csc^2(\theta) && \text{Pythagorean identity}\\1+\cot^2(\theta)&=(-3)^2\\\cot^2(\theta)&=8\\\cot(\theta)&=\pm\sqrt{8}=\pm2\sqrt{2}\end{aligned}$$

Since $-\frac{\pi}{2}\le\theta<0$, we note that $\cot(\theta)<0$ and have $\cot\left(\operatorname{arccsc}(-3)\right)=\cot(\theta)=-2\sqrt{2}$.

□

As we did with inverse tangent and cotangent functions, we finish off the inverse secant and cosecant functions by looking at compositions that contain a variable.

Example 4.2.6. Rewrite the following as algebraic expressions of x and state the values of x for which they hold.

1. $\tan(\operatorname{arcsec}(x))$
2. $\cos(\csc^{-1}(4x))$

Solution.

1. To write $\tan(\operatorname{arcsec}(x))$ as an algebraic expression of x, we begin by letting $\theta = \operatorname{arcsec}(x)$.

   Then $\sec(\theta) = x$ for θ in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and we have $\tan(\operatorname{arcsec}(x)) = \tan(\theta)$.

   To relate $\sec(\theta)$ to $\tan(\theta)$, we use a Pythagorean identity.

$$\begin{aligned} \tan^2(\theta) + 1 &= \sec^2(\theta) \qquad \text{Pythagorean identity} \\ \tan^2(\theta) + 1 &= x^2 \\ \tan(\theta) &= \pm\sqrt{x^2 - 1} \end{aligned}$$

   Either case may hold, depending on the value of the angle θ. If θ lies in $\left[0, \frac{\pi}{2}\right)$ then $\tan(\theta) \geq 0$; if, on the other hand, θ lies in $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(\theta) \leq 0$. As a result, we get a piecewise defined function for $\tan(\theta)$.

$$\tan(\theta) = \begin{cases} \sqrt{x^2 - 1}, \text{ if } 0 \leq \theta < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, \text{ if } \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

   Now, we need to determine what these conditions on θ mean for $x = \sec(\theta)$. When $0 \leq \theta < \frac{\pi}{2}$, then $x \geq 1$; when $\frac{\pi}{2} < \theta \leq \pi$, we find $x \leq -1$. Therefore,

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2 - 1}, \text{ if } x \geq 1 \\ -\sqrt{x^2 - 1}, \text{ if } x \leq -1 \end{cases}$$

2. To simplify $\cos(\csc^{-1}(4x))$, we start by letting $\theta = \csc^{-1}(4x)$. Then $\csc(\theta) = 4x$ for θ in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, and we have $\cos(\csc^{-1}(4x)) = \cos(\theta)$. From $\csc(\theta) = 4x$, we get $\sin(\theta) = \frac{1}{4x}$. Then, to find $\cos(\theta)$, we again use a Pythagorean identity.

$$\sin^2(\theta)+\cos^2(\theta)=1 \qquad \text{Pythagorean identity}$$
$$\left(\frac{1}{4x}\right)^2+\cos^2(\theta)=1$$
$$\cos^2(\theta)=1-\frac{1}{16x^2}$$
$$\cos^2(\theta)=\frac{16x^2-1}{16x^2}$$
$$\cos(\theta)=\pm\sqrt{\frac{16x^2-1}{16x^2}}$$

Recalling that $\sqrt{x^2}=|x|$, we get $\cos(\theta)=\pm\frac{\sqrt{16x^2-1}}{4|x|}$. Since θ belongs to $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, we know $\cos(\theta)\geq 0$ and so $\cos(\theta)=\frac{\sqrt{16x^2-1}}{4|x|}$. Hence, $\cos\left(\csc^{-1}(4x)\right)=\frac{\sqrt{16x^2-1}}{4|x|}$. (The absolute value here is necessary since x could be negative, as will be verified shortly.)

To find the values of x for which this equality holds, recall that $\csc^{-1}(4x)$ is defined only when $|4x|\geq 1$. We solve this inequality for x as follows.

$$|4x|\geq 1$$
$$4x\leq -1 \text{ or } 4x\geq 1$$
$$x\leq -\frac{1}{4} \text{ or } x\geq \frac{1}{4}$$

Thus, $\cos\left(\csc^{-1}(4x)\right)=\frac{\sqrt{16x^2-1}}{4|x|}$ holds for all x in $\left(-\infty,-\frac{1}{4}\right]\cup\left[\frac{1}{4},\infty\right)$.

□

4.2 Exercises

In Exercises 1 – 30, find the exact value in radians or state that it is undefined.

1. $\arctan\left(-\sqrt{3}\right)$
2. $\tan^{-1}\left(-1\right)$
3. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
4. $\arctan\left(0\right)$
5. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
6. $\tan^{-1}\left(1\right)$
7. $\tan^{-1}\left(\sqrt{3}\right)$
8. $\cot^{-1}\left(-\sqrt{3}\right)$
9. $\operatorname{arccot}\left(-1\right)$
10. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
11. $\cot^{-1}\left(0\right)$
12. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right)$
13. $\operatorname{arccot}\left(1\right)$
14. $\operatorname{arccot}\left(\sqrt{3}\right)$
15. $\operatorname{arcsec}\left(2\right)$
16. $\csc^{-1}\left(2\right)$
17. $\sec^{-1}\left(\sqrt{2}\right)$
18. $\csc^{-1}\left(\sqrt{2}\right)$
19. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
20. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
21. $\sec^{-1}\left(1\right)$
22. $\operatorname{arccsc}\left(1\right)$
23. $\operatorname{arcsec}\left(-2\right)$
24. $\sec^{-1}\left(-\sqrt{2}\right)$
25. $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$
26. $\sec^{-1}\left(-1\right)$
27. $\csc^{-1}\left(-2\right)$
28. $\operatorname{arccsc}\left(-\sqrt{2}\right)$
29. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
30. $\operatorname{arccsc}\left(-1\right)$

In Exercises 31 – 72, find the exact value or state that it is undefined.

31. $\tan\left(\tan^{-1}\left(-1\right)\right)$
32. $\tan\left(\tan^{-1}\left(\sqrt{3}\right)\right)$
33. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$
34. $\tan\left(\arctan\left(0.965\right)\right)$
35. $\tan\left(\tan^{-1}\left(3\pi\right)\right)$
36. $\cot\left(\operatorname{arccot}\left(1\right)\right)$
37. $\cot\left(\cot^{-1}\left(-\sqrt{3}\right)\right)$
38. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right)$
39. $\cot\left(\cot^{-1}\left(-0.001\right)\right)$

40. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right)$

41. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

42. $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

43. $\tan^{-1}\left(\tan\left(\pi\right)\right)$

44. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

45. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$

46. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

47. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

48. $\operatorname{arccot}\left(\cot\left(\pi\right)\right)$

49. $\cot^{-1}\left(\cot\left(\frac{\pi}{2}\right)\right)$

50. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

51. $\sec\left(\sec^{-1}\left(2\right)\right)$

52. $\sec\left(\operatorname{arcsec}\left(-1\right)\right)$

53. $\sec\left(\sec^{-1}\left(\frac{1}{2}\right)\right)$

54. $\sec\left(\sec^{-1}\left(0.75\right)\right)$

55. $\sec\left(\operatorname{arcsec}\left(117\pi\right)\right)$

56. $\csc\left(\csc^{-1}\left(\sqrt{2}\right)\right)$

57. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$

58. $\csc\left(\csc^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

59. $\csc\left(\operatorname{arccsc}\left(1.0001\right)\right)$

60. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$

61. $\sec^{-1}\left(\sec\left(\frac{\pi}{4}\right)\right)$

62. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

63. $\sec^{-1}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

64. $\sec^{-1}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

65. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

66. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

67. $\csc^{-1}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

68. $\csc^{-1}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

69. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

70. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

71. $\sec^{-1}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

72. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 73 – 100, find the exact value or state that it is undefined.

73. $\sin\left(\arctan\left(-2\right)\right)$

74. $\sin\left(\cot^{-1}\left(\sqrt{5}\right)\right)$

75. $\cos\left(\arctan\left(7\right)\right)$

76. $\cos\left(\cot^{-1}\left(3\right)\right)$

77. $\tan\left(\sin^{-1}\left(-\frac{2\sqrt{5}}{5}\right)\right)$

78. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

79. $\tan\left(\operatorname{arccot}(12)\right)$

80. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

81. $\cot\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

82. $\cot\left(\tan^{-1}(0.25)\right)$

83. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$

84. $\sin\left(2\tan^{-1}(2)\right)$

85. $\cot\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

86. $\sin\left(\frac{\arctan(2)}{2}\right)$

87. $\sin\left(\csc^{-1}(-3)\right)$

88. $\cos\left(\sec^{-1}(5)\right)$

89. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

90. $\cot\left(\csc^{-1}\left(\sqrt{5}\right)\right)$

91. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

92. $\sec\left(\sin^{-1}\left(-\frac{12}{13}\right)\right)$

93. $\sec\left(\arctan(10)\right)$

94. $\sec\left(\cot^{-1}\left(-\frac{\sqrt{10}}{10}\right)\right)$

95. $\csc\left(\operatorname{arccot}(9)\right)$

96. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

97. $\csc\left(\tan^{-1}\left(-\frac{2}{3}\right)\right)$

98. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

99. $\sin\left(2\csc^{-1}\left(\frac{13}{5}\right)\right)$

100. $\cos\left(2\sec^{-1}\left(\frac{25}{7}\right)\right)$

In Exercises 101 – 110, rewrite the quantities as equivalent algebraic expressions of x and state the values of x for which the equivalence holds.

101. $\cos\left(\tan^{-1}(x)\right)$

102. $\sin\left(2\tan^{-1}(x)\right)$

103. $\cos\left(2\arctan(x)\right)$

104. $\cos\left(\tan^{-1}(3x)\right)$

105. $\sec\left(\arctan(x)\right)$

106. $\csc\left(\arccos(x)\right)$

107. $\cos\left(\arcsin(x)+\arctan(x)\right)$

108. $\tan\left(2\sin^{-1}(x)\right)$

109. $\sin\left(\frac{1}{2}\arctan(x)\right)$

110. $\sec\left(\arctan(2x)\right)\tan\left(\arctan(2x)\right)$

111. For $\tan(\theta)=\frac{x}{7}$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, verify the identity $\frac{1}{2}\theta-\frac{1}{2}\sin(2\theta)=\frac{1}{2}\arctan\left(\frac{x}{7}\right)-\frac{7x}{x^2+49}$.

112. For $\sec(\theta)=\frac{x}{4}$, $0<\theta<\frac{\pi}{2}$, verify the identity $4\tan(\theta)-4\theta=\sqrt{x^2-16}-4\arccos\left(\frac{4}{x}\right)$.

4.3 Inverse Trigonometric Functions and Trigonometric Equations

Learning Objectives

- **Use technology to find approximate values of inverse trigonometric functions.**
- **Find domains of inverse trigonometric functions.**
- **Use inverse trigonometric functions to solve applications.**
- **Use inverse trigonometric functions to solve for angles in right triangles.**
- **Use inverse trigonometric functions to solve trigonometric equations.**
- **Find all solutions to trigonometric equations.**
- **Find solutions to trigonometric equations in a given interval.**

We begin with a look at technology (in the form of calculators) that will help us find approximate values of inverse trigonometric functions. In the next few sections this skill will prove useful, particularly in allowing us to use inverse trigonometric functions in solving applications.

Using a Calculator to Find Inverse Trigonometric Function Values

On most calculators, only the arcsine, arccosine, and arctangent functions are available, and they are labeled as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, respectively. If we are asked to find an arccotangent, arcsecant, or arccosecant value, we may need to employ some ingenuity, as the next example illustrates.

Example 4.3.1. Use a calculator to approximate the following values to four decimal places.

1. $\operatorname{arccot}(2)$ 2. $\sec^{-1}(5)$ 3. $\cot^{-1}(-2)$ 4. $\operatorname{arccsc}\left(-\frac{3}{2}\right)$

Solution.

1. We will need to change $\operatorname{arccot}(2)$ to arctangent so that we can use the inverse tangent button on a calculator. We let $\operatorname{arccot}(2)=\theta$; then $0<\theta<\pi$ and $\cot(\theta)=2$. Since $\cot(\theta)>0$, θ must be in Quadrant I. It follows that

$$\frac{1}{\tan(\theta)}=2 \quad \text{Cotangent is reciprocal of tangent.}$$

$$\tan(\theta)=\frac{1}{2}$$

Since θ is in Quadrant I, $\theta = \arctan\left(\frac{1}{2}\right)$. After verifying our calculator is in radian mode, we find $\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636$. So $\theta = \operatorname{arccot}(2) \approx 0.4636$ radians.

2. To evaluate $\sec^{-1}(5)$, we will restate the problem so we can use the button for inverse cosine on a calculator. If we let $\sec^{-1}(5) = \theta$, then $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$ and $\sec(\theta) = 5$, from which

$$\frac{1}{\cos(\theta)} = 5 \quad \text{Secant is reciprocal of cosine.}$$
$$\cos(\theta) = \frac{1}{5}$$

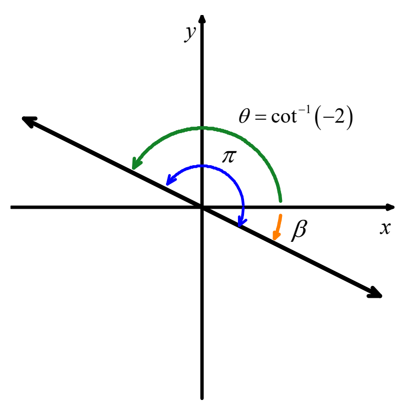
Since $\sec(\theta) > 0$, θ must be in Quadrant I. Thus, $\theta = \cos^{-1}\left(\frac{1}{5}\right)$. Using a calculator, $\theta \approx 1.3694$ or $\sec^{-1}(5) \approx 1.3694$ radians.

3. For $\cot^{-1}(-2)$, we use the inverse tangent and begin by letting $\cot^{-1}(-2) = \theta$. Then $0 < \theta < \pi$ and $\cot(\theta) = -2$. Since $\cot(\theta) < 0$, θ must be in the second quadrant; $\frac{\pi}{2} < \theta < \pi$. Also,

$$\frac{1}{\tan(\theta)} = -2 \quad \text{Cotangent is reciprocal of tangent.}$$
$$\tan(\theta) = -\frac{1}{2}$$

We must resolve the issue that $\tan^{-1}\left(-\frac{1}{2}\right)$ is in Quadrant IV while $\cot^{-1}(-2)$ is in Quadrant II. Our final answer needs to be in Quadrant II. We let $\beta = \tan^{-1}\left(-\frac{1}{2}\right)$; then $\tan(\beta) = -\frac{1}{2}$, $-\frac{\pi}{2} < \beta < 0$. Noting that the period of the tangent function is π and $\tan(\theta) = \tan(\beta)$, the angles θ and β are exactly π units apart. It follows that $\theta = \pi + \beta$, as illustrated below.

Figure 4.3. 1



Using a calculator, $\theta = \pi + \tan^{-1}\left(-\frac{1}{2}\right) \approx 2.6779$. Hence, $\theta = \cot^{-1}(-2) \approx 2.6779$ radians.

4. To evaluate $\operatorname{arccsc}\left(-\frac{3}{2}\right)$, we use the inverse sine function. We let $\theta = \operatorname{arccsc}\left(-\frac{3}{2}\right)$ and have

$$\csc(\theta) = -\frac{3}{2}$$

$$\frac{1}{\sin(\theta)} = -\frac{3}{2} \quad \text{Cosecant is reciprocal of sine.}$$

$$\sin(\theta) = -\frac{2}{3}$$

For negative arguments, both inverse cosecant and inverse sine values are between $-\frac{\pi}{2}$ and 0, so $\theta = \sin^{-1}\left(-\frac{2}{3}\right)$. Using a calculator, we find $\theta \approx -0.7297$, or $\operatorname{arccsc}\left(-\frac{3}{2}\right) \approx -0.7297$ radians.

□

Domain of Inverse Trigonometric Functions

Example 4.3.2. Find the domain of the following functions.

1. $f(x) = \frac{\pi}{2}\arccos\left(\frac{x-1}{5}\right)$ 2. $g(x) = 3\tan^{-1}(4x)$ 3. $h(x) = \operatorname{arccot}\left(\frac{2}{x}\right) + \pi$

Solution.

1. Since the domain of $y = \arccos(x)$ is $-1 \le x \le 1$, to find the domain of $f(x) = \frac{\pi}{2}\arccos\left(\frac{x-1}{5}\right)$, we set the argument of the arccosine, in this case $\frac{x-1}{5}$, to be between -1 and 1, inclusive.

$$-1 \le \frac{x-1}{5} \le 1$$

$$(-1)(5) \le \left(\frac{x-1}{5}\right)(5) \le (1)(5)$$

$$-5 \le x-1 \le 5$$

$$-5+1 \le x \le 5+1$$

$$-4 \le x \le 6$$

Thus, the domain of $f(x) = \frac{\pi}{2}\arccos\left(\frac{x-1}{5}\right)$ is $[-4,6]$.

2. To find the domain of $g(x) = 3\tan^{-1}(4x)$, we note that the domain of $y = \tan^{-1}(x)$ is the set of all real numbers. So $g(x) = 3\tan^{-1}(4x)$ is defined when $4x$ is a real number. That is true whenever x is a real number; thus the domain of g is all real numbers, or $(-\infty, \infty)$.

3. In determining the domain of $h(x)=\operatorname{arccot}\left(\frac{2}{x}\right)+\pi$, we start with the domain of $y=\operatorname{arccot}(x)$, which is all real numbers. The only restrictions on the domain of $h(x)=\operatorname{arccot}\left(\frac{2}{x}\right)+\pi$ result from the argument, $\frac{2}{x}$, having an x in its denominator. We will need to exclude 0 from the domain of h, with the result that the domain of $h(x)=\operatorname{arccot}\left(\frac{2}{x}\right)+\pi$ is $(-\infty,0)\cup(0,\infty)$.

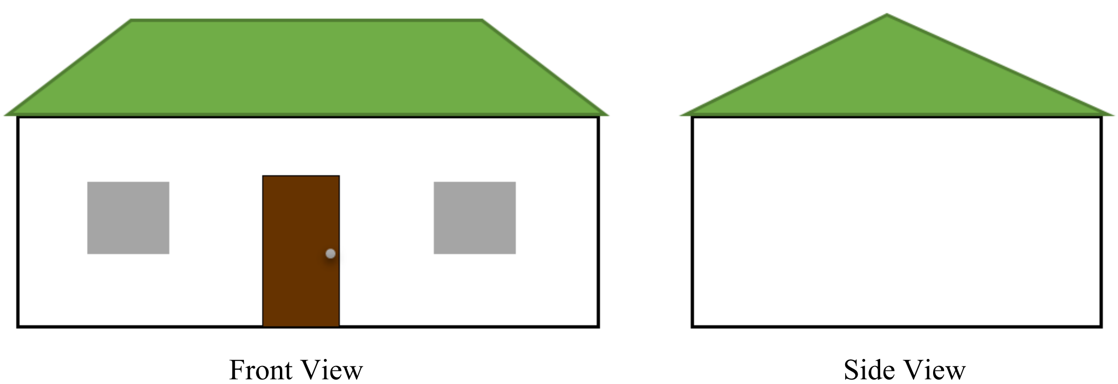
□

Applications of Inverse Trigonometric Functions

The inverse trigonometric functions are typically found in applications where the measure of an angle is required. One such scenario is presented in the following example.

Example 4.3.3.[7] The roof on the house in the following sketch has a 6/12 pitch. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of elevation from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.

Figure 4.3. 2

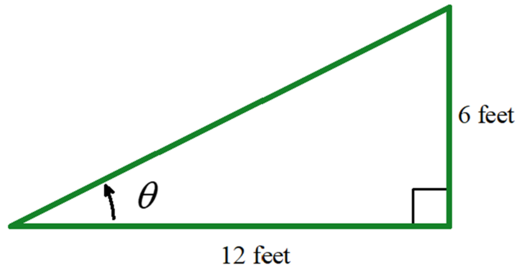


Solution. From the side view of the roof, we can create a right triangle in which the roof line forms the hypotenuse, and the legs are of lengths 6 feet and 12 feet. Using trigonometric functions of right triangles, we find that the angle of elevation, labeled θ in the following diagram, satisfies

$$\tan(\theta)=\frac{6}{12}=\frac{1}{2}.$$

[7] Thanks to Dan Stitz for this problem.

Figure 4.3. 3



Since $\tan(\theta)=\frac{1}{2}$, we can use the arctangent function, along with a calculator in degree mode, to determine the angle measure.

$$\theta=\arctan\left(\frac{1}{2}\right)$$
$$\theta\approx 26.57 \text{ degrees}$$

Note that the conversion from radians to degrees has already been completed by the calculator. The angle of elevation from the bottom of the roof to the top is approximately 26.57º.

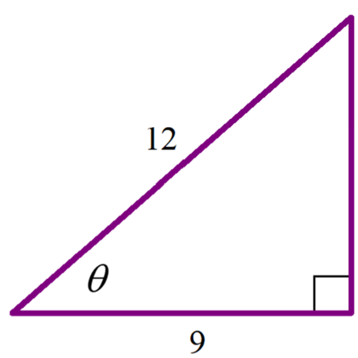
□

Solving for Angles in Right Triangles

We move on to solving for acute angles within right triangles, applying the same technique used in **Example 4.3.3**. Through inverse trigonometric functions, we can determine an angle from knowing only the value of a trigonometric function, such as cosine in the following example.

Example 4.3.4. Solve the following triangle for the angle θ.

Figure 4.3. 4



Solution. Because we know the lengths of the hypotenuse and the side adjacent to the acute angle θ, it makes sense for us to use the cosine function.

$$\cos(\theta)=\frac{9}{12}$$
$$\theta=\arccos\left(\frac{9}{12}\right) \quad \text{since } 0\le\theta\le\pi$$
$$\theta\approx 41.4^{\circ}$$

The angle θ is approximately 41.4 degrees.

□

Knowing the measure of one acute angle in a right triangle, we can easily determine the measure of the second acute angle. In the previous example, the measure of the angle opposite the side of length 9 is approximately $180^\circ - 90^\circ - 41.4^\circ = 48.6^\circ$. Note that the exact measure in radians is $\arcsin\left(\frac{9}{12}\right)$.

Solving Trigonometric Equations

In **Section 1.4**, we learned to solve equations like $\sin(\theta) = \frac{1}{2}$ or $\tan(t) = -1$. In each case, there was a standard angle with the given trigonometric function value. We used periodicity of trigonometric functions to find all such angles. However, no standard angle has the property that $\sin(\theta) = \frac{1}{3}$ or $\tan(t) = -2$. We will solve such equations using inverse trigonometric functions.

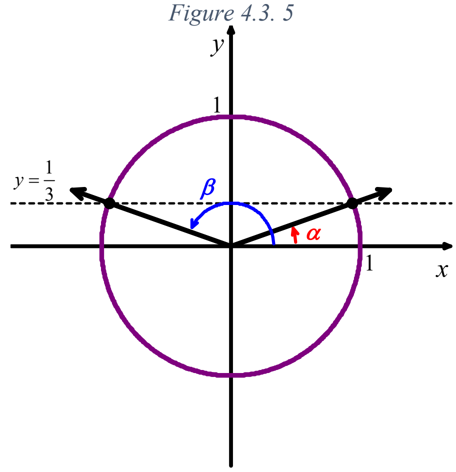
Example 4.3.5.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.
2. Find all angles θ for which $\tan(\theta) = -2$.
3. Solve $\sec(x) = -\frac{5}{3}$ for x.

Solution.

1. Let θ be any angle that satisfies $\sin(\theta) = \frac{1}{3}$. Then for θ in standard position, the y-coordinate of the point of intersection of its terminal side with the Unit Circle is $\frac{1}{3}$.

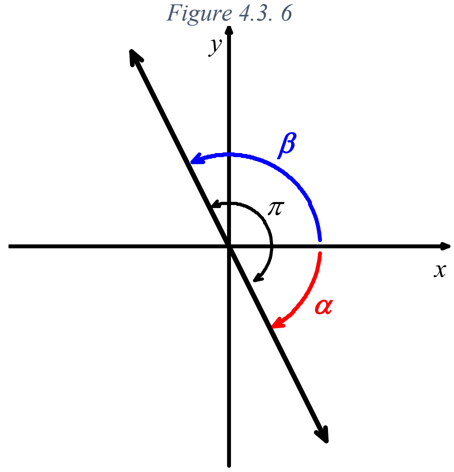
Figure 4.3. 5



As noted in **Figure 4.3.5**, one such angle is the acute angle $\alpha = \arcsin\left(\frac{1}{3}\right)$, and the second angle is $\beta = \pi - \alpha = \pi - \arcsin\left(\frac{1}{3}\right)$. All solutions to the equation $\sin(\theta) = \frac{1}{3}$ are angles with the same terminal side as the angle α or β. That is, $\theta = \alpha + 2\pi k = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ or $\theta = \beta + 2\pi k = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$, for any integer k.

2. Let θ be any angle that satisfies $\tan(\theta) = -2$. Then one such angle is $\alpha = \tan^{-1}(-2)$, which is in the fourth quadrant, $-\frac{\pi}{2} < \alpha < 0$, since $\tan(\alpha) = -2 < 0$. Noting that the period of the tangent function is π, another such angle is $\beta = \pi + \alpha$.

Figure 4.3. 6



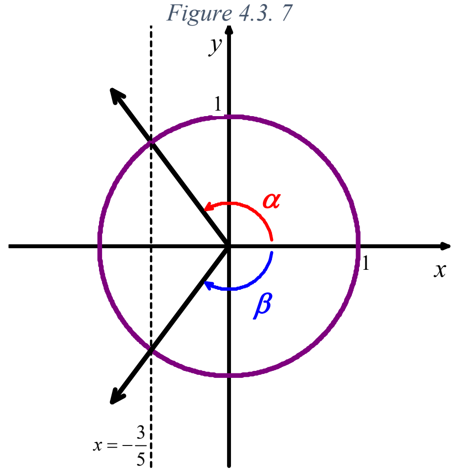
All solutions to the equation $\tan(\theta) = -2$ are angles with the same terminal side as α or β. That is, $\theta = \alpha + 2\pi k$ or $\theta = \beta + 2\pi k$, for all integers k. Due to the fact that $\beta = \pi + \alpha$, we can write the answer in the simpler form of

$$\begin{aligned} \theta &= \alpha + \pi k \\ &= \tan^{-1}(-2) + \pi k \end{aligned}$$

Thus, the solutions to $\tan(\theta) = -2$ are $\theta = \tan^{-1}(-2) + \pi k$, for any integer k.

3. Let x be any angle that satisfies $\sec(x) = -\frac{5}{3}$. Then $\cos(x) = -\frac{3}{5}$ and for x in standard position, the x-coordinate of the point of intersection of the terminal side of x with the Unit Circle is $-\frac{3}{5}$.

Figure 4.3. 7



From the above graph, one such angle is $\alpha = \arccos\left(-\frac{3}{5}\right)$ and a second angle is $\beta = -\alpha = -\arccos\left(-\frac{3}{5}\right)$. All solutions to the equation $\sec(x) = -\frac{5}{3}$ are angles with the same terminal side as α or β. That is, $x = \alpha + 2\pi k = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = \beta + 2\pi k = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$

□

We continue solving basic equations involving trigonometric functions. Below, we summarize techniques that were first introduced in **Section 1.4**. Note that we use the letter u as the argument of each trigonometric function for generality.

Strategies for Solving Trigonometric Equations

- To solve $\cos(u) = c$ or $\sin(u) = c$:
  1. If $-1 \le c \le 1$, first solve for u in the interval $[0, 2\pi)$; then add integer multiples of the period 2π.
  2. If $c < -1$ or $c > 1$, there are no real solutions.
- To solve $\sec(u) = c$ or $\csc(u) = c$:
  1. If $c \le -1$ or $c \ge 1$, convert to a cosine or sine equation, respectively, and solve as above.
  2. If $-1 < c < 1$, there are no real solutions.
- To solve $\tan(u) = c$ for any real number c, first solve for u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; then add integer multiples of the period π.
- To solve $\cot(u) = c$:
  1. If $c \ne 0$, convert to a tangent equation and solve as above.
  2. If $c = 0$, the solution to $\cot(u) = 0$ is $u = \frac{\pi}{2} + \pi k$, for integers k.

Example 4.3.6. Solve the equation $\sin(3x)=\frac{1}{2}$.

Solution. We begin by noting that the solutions to $\sin(x)=\frac{1}{2}$ are $x=\frac{\pi}{6}+2\pi k$ or $x=\frac{5\pi}{6}+2\pi k$, for integers k. The equation $\sin(3x)=\frac{1}{2}$ has the form of $\sin(u)=\frac{1}{2}$, so the solutions are of the form $u=\frac{\pi}{6}+2\pi k$ or $u=\frac{5\pi}{6}+2\pi k$, for integers k. Then, since u is representing $3x$,

$$3x=\frac{\pi}{6}+2\pi k \qquad \text{or} \qquad 3x=\frac{5\pi}{6}+2\pi k$$

$$\left(\frac{1}{3}\right)(3x)=\left(\frac{1}{3}\right)\left(\frac{\pi}{6}+2\pi k\right) \qquad \left(\frac{1}{3}\right)(3x)=\left(\frac{1}{3}\right)\left(\frac{5\pi}{6}+2\pi k\right)$$

$$x=\frac{\pi}{18}+\frac{2\pi}{3}k \qquad x=\frac{5\pi}{18}+\frac{2\pi}{3}k$$

Thus, $\sin(3x)=\frac{1}{2}$ has solutions $\frac{\pi}{18}+\frac{2\pi}{3}k$ and $\frac{5\pi}{18}+\frac{2\pi}{3}k$, for integers k.

□

In the remainder of this section, we look at examples of equations that contain a single trigonometric function. The solutions provide practice with, and extensions of, the technique applied in solving $\sin(3x)=\frac{1}{2}$. In addition to the general solution of each equation, we provide specific solutions that fall in the interval $[0,2\pi)$.

Equations Involving Sine or Cosine

Having solved an equation with a sine function, we move on to the cosine.

Example 4.3.7. Solve the equation $\cos(2x)=-\frac{\sqrt{3}}{2}$. State exact solutions, if any, that lie in the interval $[0,2\pi)$.

Solution. The solutions to $\cos(u)=-\frac{\sqrt{3}}{2}$ are $u=\frac{5\pi}{6}+2\pi k$ or $u=\frac{7\pi}{6}+2\pi k$, for integers k. Here $u=2x$; this means $2x=\frac{5\pi}{6}+2\pi k$ or $2x=\frac{7\pi}{6}+2\pi k$. Solving for x by dividing through by 2 gives $x=\frac{5\pi}{12}+\pi k$ or $x=\frac{7\pi}{12}+\pi k$, for any integer k.[8]

To determine which of our solutions lie in $[0,2\pi)$, we substitute the integer values $k=0,\pm1,\pm2,\ldots$ into $x=\frac{5\pi}{12}+\pi k$ and $x=\frac{7\pi}{12}+\pi k$, as shown in the following table.

[8] Don't forget to divide the $2\pi k$ by 2 as well!

k	…	−2	−1	0	1	2	…
$x=\frac{5\pi}{12}+\pi k$	…	$-\frac{19\pi}{12}$	$-\frac{7\pi}{12}$	$\frac{5\pi}{12}$	$\frac{17\pi}{12}$	$\frac{29\pi}{12}$	…
$x=\frac{7\pi}{12}+\pi k$	…	$-\frac{17\pi}{12}$	$-\frac{5\pi}{12}$	$\frac{7\pi}{12}$	$\frac{19\pi}{12}$	$\frac{31\pi}{12}$	…

The solutions in the interval $[0,2\pi)$ corresponding to $k=0$ and $k=1$ are $x=\frac{5\pi}{12},\frac{7\pi}{12},\frac{17\pi}{12},\frac{19\pi}{12}$.

□

In the preceding example, the solutions $x=\frac{5\pi}{12}+\pi k$ and $x=\frac{7\pi}{12}+\pi k$ can be checked analytically by substituting them into the left side of the original equation, $\cos(2x)=-\frac{\sqrt{3}}{2}$.

- Starting with $x=\frac{5\pi}{12}+\pi k$, we have

$$\begin{aligned}\cos\left(2\left(\frac{5\pi}{12}+\pi k\right)\right)&=\cos\left(\frac{5\pi}{6}+2\pi k\right)\\&=\cos\left(\frac{5\pi}{6}\right) && \text{since the period of cosine is } 2\pi\\&=-\frac{\sqrt{3}}{2}\end{aligned}$$

- Similarly, for $x=\frac{7\pi}{12}+\pi k$, we find

$$\begin{aligned}\cos\left(2\left(\frac{7\pi}{12}+\pi k\right)\right)&=\cos\left(\frac{7\pi}{6}+2\pi k\right)\\&=\cos\left(\frac{7\pi}{6}\right) && \text{since the period of cosine is } 2\pi\\&=-\frac{\sqrt{3}}{2}\end{aligned}$$

This confirms the solutions $x=\frac{5\pi}{12}+\pi k$ or $x=\frac{7\pi}{12}+\pi k$, for integers k.

Equations Involving Tangent or Cotangent

We look next at an equation that includes a cotangent function.

Example 4.3.8. Solve $\cot(3x)=0$, stating any exact solutions that lie in $[0,2\pi)$.

Solution. Since $\cot(3x)=0$ has the form $\cot(u)=0$, we know $u=\frac{\pi}{2}+\pi k$. So $3x=\frac{\pi}{2}+\pi k$ and solving for x yields $x=\frac{\pi}{6}+\frac{\pi}{3}k$, for any integer k. We move on to determining which of our solutions lie in $[0,2\pi)$.

k	…	−1	0	1	2	3	4	5	6	…
$x=\frac{\pi}{6}+\frac{\pi}{3}k$	…	$-\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	…

The solutions in $[0,2\pi)$ are $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2},\frac{11\pi}{6}$, corresponding to $k=0$ through $k=5$.

□

To check the solution of $x=\frac{\pi}{6}+\frac{\pi}{3}k$, we start with the left side of $\cot(3x)=0$.

$$\begin{aligned}\cot\left(3\left(\frac{\pi}{6}+\frac{\pi}{3}k\right)\right)&=\cot\left(\frac{\pi}{2}+\pi k\right)\\&=\cot\left(\frac{\pi}{2}\right) &&\text{since the period of cotangent is }\pi\\&=0\end{aligned}$$

This confirms our solutions $x=\frac{\pi}{6}+\frac{\pi}{3}k$, for integers k.

Equations Involving Secant or Cosecant

It is generally simpler to convert a secant or cosecant function within an equation to its reciprocal function before solving.

Example 4.3.9. Solve $\csc\left(\frac{1}{3}x-\pi\right)=\sqrt{2}$. State any exact solutions that are in the interval $[0,2\pi)$.

Solution. Noting that this equation has the form $\csc(u)=\sqrt{2}$, we rewrite it as $\sin(u)=\frac{1}{\sqrt{2}}$ and find $u=\frac{\pi}{4}+2\pi k$ or $u=\frac{3\pi}{4}+2\pi k$, for integers k. For this problem, u is $\left(\frac{1}{3}x-\pi\right)$, so $\frac{1}{3}x-\pi=\frac{\pi}{4}+2\pi k$ or $\frac{1}{3}x-\pi=\frac{3\pi}{4}+2\pi k$. We continue by solving the first equation for x.

$$\begin{aligned}
\frac{1}{3}x-\pi &= \frac{\pi}{4}+2\pi k \\
\frac{1}{3}x &= \frac{\pi}{4}+2\pi k+\pi \\
\frac{1}{3}x &= \frac{5\pi}{4}+2\pi k \\
(3)\left(\frac{1}{3}x\right) &= (3)\left(\frac{5\pi}{4}+2\pi k\right) \\
x &= \frac{15\pi}{4}+6\pi k
\end{aligned}$$

Solving the second equation, $\frac{1}{3}x-\pi=\frac{3\pi}{4}+2\pi k$, produces $x=\frac{21\pi}{4}+6\pi k$. Putting these two solutions together, we have $x=\frac{15\pi}{4}+6\pi k$ or $x=\frac{21\pi}{4}+6\pi k$, for integers k.

Despite the infinitely many solutions of $\csc\left(\frac{1}{3}x-\pi\right)=\sqrt{2}$, none of these solutions lie in $[0,2\pi)$. The reader is encouraged to verify this result. This problem has no solutions in the requested interval.

□

In **Example 4.3.9**, the solutions may be checked as was shown in the prior two examples. We continue with an equation that does not appear to fit the profile of equations presented thus far, but simply requires an additional step.

Example 4.3.10. Solve $\sec^2(x)=4$. State the exact solutions, if any, that are in the interval $[0,2\pi)$.

Solution. The complication in solving an equation like $\sec^2(x)=4$ comes not from the argument of secant, which is just x, but rather from the fact that secant is being squared. Thus, we begin by solving for $\sec(x)$: $\sec(x)=\pm\sqrt{4}=\pm 2$. Converting to cosine, we have $\cos(x)=\pm\frac{1}{2}$.

- For $\cos(x)=\frac{1}{2}$, the solutions are $x=\frac{\pi}{3}+2\pi k$ or $x=\frac{5\pi}{3}+2\pi k$, for integers k.
- For $\cos(x)=-\frac{1}{2}$, the solutions are $x=\frac{2\pi}{3}+2\pi k$ or $x=\frac{4\pi}{3}+2\pi k$, for integers k.

Since the angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$ differ by exactly π we can state both $x=\frac{\pi}{3}+2\pi k$ and $x=\frac{4\pi}{3}+2\pi k$ simultaneously as $x=\frac{\pi}{3}+\pi k$. Similarly, we can state $x=\frac{2\pi}{3}+2\pi k$ and $x=\frac{5\pi}{3}+2\pi k$ simultaneously as $x=\frac{2\pi}{3}+\pi k$. As a result, the solutions are $x=\frac{\pi}{3}+\pi k$ or $x=\frac{2\pi}{3}+\pi k$, for integers k.

The solutions in the interval $[0,2\pi)$ come from the values $k=0$ and $k=1$ as indicated in the following table.

k	…	-1	0	1	2	…
$x=\frac{\pi}{3}+\pi k$	…	$-\frac{2\pi}{3}$	$\frac{\pi}{3}$	$\frac{4\pi}{3}$	$\frac{7\pi}{3}$	…
$x=\frac{2\pi}{3}+\pi k$	…	$-\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{3}$	$\frac{8\pi}{3}$	…

The solutions in the interval $[0,2\pi)$ are $x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$.

□

Using Inverse Trigonometric Functions to Solve Equations

The remaining two examples of this section require inverse trigonometric functions in their solutions. Otherwise, the solution process is similar to that of prior examples.

Example 4.3.11. Solve $\tan\left(\frac{x}{2}\right)=-3$ and determine the solutions, if any, that lie in the interval $[0,2\pi)$.

Solution. The equation $\tan\left(\frac{x}{2}\right)=-3$ has the form $\tan(u)=-3$, which has solutions $u=\arctan(-3)+\pi k$. Then $\frac{x}{2}=\arctan(-3)+\pi k$, and we have $x=2\arctan(-3)+2\pi k$, for integers k.

To determine which of the solutions lie in the interval $[0,2\pi)$, we first need to get an idea of the value of $2\arctan(-3)$. While we could easily find an approximation using a calculator, we proceed analytically.

Since $-3<0$, it follows that $-\frac{\pi}{2}<\arctan(-3)<0$. Multiplying through by 2 gives $-\pi<2\arctan(-3)<0$. We are now in a position to argue which of the solutions $x=2\arctan(-3)+2\pi k$ lie in $[0,2\pi)$.

- For $k=0$, we get $x=2\arctan(-3)<0$, so we discard this answer and all answers $x=2\arctan(-3)+2\pi k$ where $k<0$.
- Next, we turn our attention to $k=1$ and get $x=2\arctan(-3)+2\pi$. Starting with the inequality $-\pi<2\arctan(-3)<0$, we add 2π and get $\pi<2\arctan(-3)+2\pi<2\pi$. This means that $x=2\arctan(-3)+2\pi$ lies in $[0,2\pi)$.

- Advancing k to 2 produces $x = 2\arctan(-3) + 4\pi$. Once again, by adding 4π, we get from $-\pi < 2\arctan(-3) < 0$ that $3\pi < 2\arctan(-3) + 4\pi < 4\pi$. Since this is outside the interval $[0, 2\pi)$, we discard $x = 2\arctan(-3) + 4\pi$ and all solutions of the form $x = 2\arctan(-3) + 2\pi k$ for $k > 2$.

In summary, the only (exact) solution of $\tan\left(\frac{x}{2}\right) = -3$ in the interval $[0, 2\pi)$ is $x = 2\arctan(-3) + 2\pi$. Note that, using a calculator, this solution is approximately equal to 3.7851.

□

Example 4.3.12. Solve $\sin(2x) = 0.87$ and find any solutions that lie in the interval $[0, 2\pi)$.

Solution. To solve $\sin(2x) = 0.87$, we first note that the equation has the form $\sin(u) = 0.87$, which has the family of solutions $u = \arcsin(0.87) + 2\pi k$ or $u = \pi - \arcsin(0.87) + 2\pi k$, for integers k. Since the argument of sine here is $2x$, we get

$$2x = \arcsin(0.87) + 2\pi k \text{ or } 2x = \pi - \arcsin(0.87) + 2\pi k$$

which gives

$$x = \frac{1}{2}\arcsin(0.87) + \pi k \text{ or } x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k \text{, for integers } k$$

To determine which of these solutions lie in $[0, 2\pi)$, we first need to get an idea of the value of $x = \frac{1}{2}\arcsin(0.87)$. Once again, we could use a calculator but choose an analytic approach.

$$
\begin{aligned}
&0 < \arcsin(0.87) < \frac{\pi}{2} && \text{by definition} \\
&0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4} && \text{after multiplying through by } \frac{1}{2}
\end{aligned}
$$

- Starting with the family of solutions $x = \frac{1}{2}\arcsin(0.87) + \pi k$, we use the same type of arguments as in the **Example 4.3.11** solution to find that only the solutions corresponding to $k = 0$ and $k = 1$ lie in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87)$ and $x = \frac{1}{2}\arcsin(0.87) + \pi$.

- Moving on to the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$, we need a better estimate of $\frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$.

$$0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4} \qquad \text{from above}$$

$$0 > -\frac{1}{2}\arcsin(0.87) > -\frac{\pi}{4} \qquad \text{multiply through by } -1$$

$$\frac{\pi}{2} > \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) > \frac{\pi}{4} \qquad \text{add } \frac{\pi}{2}$$

$$\frac{\pi}{4} < \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) < \frac{\pi}{2}$$

Proceeding with the usual arguments, we find the only solutions that lie in $[0, 2\pi)$ correspond to $k = 0$ and $k = 1$, namely $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$ and $x = \frac{3\pi}{2} - \frac{1}{2}\arcsin(0.87)$.

In all, we have four solutions to $\sin(2x) = 0.87$ in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87)$, $x = \frac{1}{2}\arcsin(0.87) + \pi$, $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$, $x = \frac{3\pi}{2} - \frac{1}{2}\arcsin(0.87)$.

□

We will solve more complex trigonometric equations in **Section 4.4**.

4.3 Exercises

In Exercises 1 – 6, use a calculator to evaluate each expression. Give answers in radians, correct to at least the nearest hundredth.

1. $\cos^{-1}(-0.4)$
2. $\arcsin(0.23)$
3. $\arccos\left(\frac{3}{5}\right)$
4. $\sin^{-1}(0.8)$
5. $\tan^{-1}(6)$
6. $\arctan(-6)$

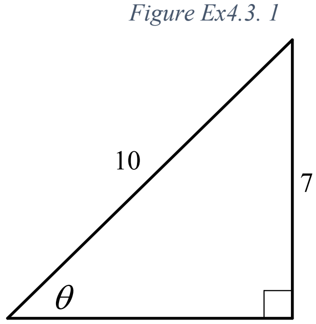
In Exercises 7 – 16, find the domain of the given function. Write your answers in interval notation.

7. $f(x)=\sin^{-1}(5x)$
8. $f(x)=\cos^{-1}\left(\frac{3x-1}{2}\right)$
9. $f(x)=\arcsin(x-3)$
10. $f(x)=\arctan(4x)$
11. $f(x)=\sec^{-1}(12x)$
12. $f(x)=\operatorname{arccsc}(x+5)$
13. $f(x)=\operatorname{arccot}(3x-2)$
14. $f(x)=\operatorname{arccot}(1-3x)$
15. $f(x)=\arccos\left(3-\frac{1}{2}x\right)$
16. $f(x)=\arctan\left(\frac{1}{x}\right)$

In Exercises 17 – 18, find the angle θ in the given right triangle. Express your answers using degree measure rounded to two decimal places.

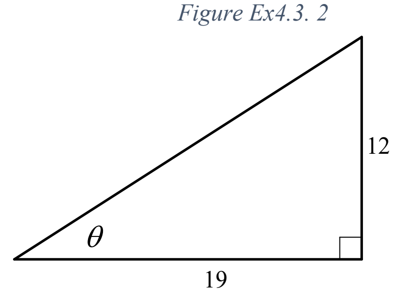
17.

Figure Ex4.3. 1



18.

Figure Ex4.3. 2



In Exercises 19 – 21, find the two acute angles in the right triangle whose sides have the given lengths. Express your answers using degree measure rounded to two decimal places.

19. lengths 3, 4, and 5
20. Lengths 5, 12, and 13
21. lengths 336, 527, and 625

22. A wire 1000 feet long is attached to the top of a tower. When pulled taut, it touches level ground 360 feet from the base of the tower. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.

23. At Cliffs of Insanity Point, the Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.

24. Shelving that is being built at the college library is to be 14 inches deep. An 18-inch rod will be attached to the wall beneath each shelf, and to the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?

25. A parasailor is being pulled by a boat on Lake Powell. The cable is 300 feet long and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.

26. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park is begun. From a 200-foot tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly toward the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquilizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundredth of a degree.

27. Suppose a 13-foot ladder leans against the side of a house, reaching to the bottom of a second-floor window 12 feet above the ground. What angle does the ladder make with the house? Round your answer to the nearest tenth of a degree.

In Exercises 28 – 47, solve the equation. Then approximate any solutions that lie in the interval $[0, 2\pi)$ to four decimal places.

28. $\sin(x) = \frac{7}{11}$

29. $\cos(x) = -\frac{2}{9}$

30. $\sin(x) = -0.569$

31. $\cos(x) = 0.117$

32. $\sin(x) = 0.008$

33. $\cos(x) = \frac{359}{360}$

34. $\tan(x) = 117$

35. $\cot(x) = -12$

36. $\sec(x) = \frac{3}{2}$

37. $\csc(x) = -\frac{90}{17}$

38. $\tan(x) = -\sqrt{10}$

39. $\sin(x) = \frac{3}{8}$

40. $\cos(x) = -\frac{7}{16}$

41. $\tan(x) = 0.03$

42. $\sin(x) = 0.3502$

43. $\sin(x) = -0.721$

44. $\cos(x) = 0.9824$

45. $\cos(x) = -0.5637$

46. $\cot(x) = \frac{1}{117}$

47. $\tan(x) = -0.6109$

In Exercises 48 – 74, solve the equation for x, where x is in radians. State the exact solutions, if any, that lie in the interval $[0, 2\pi)$.

48. $\sin(5x) = 0$

49. $\cos(3x) = \frac{1}{2}$

50. $\sin(-2x) = \frac{\sqrt{3}}{2}$

51. $\tan(6x) = 1$

52. $\csc(4x) = -1$

53. $\sec(3x) = \sqrt{2}$

54. $\cot(2x) = -\frac{\sqrt{3}}{3}$

55. $\cos(9x) = 9$

56. $\sin\left(\frac{x}{3}\right) = \frac{\sqrt{2}}{2}$

57. $\csc(x) = 0$

58. $\tan^2(x) = 3$

59. $\sec^2(x) = \frac{4}{3}$

60. $2\cos(x) = \sqrt{3}$

61. $4\sin^2(x) - 3 = 0$

62. $\cos(x) - 2 = 0$

63. $\cos\left(x + \frac{5\pi}{6}\right) = 0$

64. $\sin\left(2x - \frac{\pi}{3}\right) = -\frac{1}{2}$

65. $2\cos\left(x + \frac{7\pi}{4}\right) = \sqrt{3}$

66. $\tan(2x - \pi) = 1$

67. $\cos^2(x) = \frac{1}{2}$

68. $\sin^2(x) = \frac{3}{4}$

69. $\sqrt{3}\tan(x) + 1 = 0$

70. $3\cot^2(x) - 1 = 0$

71. $2\sin^2(x) - 4 = 0$

72. $3\tan^2(2x) - 1 = 0$

73. $2\cos\left(x + \frac{7\pi}{4}\right) = \sqrt{3}$

74. $\cot\left(2x + \frac{\pi}{3}\right) = 0$

In Exercises 75 – 82, solve the equation.

75. $\arccos(2x) = \pi$

76. $\pi - 2\arcsin(x) = 2\pi$

77. $4\arctan(3x - 1) - \pi = 0$

78. $6\operatorname{arccot}(2x) - 5\pi = 0$

79. $4\operatorname{arcsec}\left(\frac{x}{2}\right) = \pi$

80. $12\operatorname{arccsc}\left(\frac{x}{3}\right) = 2\pi$

81. $9\arcsin^2(x) - \pi^2 = 0$ 82. $9\arccos^2(x) - \pi^2 = 0$

83. Determine the number of solutions to $\sin(x) = \frac{1}{2}$ in $[0, 2\pi)$. Then find the number of solutions to $\sin(2x) = \frac{1}{2}$, $\sin(3x) = \frac{1}{2}$ and $\sin(4x) = \frac{1}{2}$ in $[0, 2\pi)$. A pattern should emerge. Explain how this pattern would help you solve equations like $\sin(11x) = \frac{1}{2}$.

84. Determine the number of solutions to $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$, $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ and $\sin\left(\frac{5x}{2}\right) = \frac{1}{2}$ in $[0, 2\pi)$. Is there a pattern that emerges? Explain. Then replace $\frac{1}{2}$ with -1 and repeat the whole exploration.

4.4 Solving General Trigonometric Equations

Learning Objectives

- **Solve equations containing different powers of the same trigonometric function.**
- **Solve equations containing multiple trigonometric functions.**
- **Solve equations containing multiple arguments of the same trigonometric function.**
- **Find solutions to trigonometric equations in a given interval.**

The general method for solving complex trigonometric equations is to reduce them to simpler trigonometric equations; that is, single trigonometric functions equal to numbers. We will do this using trigonometric identities and algebraic operations like factoring.

Equations Containing Different Powers of the Same Trigonometric Function

Example 4.4.1. Solve the equation $3\sin^3(x) = \sin^2(x)$. State the exact solutions, if any, that lie in the interval $[0, 2\pi)$.

Solution. The main steps in solving this equation are the same as solving the polynomial equation $3u^3 = u^2$. While we could use the substitution $u = \sin(x)$ to obtain the polynomial equation, we will solve without making this substitution. Note that we must resist the temptation to divide both sides of $3\sin^3(x) = \sin^2(x)$ by $\sin^2(x)$. (What goes wrong if you do?) Instead, we put all terms on one side to get zero on the other side and factor.

$$
\begin{aligned}
3\sin^3(x) &= \sin^2(x) \\
3\sin^3(x) - \sin^2(x) &= 0 \\
\sin^2(x)\left(3\sin(x) - 1\right) &= 0 \qquad \text{Factor } \sin^2(x) \text{ from both terms.}
\end{aligned}
$$

Using the Zero Product Property, we set each factor equal to zero to get $\sin^2(x) = 0$ or $3\sin(x) - 1 = 0$, from which $\sin(x) = 0$ or $\sin(x) = \frac{1}{3}$.

- The solutions to $\sin(x)=0$ are $x=\pi k$, for integers k, with 0 and π being the two solutions that lie in $[0,2\pi)$.[9]
- To solve $\sin(x)=\frac{1}{3}$, we use the arcsine function to get $x=\arcsin\left(\frac{1}{3}\right)+2\pi k$ or $x=\pi-\arcsin\left(\frac{1}{3}\right)+2\pi k$, for integers k. Of these solutions, the two that lie in $[0,2\pi)$ are $\arcsin\left(\frac{1}{3}\right)$ and $\pi-\arcsin\left(\frac{1}{3}\right)$.

To summarize, the solutions to $3\sin^3(x)=\sin^2(x)$ are $x=\pi k$, $x=\arcsin\left(\frac{1}{3}\right)+2\pi k$, or $x=\pi-\arcsin\left(\frac{1}{3}\right)+2\pi k$, for integers k. In the interval $[0,2\pi)$, the solutions are $x=0,\ \arcsin\left(\frac{1}{3}\right),\ \pi-\arcsin\left(\frac{1}{3}\right),\ \pi$.[10]

□

Equations Containing Multiple Trigonometric Functions

In the next example, we make use of a Pythagorean identity to arrive at an equation involving just one trigonometric function.

Example 4.4.2. Solve the equation $\sec^2(x)=\tan(x)+3$. State the exact solutions, if any, that lie in the interval $[0,2\pi)$.

Solution. Since $\sec^2(x)=\tan(x)+3$ contains two different trigonometric functions, an identity is in order so that we may rewrite the equation in terms of all secants or all tangents. We use the Pythagorean identity $\tan^2(x)+1=\sec^2(x)$.

$$\begin{aligned} \sec^2(x)&=\tan(x)+3 \\ \tan^2(x)+1&=\tan(x)+3 \quad \text{Pythagorean identity} \end{aligned}$$

Now we have an equation like in the last example. To simplify the process, we will make a substitution.

$$\begin{aligned} \tan^2(x)+1&=\tan(x)+3 \quad \text{Let } u=\tan(x). \\ u^2+1&=u+3 \\ u^2-u-2&=0 \\ (u+1)(u-2)&=0 \end{aligned}$$

[9] In this section, the reader should verify solutions in $[0,2\pi)$ by plugging in values for *k*.

[10] Note that we are not counting $x=2\pi$ as a solution since it is not in the interval $[0,2\pi)$.

Using the Zero Product Property, we have $u+1=0$ or $u-2=0$, from which $u=-1$ or $u=2$. Then, since $u=\tan(x)$, $\tan(x)=-1$ or $\tan(x)=2$.

- From $\tan(x)=-1$, we get $x=-\frac{\pi}{4}+\pi k$, for integers k. The solutions that lie in $[0,2\pi)$ are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.
- To solve $\tan(x)=2$, we employ the arctangent function and get $x=\arctan(2)+\pi k$, for integers k. Using the same sort of argument that we saw in **Example 4.3.11**, we get $x=\arctan(2)$ and $x=\pi+\arctan(2)$ as solutions in $[0,2\pi)$.

In summary, the solutions to $\sec^2(x)=\tan(x)+3$ are $x=-\frac{\pi}{4}+\pi k$ or $x=\arctan(2)+\pi k$, for any integer k. The solutions in the interval $[0,2\pi)$ are $x=\frac{3\pi}{4}, \frac{7\pi}{4}, \arctan(2), \pi+\arctan(2)$.

□

Equations Containing Multiple Arguments of the Same Trigonometric Function

As in **Example 4.4.2**, some trigonometric equations can be solved by treating them as quadratic equations. The next example is another equation that is quadratic in form after applying a trigonometric identity.

Example 4.4.3. Solve the equation $\cos(2x)=1-3\cos(x)$. State the exact solutions that lie in the interval $[0,2\pi)$.

Solution. The equation $\cos(2x)=1-3\cos(x)$ has just one type of trigonometric function, but two different arguments, x and $2x$. We will use the trigonometric identity $\cos(2x)=2\cos^2(x)-1$ to rewrite the equation with the single argument x, and then proceed as in the last example.

$$\begin{aligned} \cos(2x)&=1-3\cos(x) \\ 2\cos^2(x)-1&=1-3\cos(x) && \text{double angle identity} \\ 2\cos^2(x)+3\cos(x)-2&=0 \\ 2u^2+3u-2&=0 && \text{Let } u=\cos(x). \\ (2u-1)(u+2)&=0 \end{aligned}$$

This gives $u=\frac{1}{2}$ or $u=-2$. Since $u=\cos(x)$, we have $\cos(x)=\frac{1}{2}$ or $\cos(x)=-2$.

- Solving $\cos(x)=\frac{1}{2}$, we get $x=\frac{\pi}{3}+2\pi k$ or $x=\frac{5\pi}{3}+2\pi k$, for integers k.

- Since -2 is outside of the range of cosine, there are no values of x for which $\cos(x) = -2$.

The solutions to $\cos(2x) = 1 - 3\cos(x)$ are $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{5\pi}{3} + 2\pi k$, for integers k. The solutions that lie in $[0, 2\pi)$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

□

Equations Containing Multiple Trigonometric Functions and Multiple Arguments

In the following example, we cannot use trigonometric identities to obtain just one trigonometric function and one argument. However, we can still solve using factoring.

Example 4.4.4. Solve the equation $\sin(2x) = \sqrt{3}\cos(x)$. State the exact solutions, if any, that lie in the interval $[0, 2\pi)$.

Solution. In examining the equation $\sin(2x) = \sqrt{3}\cos(x)$, not only do we have multiple functions involved, namely sine and cosine, we also have multiple arguments to contend with, namely $2x$ and x, respectively. Using the double angle identity $\sin(2x) = 2\sin(x)\cos(x)$ makes all of the arguments the same. We then rewrite the equation so we have an expression equal to zero, and proceed by factoring.

$$\begin{aligned}
\sin(2x) &= \sqrt{3}\cos(x) \\
2\sin(x)\cos(x) &= \sqrt{3}\cos(x) \\
2\sin(x)\cos(x) - \sqrt{3}\cos(x) &= 0 \\
\cos(x)\left(2\sin(x) - \sqrt{3}\right) &= 0
\end{aligned}$$

Then $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$. From $\cos(x) = 0$, we obtain $x = \frac{\pi}{2} + \pi k$, for integers k. From $\sin(x) = \frac{\sqrt{3}}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$, for integers k.

The answers that lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$.

□

The last example in this section, which also includes multiple functions and arguments, tests our memory a bit and introduces another solution technique.

Example 4.4.5. Solve the equation $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$. State the exact solutions that lie in the interval $[0,2\pi)$.

Solution. Unlike the equation in the previous example, there seems to be no quick way to get the trigonometric functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$. If we stare at it long enough, however, we realize that the left side is the expanded form of the sum formula for $\sin\left(x+\frac{x}{2}\right)$.

$$\begin{aligned}\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)&=1\\ \sin\left(x+\frac{x}{2}\right)&=1\\ \sin\left(\frac{3}{2}x\right)&=1\end{aligned}$$

Consequently, our original equation is equivalent to $\sin\left(\frac{3}{2}x\right)=1$. We proceed to solve for x.

$$\begin{aligned}\sin\left(\frac{3}{2}x\right)&=1\\ \frac{3}{2}x&=\frac{\pi}{2}+2\pi k && \text{since sine is equal to 1}\\ \left(\frac{2}{3}\right)\left(\frac{3}{2}x\right)&=\left(\frac{2}{3}\right)\left(\frac{\pi}{2}+2\pi k\right)\\ x&=\frac{\pi}{3}+\frac{4\pi}{3}k\end{aligned}$$

The solution to the original equation is $x=\frac{\pi}{3}+\frac{4\pi}{3}k$, for integers k. The following solutions lie in the interval $[0,2\pi)$: $x=\frac{\pi}{3},\frac{5\pi}{3}$.

□

As demonstrated in the examples, solutions in this section require recognizing the correct identity and/or factoring. With a little practice, you will become proficient in solving trigonometric equations.

4.4 Exercises

In Exercises 1 – 14, solve the equation for x, where x is in radians. State the exact solutions, if any, that lie in the interval $[0,2\pi)$.

1. $2\sin^2(x)-\sin(x)=0$
2. $\tan^3(x)=3\tan(x)$
3. $\cos^2(x)-\cos(x)-2=0$
4. $\cos^3(x)=-\cos(x)$
5. $2\cos^2(x)+\cos(x)=1$
6. $\tan^3(x)=\tan(x)$
7. $\tan^2(x)-\sqrt{3}\tan(x)=0$
8. $\sin^2(x)+\sin(x)-2=0$
9. $\cot^2(x)=-\cot(x)$
10. $5\cos^2(x)+4\cos(x)-1=0$
11. $\csc^3(x)+\csc^2(x)=4\csc(x)+4$
12. $8\sin^2(x)-6\sin(x)+1=0$
13. $8\cos^2(x)=3-2\cos(x)$
14. $4\cos^2(x)-4=15\cos(x)$

In Exercises 15 – 44, use trigonometric identities in solving each equation for x, where x is in radians. State the exact solutions, if any, that lie in the interval $[0,2\pi)$.

15. $\sin(2x)=\sin(x)$
16. $\sin(2x)=\cos(x)$
17. $\cos(2x)=\sin(x)$
18. $\cos(2x)=\cos(x)$
19. $\cos(2x)=2-5\cos(x)$
20. $3\cos(2x)+\cos(x)+2=0$
21. $\cos(2x)=5\sin(x)-2$
22. $3\cos(2x)=\sin(x)+2$
23. $2\sec^2(x)=3-\tan(x)$
24. $2\tan^2(x)=3\sec(x)$
25. $\tan(x)-2\sin(x)\tan(x)=0$
26. $\sin^2(x)-\cos^2(x)-\sin(x)=0$
27. $10\sin(x)\cos(x)=6\cos(x)$
28. $9\cos(2x)=9\cos^2(x)-4$
29. $12\sin^2(x)+\cos(x)-6=0$
30. $\tan^2(x)=1-\sec(x)$
31. $\cot^2(x)=3\csc(x)-3$
32. $\tan^2(x)=\frac{3}{2}\sec(x)$

33. $\tan(x) = \sec(x)$

34. $\sin(x) = \cos(x)$

35. $\sec(x) = 2\csc(x)$

36. $\sin(2x) = \tan(x)$

37. $\cot^4(x) = 4\csc^2(x) - 7$

38. $\cos(2x) + \csc^2(x) = 0$

39. $\tan(2x) - 2\cos(x) = 0$

40. $\cos(x)\csc(x)\cot(x) = 6 - \cot^2(x)$

41. $\sin(6x)\cos(x) = -\cos(6x)\sin(x)$

42. $\sin(3x)\cos(x) = \cos(3x)\sin(x)$

43. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

44. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$

CHAPTER 5
BEYOND RIGHT TRIANGLES

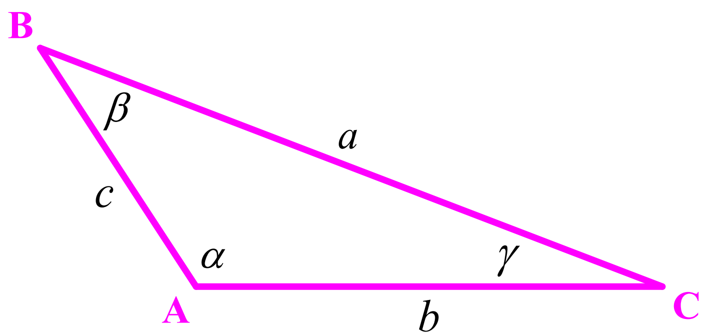


Figure 5.0. 1

Chapter Outline

Introduction

By now, you can use the Pythagorean Theorem and trigonometric functions to fully determine a right triangle if you know the measure of one of its acute angles and the length of one of its sides, or the lengths of two of its sides. We would like to expand this capability to general planar shapes. Since any straight-sided geometric shape can be subdivided into triangles, but not necessarily right triangles, we will discuss properties of non-right (oblique) triangles. In particular, for an oblique triangle we seek generalizations of right triangle trigonometry.

In Section 5.1, we will develop the Law of Sines, which can be used to fully determine a triangle if we know certain information; for example, the measure of two of its angles and the length of one of its sides. Then, in Section 5.2, we will develop the Law of Cosines, which is a generalization of the Pythagorean Theorem to oblique triangles. We can use the Law of Cosines to fully determine a triangle if we know the lengths of all of its sides or the lengths of two of its sides and the measure of the angle between the known sides.

More generally, using the Laws of Sines and Cosines, we will determine if there is no triangle, one triangle, or two triangles that may satisfy a given set of conditions.

5.1 The Law of Sines

Learning Objectives

- **Use the Law of Sines to solve triangles.**
- **Distinguish between ASA, AAS, and SSA triangles.**
- **Determine when given criteria will result in one, two, or no triangles.**
- **Find the area of a triangle using the sine function.**
- **Use the Law of Sines to solve applied problems.**

Trigonometry literally means 'measuring triangles', and with **Chapters 1 – 4** under our belts we are prepared to do just that. The main goal of this chapter is to develop theorems that allow us to solve triangles; that is, to find the length of each side of a triangle and the measure of each of its angles.

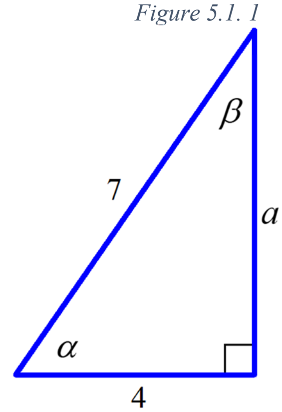
Solving Triangles

We have had some experience solving right triangles. The following example reviews what we know.

Example 5.1.1. Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, find the length of the remaining side and the measures of the remaining angles. Express the angle measures in degrees, rounded to the nearest hundredth of a degree.

Solution. For definitiveness, we label the triangle below.

Figure 5.1. 1



- To find the length of the missing side a, we use the Pythagorean Theorem to get $a^2 + 4^2 = 7^2$, which yields $a = \sqrt{33}$ units.

Now that all three sides of the triangle are known, there are several ways we can find α and β using the inverse trigonometric functions. To decrease the chances of propagating error, however, we stick to the data given to us in the problem. In this case, the lengths 4 and 7 were given.

- We want to relate the lengths 4 and 7 to α. Since α is one of the acute angles in a right triangle, and $\cos(\alpha)=\frac{4}{7}$, we find $\alpha=\arccos\left(\frac{4}{7}\right)$ radians. Using a calculator and converting to degrees, α is approximately equal to 55.15º.
- Similarly, since β is one of the acute angles in a right triangle, and $\sin(\beta)=\frac{4}{7}$, then $\beta=\arcsin\left(\frac{4}{7}\right)$ radians, and we have $\beta\approx 34.85^\circ$.

Note that we could have used the measure of angle α to find the measure of angle β, using the fact that α and β are complements ($\alpha+\beta=90^\circ$).

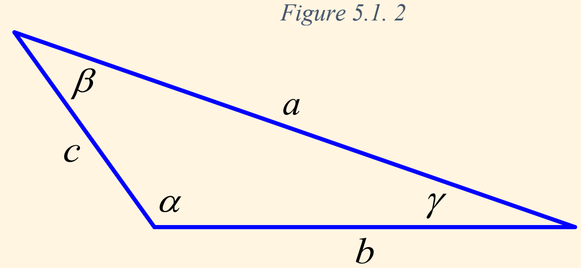
□

A few remarks about **Example 5.1.1** are in order:

1. As mentioned earlier, we strive to solve for quantities using the original data given in the problem. To avoid the accumulation of rounding errors, whenever possible, we will use exact values. While this is not always the easiest or fastest way to proceed, it minimizes the chances of propagated error.
2. We will use angle measure in degrees for the time being, since degrees are used in both design and manufacturing.[1]

For a triangle with sides a, b, c and angles α, β, γ, we adhere to the convention that a is the side opposite α, b is the side opposite β, and c is the side opposite γ. Taken together, the pairs (α,a), (β,b), and (γ,c) are called **angle-side opposite pairs**.

Figure 5.1. 2



The Law of Sines

The Pythagorean Theorem (along with the definitions of the trigonometric functions) allows us to easily handle any given right triangle problem, but what if the triangle is not a right triangle? Any triangle that is not a right triangle is called an **oblique triangle**. In oblique triangles, the Pythagorean Theorem no

[1] Don't worry! Radians will be back before you know it!

longer applies so we need something else. In certain cases, we can use the Law of Sines to solve such triangles.

Theorem 5.1. The Law of Sines: In any triangle with angle-side opposite pairs (α, a), (β, b), and (γ, c), the following equality of ratios holds:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

or, equivalently,

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

The proof of the Law of Sines can be broken into three cases.

1. For our first case, consider $\triangle ABC$, shown below, having all acute angles and angle-side opposite pairs (α, a), (β, b), and (γ, c). Note that A, B, and C identify the vertices corresponding to angles α, β, and γ, respectively.

Figure 5.1. 3

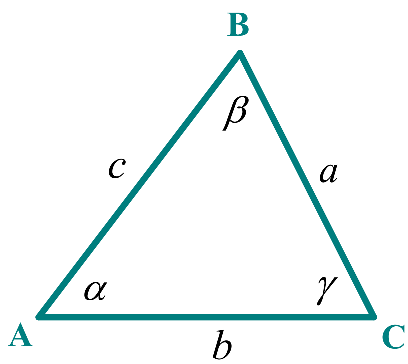
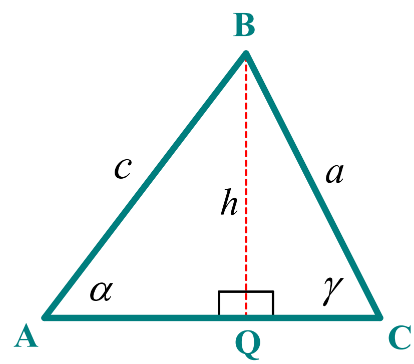
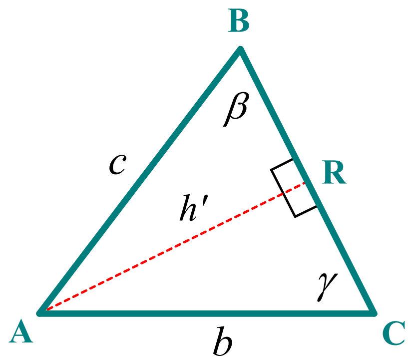


Figure 5.1. 4



If we drop an altitude from vertex B, we divide the triangle into two right triangles: $\triangle ABQ$ and $\triangle BCQ$. We call the altitude h (for height). Then $\sin(\alpha) = \frac{h}{c}$ and $\sin(\gamma) = \frac{h}{a}$, so that $h = c\sin(\alpha) = a\sin(\gamma)$. Then, from $c\sin(\alpha) = a\sin(\gamma)$, we get $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

Figure 5.1. 5



If we drop an altitude from vertex A, we can proceed as above, using $\triangle\ ABR$ and $\triangle\ ACR$ to get $\frac{\sin(\beta)}{b}=\frac{\sin(\gamma)}{c}$, completing the proof for this case.

2. For the next case, consider $\triangle\ ABC$, shown below, with obtuse angle α. Extending an altitude from vertex A gives two right triangles: $\triangle\ ABQ$ and $\triangle\ ACQ$. Proceeding as before, $h=c\sin(\beta)$ and $h=b\sin(\gamma)$, so that $\frac{\sin(\beta)}{b}=\frac{\sin(\gamma)}{c}$.

Figure 5.1. 6

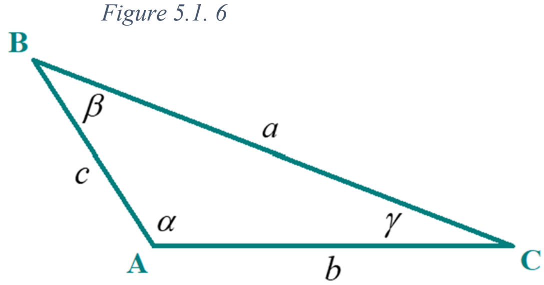
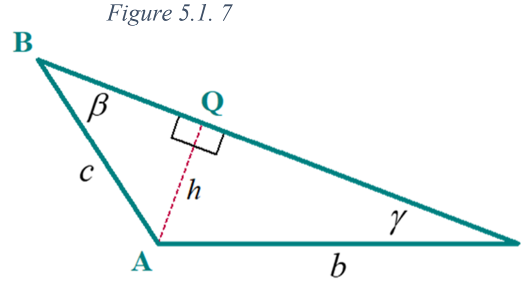
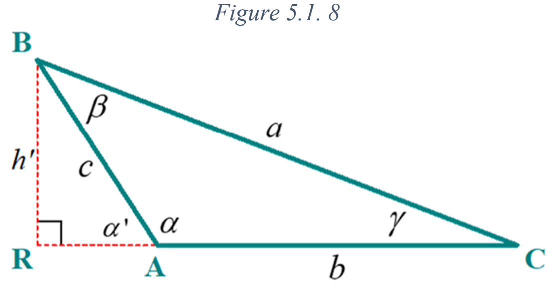


Figure 5.1. 7



Dropping an altitude from vertex B also generates two right triangles: $\triangle\ ABR$ and $\triangle\ BCR$.

Figure 5.1. 8



We know that $\sin(\alpha')=\frac{h'}{c}$. Since $\alpha'=180^\circ-\alpha$, $\sin(\alpha')=\sin(\alpha)$, so in fact we have $h'=c\sin(\alpha)$. Proceeding to $\triangle\ BCR$, we get $\sin(\gamma)=\frac{h'}{a}$, so $h'=a\sin(\gamma)$. Putting this together with the previous equation results in $\frac{\sin(\gamma)}{c}=\frac{\sin(\alpha)}{a}$, and we are finished with this case.

3. The remaining case is when $\triangle ABC$ is a right triangle. In this case, the definitions of trigonometric functions can be used to verify the Law of Sines. This verification is left to the reader.

In order to use the Law of Sines to solve a triangle, we need at least three measurements of angles and/or sides, including at least one of the sides. Also note that we need to be given, or be able to find, at least one angle-side opposite pair. We will investigate three possible scenarios.

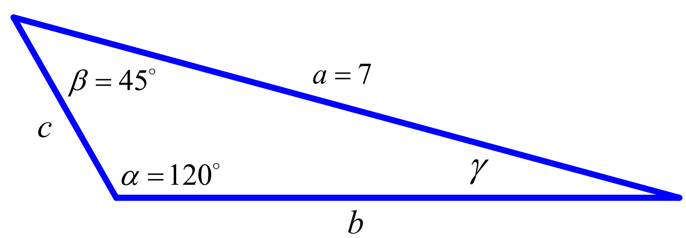
AAS (Angle-Angle-Side)

Given the measures of two angles in a triangle (the sum of which is less than 180 degrees) and the length of a side not connecting the angles, the triangle is unique. Although we will not prove this fact, we will demonstrate it in the next example.

Example 5.1.2. Solve the triangle with $\alpha = 120^\circ$, $a = 7$ units, and $\beta = 45^\circ$. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

Solution. We begin by sketching a representative triangle. Then, noting the existence of an angle-side opposite pair, namely $\alpha = 120^\circ$ and $a = 7$ units, we proceed in using the Law of Sines.

Figure 5.1. 9



$$\frac{\sin(45^\circ)}{b} = \frac{\sin(120^\circ)}{7} \qquad \text{since } \beta = 45^o$$

$$b = \frac{7\sin(45^\circ)}{\sin(120^\circ)}$$

$$b = \frac{7\left(\sqrt{2}/2\right)}{\sqrt{3}/2}$$

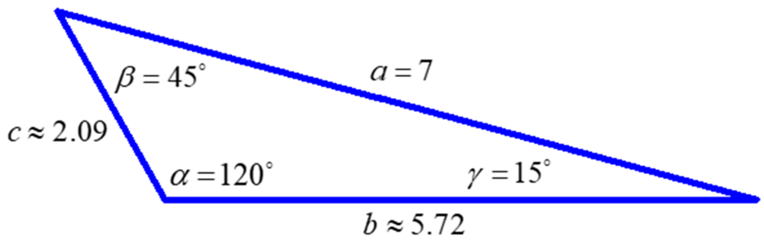
$$b = \frac{7\sqrt{2}}{\sqrt{3}} \approx 5.72 \text{ units } (\text{rounded to hundredths})$$

Now that we have two angle-side opposite pairs, it is time to find the third. To find γ, we use the fact that the sum of the measures of angles in a triangle is 180º. Hence, $\gamma = 180^\circ - 120^\circ - 45^\circ = 15^\circ$. To find c, we use the derived value $\gamma = 15^\circ$ and, to minimize the propagation of error, we use the given angle-side opposite pair (α, a). The Law of Sines gives us

$$\frac{\sin\left(15^\circ\right)}{c} = \frac{\sin\left(120^\circ\right)}{7}$$
$$c = \frac{7\sin\left(15^\circ\right)}{\sin\left(120^\circ\right)}$$
$$c \approx 2.09 \text{ units}$$

The exact value of $\sin\left(15^\circ\right)$ could be found using the difference identity for sine or a half-angle formula, but that becomes unnecessarily messy for the discussion at hand. Thus, 'exact' here means $c = \dfrac{7\sin\left(15^\circ\right)}{\sin\left(120^\circ\right)}$.

Figure 5.1. 10



□

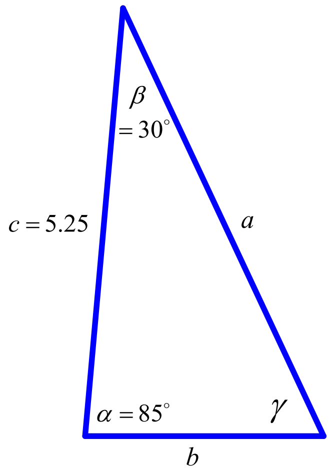
ASA (Angle-Side-Angle)

Given the measures of two angles in a triangle (the sum of which is less than 180 degrees) and the length of the side connecting the angles, the triangle is unique. Although we will not prove this fact, we will demonstrate it in the next example.

Example 5.1.3. Solve the triangle with $\alpha = 85^\circ$, $\beta = 30^\circ$, and $c = 5.25$ units. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

Solution. We begin with a sketch.

Figure 5.1. 11



Having the measures of α and β, we can solve for γ:

$$\begin{aligned}\gamma &= 180^\circ - 85^\circ - 30^\circ \\ &= 65^\circ\end{aligned}$$

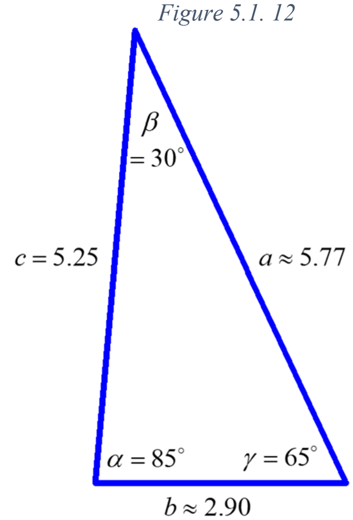
The Law of Sines gives

$$\begin{aligned}\frac{\sin(85^\circ)}{a} &= \frac{\sin(65^\circ)}{5.25} \\ a &= \frac{5.25\sin(85^\circ)}{\sin(65^\circ)} \\ a &\approx 5.77 \text{ units}\end{aligned}$$

To find b, we again use the angle-side opposite pair (γ, c), which yields

$$\begin{aligned}\frac{\sin(30^\circ)}{b} &= \frac{\sin(65^\circ)}{5.25} \\ b &= \frac{5.25\sin(30^\circ)}{\sin(65^\circ)} \\ b &\approx 2.90 \text{ units}\end{aligned}$$

Figure 5.1. 12



□

SSA (Side-Side-Angle)

In the third scenario, we are given the lengths of two sides and the measure of an angle (less than 180°) that is not between the two sides. In this case, there can be three possible outcomes: one triangle, two triangles, or no triangle that satisfies the given measures. The case in which there are two triangles is referred to as the **ambiguous case**. We will demonstrate these three cases in the following examples.

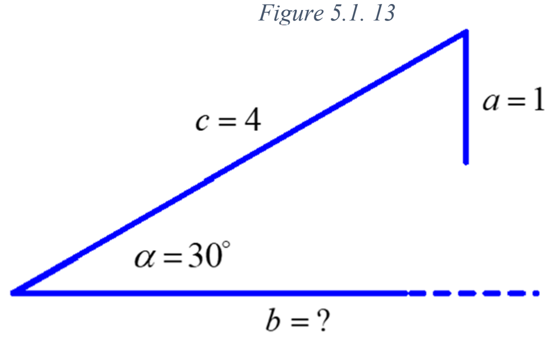
Example 5.1.4. Solve the triangle with $\alpha = 30^\circ$, $a = 1$ unit, and $c = 4$ units.

Solution. Since we are given (α, a) and c, we use the Law of Sines to find the measure of γ.

$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{1}$$
$$\sin(\gamma) = 4\sin(30^\circ)$$
$$\sin(\gamma) = 2$$

Since the range of the sine function is $[-1,1]$, there is no angle γ with $\sin(\gamma) = 2$, and thus no triangle that meets the given criteria. Geometrically, we see that side a is just too short to make a triangle.

Figure 5.1. 13



□

The following examples keep the same value for the measure of α and the length of c, while varying the length of a. We will discuss the preceding case in more detail after we see what happens in the next three examples.

Example 5.1.5. Solve the triangle with $\alpha = 30^\circ$, $a = 2$ units, and $c = 4$ units.

Solution. Using the Law of Sines, we get

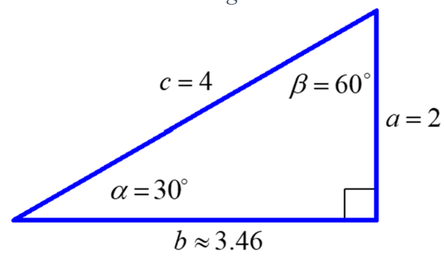
$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{2}$$
$$\sin(\gamma) = 2\sin(30^\circ)$$
$$\sin(\gamma) = 1$$

Since γ is an angle in a triangle (its measure is less than 180º), we must have $\gamma = 90^\circ$. In other words, we have a right triangle. We find the measure of β to be $\beta = 180^\circ - 30^\circ - 90^\circ = 60^\circ$, and use the Law of Sines to determine b:

$$\frac{\sin(30^\circ)}{2} = \frac{\sin(60^\circ)}{b}$$
$$b = \frac{2\sin(60^\circ)}{\sin(30^\circ)}$$
$$b = \frac{2\left(\sqrt{3}/2\right)}{1/2}$$
$$b = 2\sqrt{3} \approx 3.46 \text{ units}$$

In this case, the side a is precisely long enough to form a unique right triangle.

Figure 5.1. 14



Note that we could also solve for b using the Pythagorean Theorem.

□

Example 5.1.6. Solve the triangle with $\alpha=30^\circ$, $a=3$ units, and $c=4$ units.

Solution. Proceeding as in the previous two examples, we use the Law of Sines to find γ. In this case, we have

$$\frac{\sin(\gamma)}{4}=\frac{\sin(30^\circ)}{3}$$
$$\sin(\gamma)=\frac{4\sin(30^\circ)}{3}$$
$$\sin(\gamma)=\frac{2}{3}$$

Since γ lies in a triangle with $\alpha=30^\circ$, we must have $0^\circ<\gamma<150^\circ$. There are two angles γ that fall in this range and have $\sin(\gamma)=\frac{2}{3}$: $\gamma=\arcsin\left(\frac{2}{3}\right)$ radians, approximately 41.81º, and $\gamma=\pi-\arcsin\left(\frac{2}{3}\right)$ radians, approximately 138.19º.

- In the case $\gamma=\arcsin\left(\frac{2}{3}\right)$ radians ≈ 41.81º, we find [2] $\beta\approx 180^\circ-30^\circ-41.81^\circ=108.19^\circ$. Using the Law of Sines with the angle-side opposite pair (α,a) and β, we find

  $b\approx\dfrac{3\sin(108.19^\circ)}{\sin(30^\circ)}\approx 5.70$ units.

---

[2] To find an exact expression for β, we convert everything back to radians: $\alpha=30^\circ=\frac{\pi}{6}$ radians, $\gamma=\arcsin\left(\frac{2}{3}\right)$ radians, and $180^\circ=\pi$ radians. Hence, $\beta=\pi-\frac{\pi}{6}-\arcsin\left(\frac{2}{3}\right)=\frac{5\pi}{6}-\arcsin\left(\frac{2}{3}\right)$ radians ≈ 108.19º.

- In the case $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians ≈ 138.19º, we repeat the exact same steps and find $\beta \approx 11.81^\circ$ and $b \approx 1.23$ units.[3]

Both triangles are drawn below.

Figure 5.1. 15

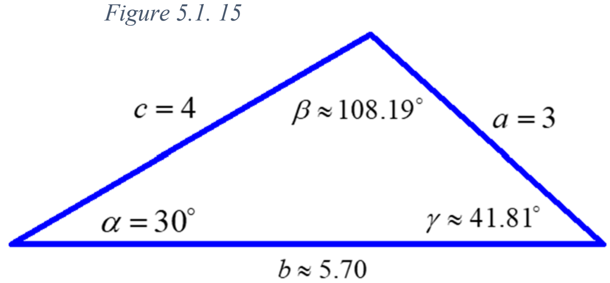
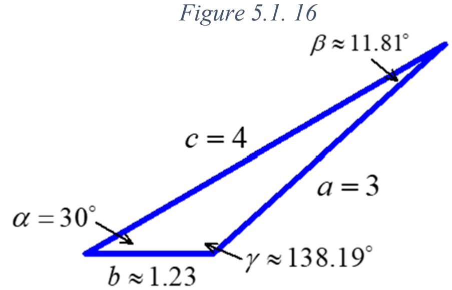


Figure 5.1. 16



□

Example 5.1.7. Solve the triangle with $\alpha = 30^\circ$, $a = 4$ units, and $c = 4$ units.

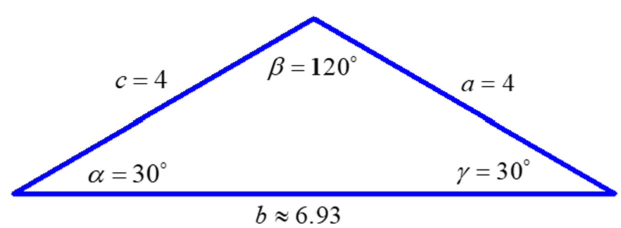
Solution. For this problem, we repeat the usual Law of Sines routine to find that

$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{4}$$
$$\sin(\gamma) = \sin(30^\circ)$$
$$\sin(\gamma) = \frac{1}{2}$$

Then γ must inhabit a triangle with $\alpha = 30^\circ$, so that $0^\circ < \gamma < 150^\circ$. Since the measure of γ must be strictly less than 150º, there is only one angle that satisfies both required conditions, namely $\gamma = 30^\circ$. Then $\beta = 180^\circ - 30^\circ - 30^\circ = 120^\circ$ and, using the Law of Sines one last time,

$$\frac{\sin(30^\circ)}{4} = \frac{\sin(120^\circ)}{b}$$
$$b = \frac{4\sin(120^\circ)}{\sin(30^\circ)}$$
$$b = \frac{4\left(\sqrt{3}/2\right)}{1/2}$$
$$b = 4\sqrt{3} \approx 6.93 \text{ units}$$

Figure 5.1. 17



□

Before moving on, we review the information needed to determine a triangle.

[3] An exact answer for β in this case is $\beta = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{6}$ radians ≈ 11.81º.

1. If we know the measures of two angles in a triangle, the measure of the third angle can be uniquely determined using the fact that the sum of the angles in a triangle is always 180 degrees. Having the measures of all three angles allows us to completely determine the shape of a triangle.
2. In addition to knowing the measures of two angles in a triangle, if we are given the length of one side, we can utilize the Law of Sines. This enables us to find the lengths of the remaining two sides, allowing us to determine the size of the triangle. This holds true for both the AAS and the ASA cases.
3. If we are provided with the measure of only one angle and the lengths of two sides in a triangle, where only one of these two sides is adjacent to the given angle, we have the SSA case. In this situation, it is possible to encounter one triangle, two triangles, or no triangle at all, as demonstrated in **Examples 5.1.4 – 5.1.7**.

The possibilities in the SSA case are summarized in the following theorem.

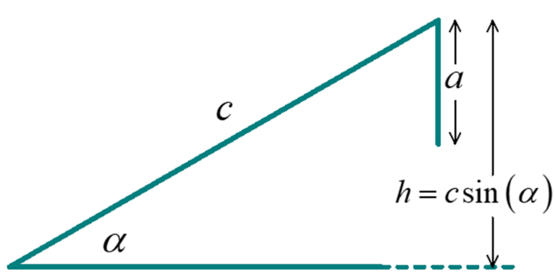
Theorem 5.2. Suppose (α, a) and (γ, c) are intended to be angle-side opposite pairs in a triangle where α, a, and c are given. Let $h = c\sin(\alpha)$.

- If $a < h$, then no triangle exists that satisfies the given criteria.
- If $a = h$, then $\gamma = 90^\circ$ so exactly one (right) triangle exists that satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist that satisfy the criteria.
- If $a \geq c$, then γ is acute and exactly one triangle exists that satisfies the given criteria.

Theorem 5.2 is proved on a case-by-case basis.

- If $a < h$ then $a < c\sin(\alpha)$. If a triangle were to exist, the Law of Sines would have $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} > \frac{a}{a} = 1$, which is impossible. In the figure below, we see geometrically why this is the case.

Figure 5.1. 18

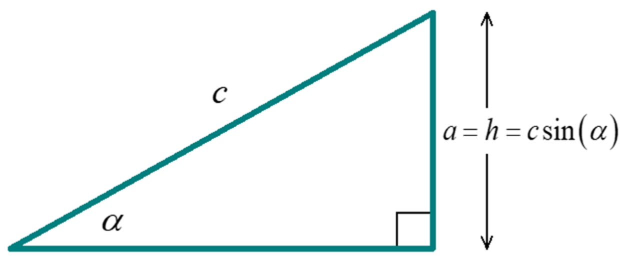


$a < h$, no triangle

Simply put, if $a<h$ the side a is too short to connect so that a triangle may be formed. On the other hand, if $a \geq h$, we are always guaranteed to have at least one triangle, and the remaining parts of the theorem tell us what kind and how many triangles to expect in each case.

- If $a=h$, then $a=c\sin(\alpha)$ and the Law of Sines gives $\dfrac{\sin(\alpha)}{a}=\dfrac{\sin(\gamma)}{c}$ so that $\sin(\gamma)=\dfrac{c\sin(\alpha)}{a}=\dfrac{a}{a}=1$. Here, $\gamma=90^\circ$ as required.

Figure 5.1. 19



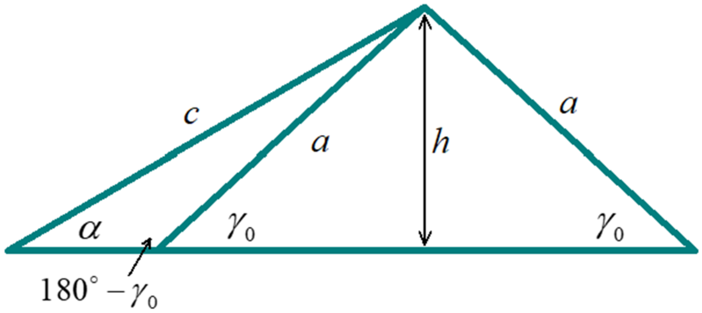
$a=h$, $\gamma=90^\circ$

- Moving along, now suppose $h<a<c$. As before, the Law of Sines gives $\sin(\gamma)=\dfrac{c\sin(\alpha)}{a}$.

Since $h<a$, $c\sin(\alpha)<a$ or $\dfrac{c\sin(\alpha)}{a}<1$, which means there are two solutions to $\sin(\gamma)=\dfrac{c\sin(\alpha)}{a}$: an acute angle we'll call γ_0, and its supplement $180^\circ-\gamma_0$. We need to argue that each of these angles 'fits' into a triangle with a.

  - Since (α,a) and (γ_0,c) are angle-side opposite pairs, the assumption $c>a$ in this case gives $\gamma_0>\alpha$. Then, knowing that γ_0 is an acute angle, it must be true that α is acute as well. This means that one triangle can contain both α and γ_0, giving us one of the triangles promised in the theorem.
  - By manipulating the inequality $\gamma_0>\alpha$, we get $180^\circ-\gamma_0<180^\circ-\alpha$, which gives $(180^\circ-\gamma_0)+\alpha<180^\circ$. This proves a triangle can contain both of the angles α and $(180^\circ-\gamma_0)$, giving us the second triangle predicted in the theorem.

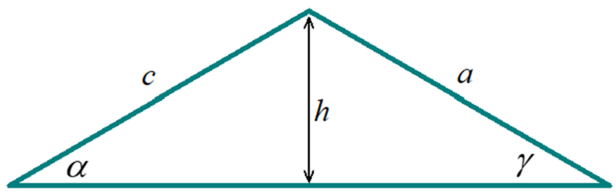
Figure 5.1. 20



$h < a < c$, two triangles

- To prove the last case in the theorem, we assume $a \geq c$. Then $\alpha \geq \gamma$, which forces γ to be an acute angle. Hence, we get only one triangle in this case, completing the proof.

Figure 5.1. 21



$a \geq c$, one triangle

One last comment before we end this discussion. In the Side-Side-Angle case, if you are given an obtuse angle to begin with then it is impossible to have the two triangle case.

Finding the Area of a Triangle

The following theorem introduces a new formula to compute the area enclosed by a triangle. Its proof uses the same cases and diagrams as the proof of the Law of Sines and is left to the reader as an exercise.

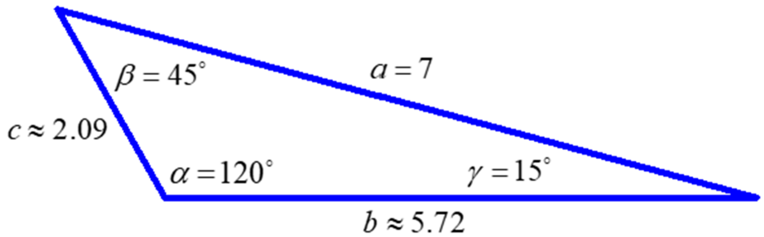
Theorem 5.3. Suppose (α, a), (β, b), and (γ, c) are the angle-side opposite pairs of a triangle. Then the area enclosed by the triangle is given by

$$A = \frac{1}{2} bc \sin(\alpha) = \frac{1}{2} ac \sin(\beta) = \frac{1}{2} ab \sin(\gamma)$$

Example 5.1.8. Find the area of the triangle in which $\alpha = 120^\circ$, $a = 7$ units and $\beta = 45^\circ$.

Solution. This is the triangle from **Example 5.1.2** in which we found all three angles and all three sides.

Figure 5.1. 22



To minimize propagated error, we choose $A = \frac{1}{2}ac\sin(\beta)$, from **Theorem 5.3**, because it uses the most pieces of given information. We are given $a = 7$ and $\beta = 45^\circ$, and we calculated $c = \frac{7\sin(15^\circ)}{\sin(120^\circ)}$ in **Example 5.1.2**. Using these values, we find

$$A = \frac{1}{2}(7)\left(\frac{7\sin(15^\circ)}{\sin(120^\circ)}\right)\sin(45^\circ)$$
$$\approx 5.18 \text{ square units}$$

The reader is encouraged to check this answer against the results obtained using the other formulas in **Theorem 5.3**.

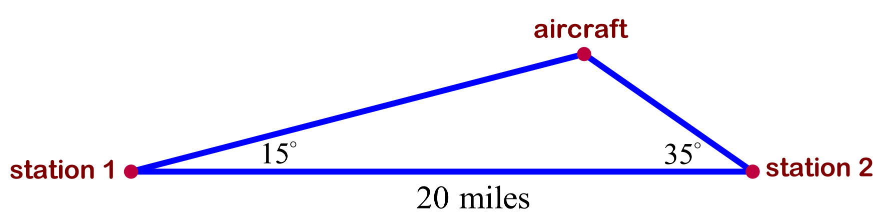
□

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 5.1.9. Suppose two radar stations located 20.0 miles apart each detect an aircraft between them, as indicated in the following figure.

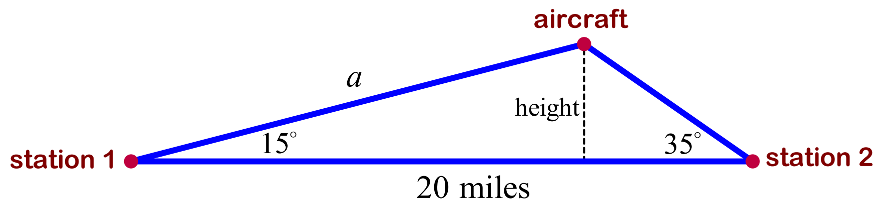
Figure 5.1. 23



The angle of elevation measured by the first station is 15 degrees, whereas the angle of elevation measured by the second station is 35 degrees. Find the altitude of the aircraft and round your answer to the nearest tenth of a mile.

Solution. To find the altitude (height) of the aircraft, we first determine the distance from one of the radar stations to the aircraft.

Figure 5.1. 24

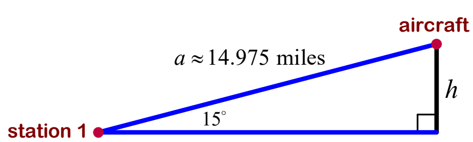


Letting a represent the distance from the first station to the aircraft, we look for an angle-side opposite pair from which we can determine the distance a. We know the measure of two angles in the triangle, but the measure of the angle opposite the side of length 20 miles is missing. Noting that the angles in a triangle add up to 180 degrees, we find the unknown angle measure to be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This gives an angle-side opposite pair with known values and allows us to set up a Law of Sines relationship.

$$\frac{\sin\left(130^\circ\right)}{20} = \frac{\sin\left(35^\circ\right)}{a}$$
$$a\sin\left(130^\circ\right) = 20\sin\left(35^\circ\right)$$
$$a = \frac{20\sin\left(35^\circ\right)}{\sin\left(130^\circ\right)}$$
$$a = 14.975\ldots$$

The distance a, from the first station to the aircraft, is about 14.975 miles. Now that we know an approximate value for a, we can use right triangle relationships to find an approximate height, h, of the aircraft.

Figure 5.1. 25



$$\sin\left(15^\circ\right) = \frac{h}{a}$$
$$h = a\sin\left(15^\circ\right)$$
$$h = \left(\frac{20\sin\left(35^\circ\right)}{\sin\left(130^\circ\right)}\right)\sin\left(15^\circ\right)$$
$$h \approx 3.9 \qquad \text{(rounded to nearest tenth)}$$

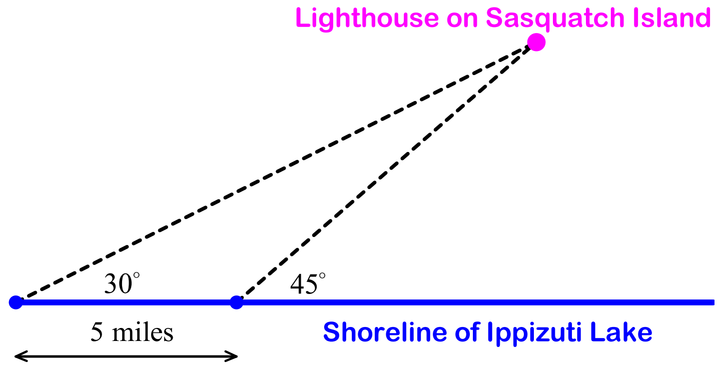
The aircraft is at an altitude of approximately 3.9 miles.

□

Note that when we do not use exact values in calculations, it is wise to include more decimal places in intermediate steps, since it may result in better accuracy for the final answer.

Example 5.1.10. Sasquatch Island lies off the shore of Ippizuti Lake. As indicated in the following figure, two sightings from the shoreline of Ippizuti Lake to the lighthouse on Sasquatch Island are taken 5 miles apart.

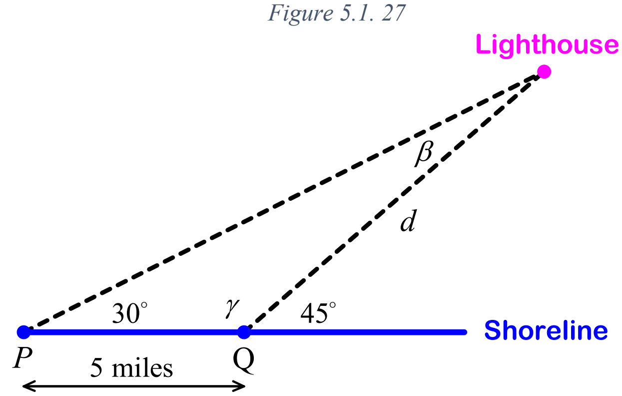
Figure 5.1. 26



The angle between the shoreline and the lighthouse at the first observation point is 30º and at the second point is 45º. Assuming a straight shoreline, find the distance from the second observation point to the lighthouse. What point on the shoreline, relative to the second observation point, is closest to the lighthouse? How far is the lighthouse from this point?

Solution. We sketch the scenario, labeling the first observation point P and the second observation point Q.

Figure 5.1. 27



In order to use the Law of Sines to find the distance d from Q to the lighthouse, we first need to find the measure of β, which is the angle opposite the side of length 5 miles. To that end, we note that the angles γ and 45º are supplementary, so that $\gamma = 180^\circ - 45^\circ = 135^\circ$. We can now find β.

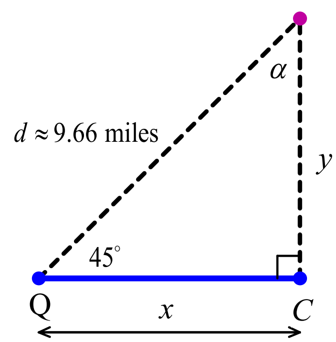
$$\begin{aligned}\beta &= 180^\circ - 30^\circ - \gamma \\ &= 180^\circ - 30^\circ - 135^\circ \\ &= 15^\circ\end{aligned}$$

By the Law of Sines, we have

$$\frac{\sin(30^\circ)}{d} = \frac{\sin(15^\circ)}{5}$$
$$d = \frac{5\sin(30^\circ)}{\sin(15^\circ)}$$
$$d \approx 9.66 \text{ miles}$$

Next, to find the point on the shoreline closest to the lighthouse, which we have labeled as C, we need to find the perpendicular distance from the lighthouse to the shoreline.[4]

Figure 5.1. 28



Let x denote the distance from the second observation point Q to the point C and let y denote the distance from C to the lighthouse. Using the right triangle definition of sine, we get

$$\sin(45^\circ) = \frac{y}{d}$$
$$y = d\sin(45^\circ)$$
$$y = \left(\frac{5\sin(30^\circ)}{\sin(15^\circ)}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$y \approx 6.83 \text{ miles}$$

The lighthouse is approximately 6.83 miles from the coast. To find the distance from Q to C, we note that $\alpha = 180^\circ - 90^\circ - 45^\circ = 45^\circ$, showing that we have an isosceles triangle, so $x = y \approx 6.83$ miles. Hence, the point on the shoreline closest to the lighthouse is approximately 6.83 miles down the shoreline from the second observation point.

□

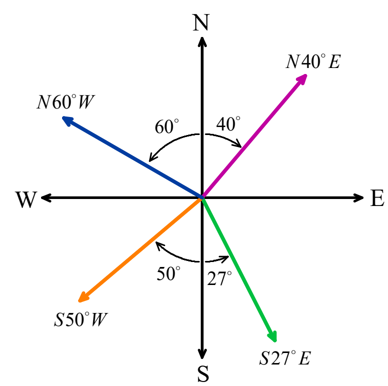
[4] Do you see why C must lie to the right of Q?

Bearings

We next introduce the navigation tool known as bearings. Simply put, a bearing is the direction you are heading according to a compass. The classic nomenclature for bearings, however, is not given as an angle in standard position, so we must first understand the notation.

A bearing is given as an acute angle of rotation (to the east or to the west) away from the north-south (up and down) line of a compass rose. For example, N40ºE (read '40º east of north') is a bearing that is rotated clockwise 40º east from due north. Similarly, for the bearing S50ºW we start out pointing due south and then rotate west (clockwise) 50º. Counter-clockwise rotations would be found in the bearings N60ºW (which is the rotation of 60° west from due north) and S27ºE (which is the rotation of 27° east from due south). These four bearings are drawn in the plane in the following sketch.

Figure 5.1. 29

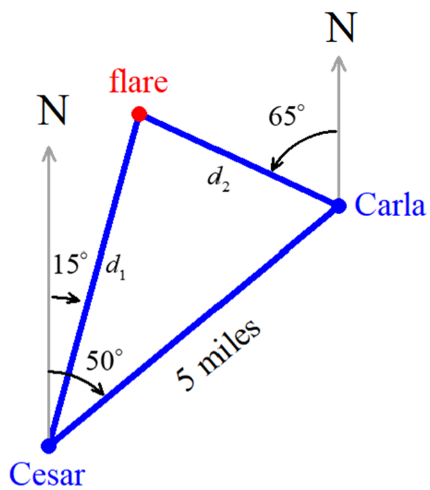


The cardinal directions north, south, east, and west are usually not given as bearings in the fashion described above, but rather referred to as 'due north', 'due south', 'due east', and 'due west', respectively. It is assumed that you know which quadrantal angle goes with each cardinal direction.

Example 5.1.11. Two hikers, Cesar and Carla, are 5 miles apart when they each sight a signal flare. Cesar observes the signal flare at a bearing of N15ºE from his current location. From her position, Carla finds the signal flare to be at a bearing of N65ºW. If the bearing from Cesar to Carla is N50ºE, find the distance from each hiker to the flare, rounded to the nearest tenth of a mile.

Solution. We sketch the problem below, labeling the distance from Cesar to the flare as d_1 and the distance from Carla to the flare as d_2.

Figure 5.1. 30



We will use the Law of Sines to find d_1 and d_2, but first need to determine the measure of the three angles: the angle γ, opposite the side of length 5; the angle α, opposite the side of length d_1; and the angle β, opposite the side of length d_2.

Figure 5.1. 31

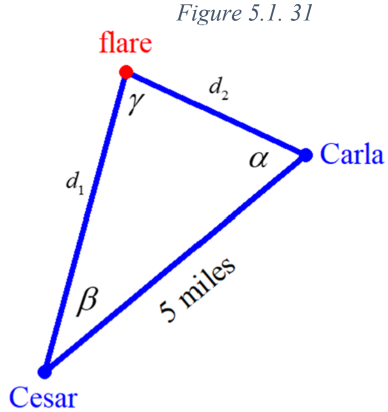
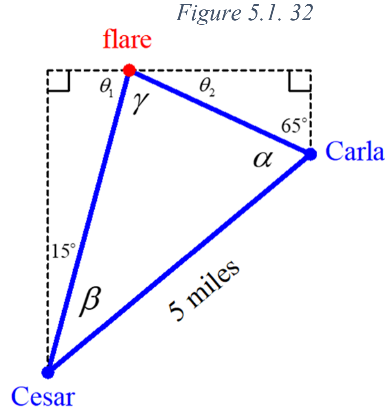
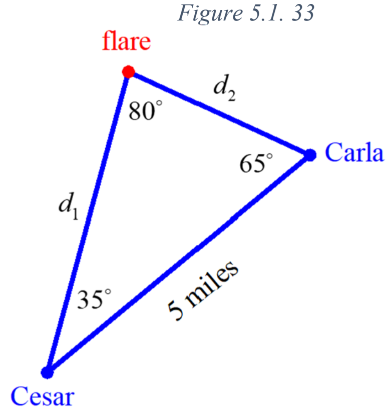


Figure 5.1. 32



From the original sketch, $\beta = 50^\circ - 15^\circ = 35^\circ$. Then, using the bearing angle measures of 15º and 65º, we find angles θ_1 and θ_2, as shown above, to have measures $90^\circ - 15^\circ = 75^\circ$ and $90^\circ - 65^\circ = 25^\circ$, respectively. Thus, $\gamma = 180^\circ - 75^\circ - 25^\circ = 80^\circ$. It follows that $\alpha = 180^\circ - 35^\circ - 80^\circ = 65^\circ$. We are ready to use the Law of Sines to determine distances d_1 and d_2.

Figure 5.1. 33



$$\frac{\sin(80^\circ)}{5} = \frac{\sin(65^\circ)}{d_1} \qquad \frac{\sin(80^\circ)}{5} = \frac{\sin(35^\circ)}{d_2}$$

$$d_1 \sin(80^\circ) = 5\sin(65^\circ) \qquad d_2 \sin(80^\circ) = 5\sin(35^\circ)$$

$$d_1 = \frac{5\sin(65^\circ)}{\sin(80^\circ)} \approx 4.6 \qquad d_2 = \frac{5\sin(35^\circ)}{\sin(80^\circ)} \approx 2.9$$

The distance from Cesar to the flare is approximately 4.6 miles, while the distance from Carla to the flare is approximately 2.9 miles.

□

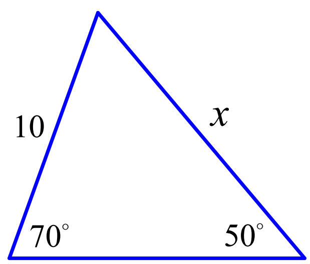
We close this section with the encouragement that, by working through the many problems in the Exercises, you will become proficient in applying the Law of Sines to real-world applications and will be ready to move on to the Law of Cosines in **Section 5.2**.

5.1 Exercises

In Exercises 1 – 12, find the value of x. Round your answers to the nearest tenth.

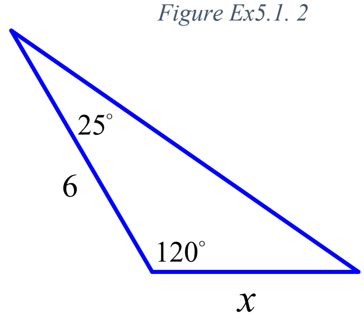
1.

Figure Ex5.1. 1



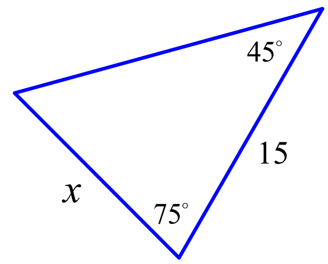
2.

Figure Ex5.1. 2



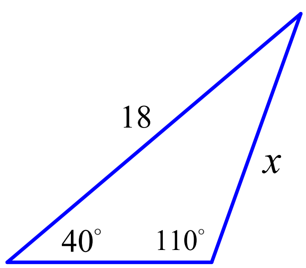
3.

Figure Ex5.1. 3



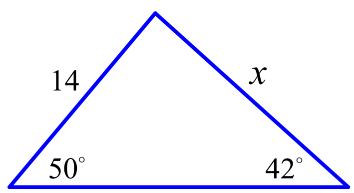
4.

Figure Ex5.1. 4



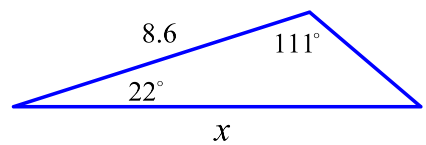
5.

Figure Ex5.1. 5



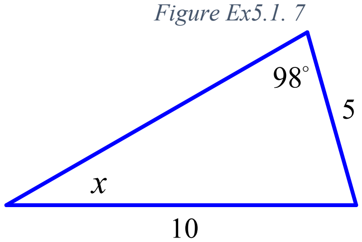
6.

Figure Ex5.1. 6



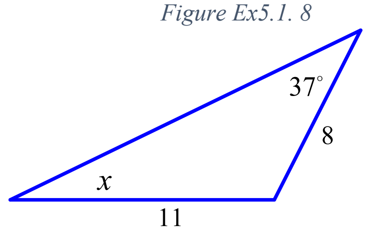
7.

Figure Ex5.1. 7



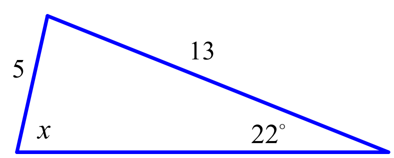
8.

Figure Ex5.1. 8



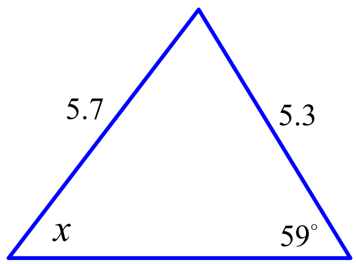
9.

Figure Ex5.1. 9



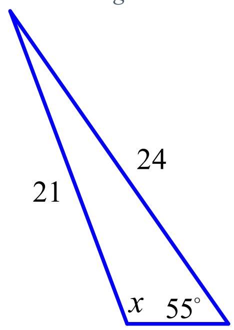
10.

Figure Ex5.1. 10



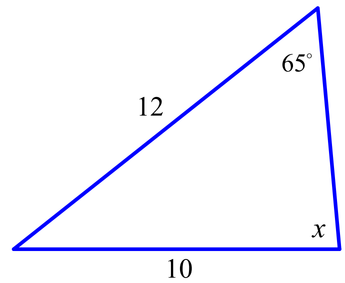
11. Note: x is an obtuse angle.

Figure Ex5.1. 11



12.

Figure Ex5.1. 12



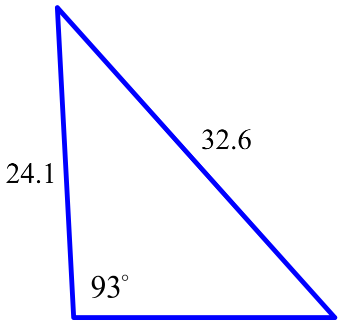
In Exercises 13 – 32, solve for the remaining side(s) and angle(s) if possible. As in the text, (α, a), (β, b), and (γ, c) are angle-side opposite pairs. Round answers to the nearest hundredth.

13. $\alpha = 13^\circ$, $\beta = 17^\circ$, $a = 5$
14. $\alpha = 73.2^\circ$, $\beta = 54.1^\circ$, $a = 117$
15. $\alpha = 95^\circ$, $\beta = 85^\circ$, $a = 33.33$
16. $\alpha = 95^\circ$, $\beta = 62^\circ$, $a = 33.33$
17. $\alpha = 117^\circ$, $a = 35$, $b = 42$
18. $\alpha = 117^\circ$, $a = 45$, $b = 42$
19. $\alpha = 68.7^\circ$, $a = 88$, $b = 92$
20. $\alpha = 42^\circ$, $a = 17$, $b = 23.5$
21. $\alpha = 68.7^\circ$, $a = 70$, $b = 90$
22. $\alpha = 30^\circ$, $a = 7$, $b = 14$
23. $\alpha = 42^\circ$, $a = 39$, $b = 23.5$
24. $\gamma = 53^\circ$, $\alpha = 53^\circ$, $c = 28.01$
25. $\alpha = 6^\circ$, $a = 57$, $b = 100$
26. $\gamma = 74.6^\circ$, $c = 3$, $a = 3.05$
27. $\beta = 102^\circ$, $b = 16.75$, $c = 13$
28. $\beta = 102^\circ$, $b = 16.75$, $c = 18$
29. $\beta = 102^\circ$, $\gamma = 35^\circ$, $b = 16.75$
30. $\beta = 29.13^\circ$, $\gamma = 83.95^\circ$, $b = 314.15$
31. $\gamma = 120^\circ$, $\beta = 61^\circ$, $c = 4$
32. $\alpha = 50^\circ$, $a = 25$, $b = 12.5$

In Exercises 33 – 38, find the area of each triangle. Round answers to the nearest tenth.

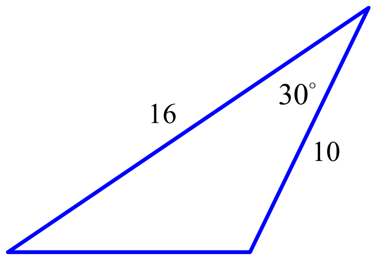
33.

Figure Ex5.1. 13



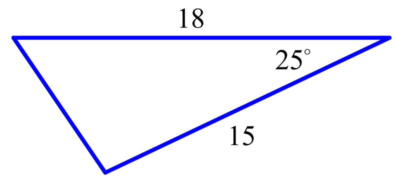
34.

Figure Ex5.1. 14



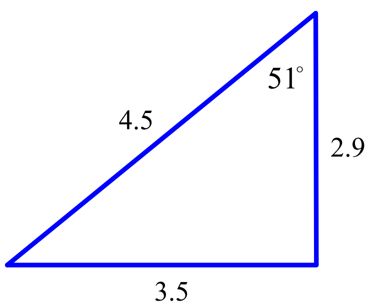
35.

Figure Ex5.1. 15



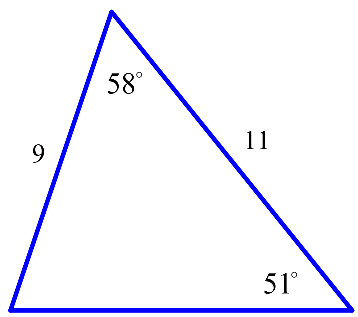
36.

Figure Ex5.1. 16



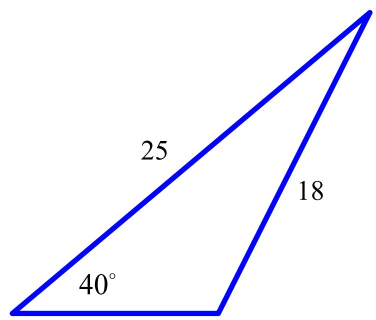
37.

Figure Ex5.1. 17



38. Assume the triangle has an obtuse angle

Figure Ex5.1. 18



39. Find the area of the triangles. As in the text, (α, a), (β, b), and (γ, c) are angle-side opposite pairs. Round to the nearest tenth.

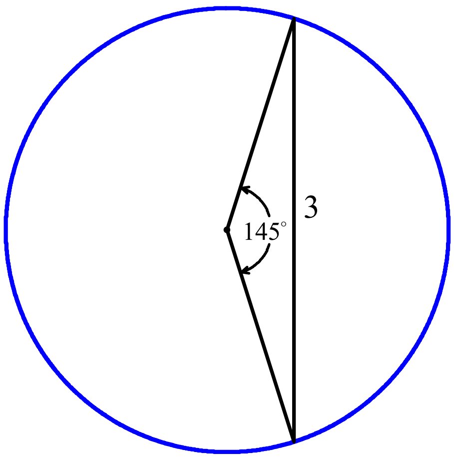
a) $\alpha = 13^\circ$, $\beta = 17^\circ$, $a = 5$ units

b) $\gamma = 53^\circ$, $\alpha = 53^\circ$, $c = 28.01$ units

c) $\alpha = 50^\circ$, $a = 25$ units, $b = 12.5$ units

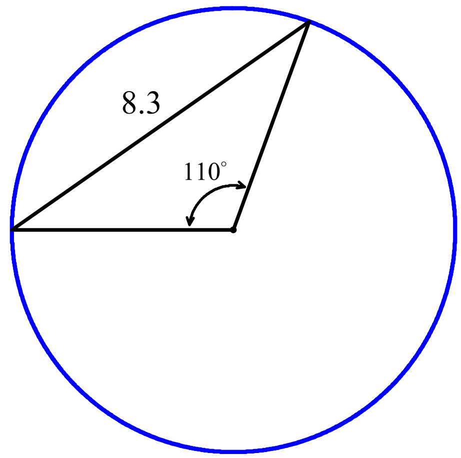
40. Find the radius of the circle.
    Round to the nearest tenth.

Figure Ex5.1. 19



41. Find the diameter of the circle.
    Round to the nearest tenth.

Figure Ex5.1. 20

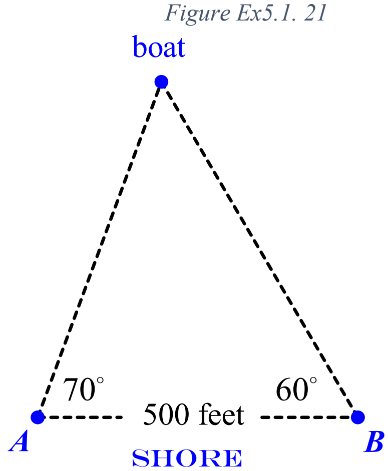


42. In order to estimate the height of a building, Sadie observes that from her current position the angle of elevation from the street to the top of the building is 39°. She moves 300 feet closer to the building and finds the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

43. A man and a woman standing 3.5 miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27° and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

44. Two radar stations, *A* and *B*, are 500 feet apart. From the radar stations, the angles out to a boat are measured to be 70° and 60°, respectively, as shown in the figure. Determine the distance of the boat from station *A* and the distance of the boat from shore. Round your answers to the nearest whole foot.

Figure Ex5.1. 21



45. Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first

search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round your answer to the nearest tenth of a mile.

46. The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1040 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°. Round to the nearest square mile.
47. Michael and Brenda decide to hunt UFOs. One night, they position themselves 2 miles apart on an abandoned stretch of desert runway. An hour into their investigation, Michael spies a UFO hovering over a spot on the runway directly between him and Brenda. He records the angle of inclination from the ground to the craft to be 75° and radios Brenda immediately to find that the angle of inclination from her position to the craft is 50°. How high off the ground is the UFO at this point? Round your answer to the nearest foot. (Recall: 1 mile is 5280 feet.)
48. A yield sign measures 30 inches on each of its three sides. What is the area of the sign?

Grade: The grade of a road is much like the pitch of a roof in that it expresses the ratio of rise/run. In the case of a road, this ratio is always positive because it is measured going uphill and it is usually given as a percentage. For example, a road that rises 7 feet for every 100 feet of (horizontal) forward progress is said to have a 7% grade. However, if we want to apply any trigonometry to a story problem involving roads going uphill or downhill, we need to view the grade as an angle with respect to the horizontal.

In Exercises 49 – 51, begin by changing road grades into angles and then apply the Law of Sines.

49. Using a right triangle with a horizontal leg of length 100 and a vertical leg of length 7, show that a 7% grade means that the road (hypotenuse) makes about a 4° angle with the horizontal. (It will not be exactly 4° but it is pretty close.)
50. What grade is given by a 9.65° angle made by the road and the horizontal?
51. Along a long, straight stretch of mountain road with a 7% grade, you see a tall tree standing perfectly plumb alongside the road.[5] From a point 500 feet downhill from the tree, the angle of inclination from the road to the top of the tree is 6°. Use the Law of Sines to find the height of the tree. (Hint: First show that the tree makes a 94° angle with the road.)

[5] The word 'plumb' here means that the tree is perpendicular to the horizontal.

52. Find the angle θ in standard position with $0^\circ \le \theta < 360^\circ$ that corresponds to each of the bearings given below.

(a) due west (b) S83°E (c) N5.5°E (d) due south

(e) N31.25°W (f) S72°W (g) N45°E (h) S45°W

53. Bozena spots a campfire at a bearing of N42°E from her current position. Piotr, who is positioned 3000 feet due east of Bozena, reckons the bearing to the fire to be N20°W from his current position. Determine the distance from the campfire to Bozena and the distance from the campfire to Piotr, rounded to the nearest foot.

54. A hiker starts walking due west from Sasquatch Point and gets to the Chupacabra Trailhead before she realizes that she hasn't reset her pedometer. From the Chupacabra Trailhead she hikes for 5 miles along a bearing of N53°W, which brings her to the Muffin Ridge Observatory. From there, she knows a bearing of S65°E will take her straight back to Sasquatch Point. How far will she have to walk to get from the Muffin Ridge Observatory to Sasquatch Point? What is the distance between Sasquatch Point and the Chupacabra Trailhead?

55. The captain of the SS Bigfoot sees a signal flare at a bearing of N15°E from her current location. From his position, the captain of the HMS Sasquatch finds the signal flare to be at a bearing of N75°W. If the SS Bigfoot is 5 miles from the HMS Sasquatch and the bearing from the SS Bigfoot to the HMS Sasquatch is N50°E, find the distances from the flare to each vessel, rounded to the nearest tenth of a mile.

56. Carl spies a potential Sasquatch nest at a bearing of N10°E and radios Jeff, who is at a bearing of N50°E from Carl's position. From Jeff's postion, the nest is at a bearing of S70°W. If Jeff and Carl are 500 feet apart, how far is Jeff from the Sasquatch nest? Round your answer to the nearest foot.

57. Lars determines that the bearing to a lodge from his current position is S40°W. He proceeds to hike 2 miles at a bearing of S20°E at which point he determines that the bearing to the lodge is S75°W. How far is he from the lodge at this point? Round your answer to the nearest hundredth of a mile.

58. A watchtower spots a ship offshore at a bearing of N70°E. A second tower, which is 50 miles from the first at a bearing of S80°E from the first tower, determines the bearing to the ship to be N25°W. How far is the boat from the second tower? Round your answer to the nearest tenth of a mile.

59. The angle of depression from an observer in an apartment complex to a gargoyle on the building next door is 55°. From a point five stories below the original observer, the angle of inclination to the gargoyle is 20°. Find the distance from each observer to the gargoyle and the distance from the gargoyle to the apartment complex. Round your answers to the nearest foot. (Use the rule of thumb that one story of a building is 9 feet.)

60. Prove that the Law of Sines holds for right triangles.

61. Why is knowing only the three angles of a triangle not enough information to determine any of the side lengths?

62. Why can the Law of Sines not be used to find the angles in a triangle when only the three side lengths are given? What about when only the lengths of two sides and the angle between them are given? (Said another way, explain why the Law of Sines cannot be used in the SSS and SAS cases.)

63. Given $\alpha = 30^\circ$ and $b = 10$, for each of the following choose a value for a so that

a) the information yields no triangle;
b) the information yields exactly one right triangle;
c) the information yields two distinct triangles;
d) the information yields exactly one obtuse triangle.

Explain why no value of a yields exactly one triangle with three acute angles.

5.2 The Law of Cosines

Learning Objectives

- **Use the Law of Cosines to solve triangles.**
- **Solve SAS and SSS triangles.**
- **Use Heron's Formula to find the area of a triangle.**
- **Use the Law of Cosines to solve applied problems.**

In **Section 5.1**, we developed the Law of Sines to enable us to solve triangles in the 'Angle-Side-Angle' (ASA), the 'Angle-Angle-Side' (AAS), and the 'Side-Side-Angle' (SSA) cases. In this section, we develop the Law of Cosines, which enables us to solve triangles in two additional cases: the 'Side-Angle-Side' (SAS) case and the 'Side-Side-Side' (SSS) case.

The Law of Cosines

We state and prove the Law of Cosines theorem below.

Theorem 5.4. The Law of Cosines: In any triangle with angle-side opposite pairs (α, a), (β, b), and (γ, c), the following relationship holds:

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

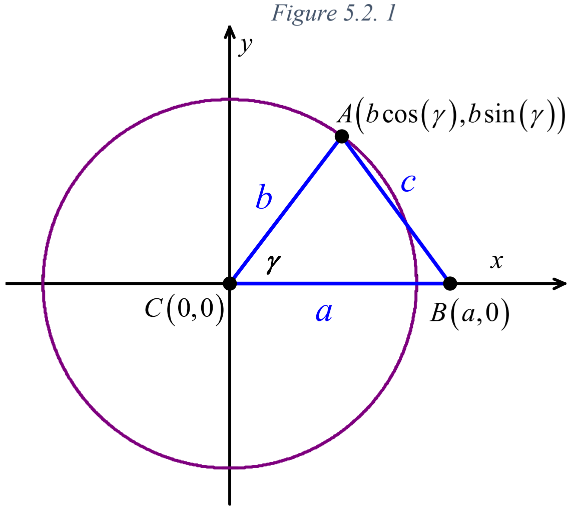
By solving for the cosine, this equation may be written as $\cos(\gamma) = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Note that the Law of Cosines reduces to the Pythagorean Theorem if γ, the angle opposite the side of length c, is 90° (since cosine of 90° is zero). We can rewrite this law in the alternate forms:

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$
$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$

To prove the theorem, we consider a generic triangle ABC, having angle-side opposite pairs (α, a), (β, b), and (γ, c). With vertices A, B, and C positioned at angles α, β, and γ, respectively, we place vertex C at the origin, and side a along the positive *x*-axis.

Figure 5.2. 1



From this set-up, we find that the coordinates of C are $(0,0)$ and that the coordinates of B are $(a,0)$. Since the point A lies on a circle of radius b, the coordinates of A are $(x,y)=\left(b\cos(\gamma),b\sin(\gamma)\right)$. (This would be true even if γ was an obtuse or right angle, so although we have drawn the case where γ is acute, the following computations hold for any angle γ where $0^\circ<\gamma<180^\circ$.) We note that the length between points A and B is the length of side c. Using the distance formula, we get

$$c=\sqrt{\left(b\cos(\gamma)-a\right)^2+\left(b\sin(\gamma)-0\right)^2}$$
$$c^2=\left(b\cos(\gamma)-a\right)^2+\left(b\sin(\gamma)\right)^2$$
$$c^2=b^2\cos^2(\gamma)-2ab\cos(\gamma)+a^2+b^2\sin^2(\gamma)$$
$$c^2=a^2+b^2\left(\cos^2(\gamma)+\sin^2(\gamma)\right)-2ab\cos(\gamma)$$
$$c^2=a^2+b^2(1)-2ab\cos(\gamma) \qquad \text{Pythagorean identity}$$
$$c^2=a^2+b^2-2ab\cos(\gamma)$$

The remaining formulas, $a^2=b^2+c^2-2bc\cos(\alpha)$ and $b^2=a^2+c^2-2ac\cos(\beta)$, can be verified by simply reorienting the triangle to place a different vertex at the origin. We leave the details to the reader. It is important to note in the preceding proof that (γ,c) is an angle-side opposite pair with a and b the sides adjacent to c. The same can be said of any other angle-side opposite pair in the triangle. In general, for an angle θ in a triangle with sides p, q, and r, where r is opposite θ, $r^2=p^2+q^2-2pq\cos(\theta)$.

The proof of the Law of Cosines relies on the distance formula, which has its roots in the Pythagorean Theorem. As noted earlier, if we have a triangle in which $\gamma=90^\circ$, then $\cos(\gamma)=\cos\left(90^\circ\right)=0$, and we get the familiar relationship $c^2 = a^2 + b^2$. What this means is that in the larger mathematical sense, the Law of Cosines is the generalization of the Pythagorean Theorem.[6] To utilize the Law of Cosines for solving a triangle, it is necessary to have a minimum of three measurements of angles and/or sides, with at least two sides included. We will explore two potential scenarios in this context.

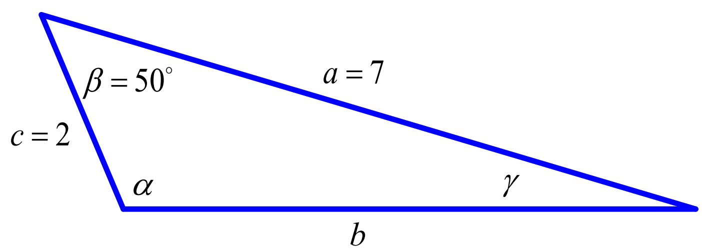
SAS (Side-Angle-Side)

Given two lengths and an angle measure less than 180 degrees for the angle formed by the two sides with the given lengths, there exists a unique triangle that satisfies these properties. We call this specific scenario the SAS case. While we will not provide a formal proof of this fact, we will illustrate it in the next example.

Example 5.2.1. Solve the triangle in which $\beta = 50^\circ$, $a = 7$ units, and $c = 2$ units. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

Solution. We are given the lengths of two sides, $a = 7$ units and $c = 2$ units, and the measure of the included angle $\beta = 50^\circ$. We begin with a representative triangle.

Figure 5.2. 2



While our first choice is generally to apply the Law of Sines, with no angle-side opposite pair to use, we apply the Law of Cosines to find b.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\ b^2 &= 7^2 + 2^2 - 2(7)(2)\cos(50^\circ) \\ b &= \sqrt{53 - 28\cos(50^\circ)} \\ b &= 5.9162\ldots \end{aligned}$$

We proceed with the Law of Cosines to solve for α by selecting the form $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$, with $a = 7$, $b \approx 5.9162$ and $c = 2$.

[6] This shouldn't come as too much of a shock. All theorems in Trigonometry can ultimately be traced back to the definition of the trigonometric functions along with the distance formula and, hence, the Pythagorean Theorem.

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$
$$(7)^2 \approx (5.9162)^2 + (2)^2 - 2(5.9162)(2)\cos(\alpha)$$
$$49 - (5.9162)^2 - 4 \approx -4(5.9162)\cos(\alpha)$$
$$\frac{49 - (5.9162)^2 - 4}{-4(5.9162)} \approx \cos(\alpha)$$

Since α is an angle in a triangle, it must be between 0 and π radians, so

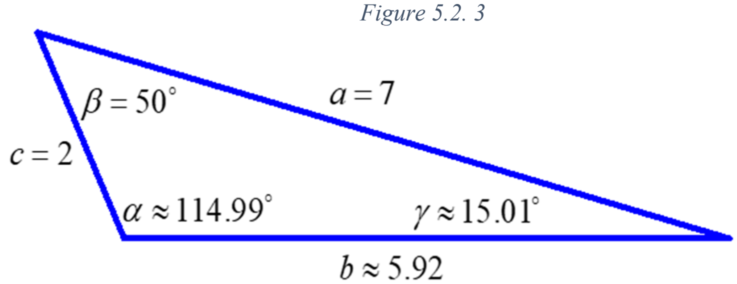
$$\alpha \approx \arccos\left(\frac{49 - (5.9162)^2 - 4}{-4(5.9162)}\right) \text{ radians}$$
$$\approx 114.99^\circ$$

Now, the simplest way to find the remaining angle is to use the fact that the sum of the angles in a triangle is 180 degrees.

$$\gamma = 180^\circ - \alpha - \beta$$
$$\approx 180^\circ - 114.99^\circ - 50^\circ$$
$$\approx 15.01^\circ$$

Below is a sketch of the triangle.

Figure 5.2. 3



□

Note that in the last example we could have used the Law of Sines to find the angle α. However, that approach would lead to two possibilities for α and requires knowing more properties of triangles to arrive at the correct answer. This is shown below.

Returning to the point in the solution where we found $b \approx 5.9162$, and having $\beta = 50^\circ$ and $a = 7$, we could use the Law of Sines to determine α:

$$\frac{\sin(\alpha)}{7} \approx \frac{\sin(50^\circ)}{5.9162}$$

The usual calculations then produce $\alpha \approx 65.01^\circ$ or $\alpha \approx 114.99^\circ$. Neither of these values can be eliminated until γ is calculated. With $c = 2$, we find

$$\frac{\sin(\gamma)}{2} \approx \frac{\sin(50^\circ)}{5.9162}$$

This results in $\gamma \approx 15.01^\circ$ or $\gamma \approx 164.99^\circ$, with the second value being too large since $\beta + 164.99^\circ > 180^\circ$. To finish off this solution,

$$\begin{aligned}\alpha &= 180^\circ - \beta - \gamma \\ &\approx 180^\circ - 50^\circ - 15.01^\circ \\ &\approx 114.99^\circ\end{aligned}$$

For future reference, although the Law of Sines may simplify the solution process, this should be restricted to the situation where the angle being determined is clearly an acute angle. Otherwise, using the Law of Cosines is generally less cumbersome.

SSS (Side-Side-Side)

Given the lengths of all three sides of a triangle, where the sum of any two lengths is greater than the remaining length, there exists a unique triangle with sides of these lengths. This specific scenario is known as the SSS case. Although we will not prove this fact, we will demonstrate it in the next example.

Example 5.2.2. Solve for the angles in the triangle with sides of lengths $a = 4$ units, $b = 7$ units, and $c = 5$ units.

Solution. Since all three sides and no angles are given, we are forced to use the Law of Cosines. While we could start by finding any of the three angles, we will solve for β first, using the formula $b^2 = a^2 + c^2 - 2ac\cos(\beta)$ along with $a = 4$, $b = 7$, and $c = 5$.

$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\ (7)^2 &= (4)^2 + (5)^2 - 2(4)(5)\cos(\beta) \\ 49 &= 41 - 40\cos(\beta) \\ \frac{49-41}{-40} &= \cos(\beta)\end{aligned}$$

Simplifying, we find $\cos(\beta) = -\frac{1}{5}$, from which

$$\begin{aligned}\beta &= \arccos\left(-\frac{1}{5}\right) \text{ radians} \\ &\approx 101.54^\circ\end{aligned}$$

Noting that β is an obtuse angle, the remaining angles α and γ must be acute, so we can use the Law of Sines without any ambiguity. To make calculations even simpler,[7] we can use $\cos(\beta) = -\frac{1}{5}$ to get $\sin(\beta) = \frac{2\sqrt{6}}{5}$. Then we have

$$\frac{\sin(\beta)}{b} = \frac{\sin(\alpha)}{a} \qquad \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

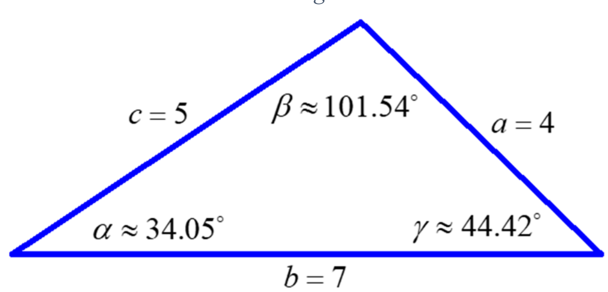
$$\frac{\left(\frac{2\sqrt{6}}{5}\right)}{7} = \frac{\sin(\alpha)}{4} \qquad \frac{\left(\frac{2\sqrt{6}}{5}\right)}{7} = \frac{\sin(\gamma)}{5}$$

$$7\sin(\alpha) = \frac{8\sqrt{6}}{5} \qquad 7\sin(\gamma) = \frac{10\sqrt{6}}{5}$$

$$\sin(\alpha) = \frac{8\sqrt{6}}{35} \qquad \sin(\gamma) = \frac{2\sqrt{6}}{7}$$

(Note that in practice, we will apply Law of Sines only once.) From these results, we find $\alpha = \arcsin\left(\frac{8\sqrt{6}}{35}\right)$ radians, which is approximately 34.05º, and $\gamma = \arcsin\left(\frac{2\sqrt{6}}{7}\right)$ radians, approximately 44.42º. A sketch of the triangle with labeled side lengths and angle measures follows.

Figure 5.2. 4



□

We note that when rounded values are carried from one step to the next, depending on how many decimal places are carried through the successive calculations, the approximate answers you obtain may differ slightly from those posted in examples and exercises. The different approaches used to solve problems may also result in slightly different answers. **Example 5.2.2** is a great example of this in that the approximate values we record for the measures of the angles sum to 180.01º, which is geometrically impossible.

We continue the discussion from **Section 5.1** on information needed to determine a triangle, with references to the two additional cases introduced in this section.

[7] Refer to **Section 1.3** if necessary, noting that β is a Quadrant II angle.

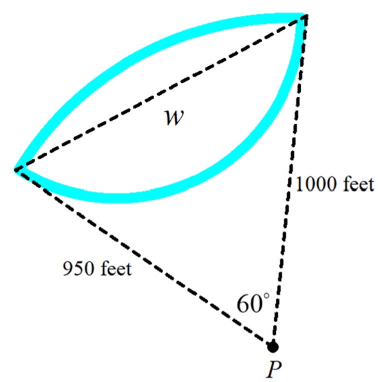
1. Given the lengths of two sides and an angle measure less than 180 degrees, where the angle with known measure is formed by the two sides with given lengths, we refer to this as the SAS case. We note that there exists a unique triangle with these properties.
2. Given the lengths of three sides of a triangle, where the sum of any two lengths is greater than the remaining length, we refer to this as the SSS case. In this case, there exists a unique triangle having sides of these lengths.

Solving Applied Problems Using the Law of Cosines

Next, we have an application of the Law of Cosines.

Example 5.2.3. A researcher wishes to determine the width of a vernal pond as drawn below. From a point P, she finds the distance to the western-most point of the pond to be 950 feet, while the distance to the northern-most point of the pond from P is 1000 feet. If the angle between the two lines of sight is 60º, find the width of the pond.

Figure 5.2. 5



Solution. We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle. Calling this length w for width (as already indicted in the drawing), we get

$$\begin{aligned} w^2 &= 950^2 + 1000^2 - 2(950)(1000)\cos\left(60^\circ\right) \\ &= 952{,}500 \end{aligned}$$

Then $w = \pm\sqrt{952500}$ and, since the distance is positive, $w = +\sqrt{952500}$, from which the width of the pond is approximately 976 feet.

□

Heron's Formula

In **Section 5.1**, the proof of the Law of Sines may be used to develop **Theorem 5.3** as a method for finding the area enclosed by a triangle. In this section, we use the Law of Cosines to derive another such formula, known as Heron's Formula.

Theorem 5.5. Heron's Formula. Suppose a, b, and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle; that is, let $s=\frac{1}{2}(a+b+c)$. Then the area A, enclosed by the triangle, is given by

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

We use **Theorem 5.3** to prove **Theorem 5.5**. Using the convention that the angle γ is opposite the side c, we have $A=\frac{1}{2}ab\sin(\gamma)$ from **Theorem 5.3**. In order to simplify computations, we start by manipulating the expression for A^2.

$$\begin{aligned} A^2 &= \left(\frac{1}{2}ab\sin(\gamma)\right)^2 \\ &= \frac{1}{4}a^2b^2\sin^2(\gamma) \\ &= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \quad \text{Pythagorean identity} \end{aligned}$$

From the Law of Cosines, we have $\cos(\gamma)=\frac{a^2+b^2-c^2}{2ab}$. Substituting this into our equation for A^2 gives

$$\begin{aligned} A^2 &= \frac{a^2b^2}{4}\left(1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right) \\ &= \frac{a^2b^2}{4}\left(1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right) \\ &= \frac{a^2b^2}{4}\left(\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right) \\ &= \frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16} \end{aligned}$$

Noting that we have a difference of squares, we factor and simplify.

$$\begin{aligned}
A^2 &= \frac{\left(2ab-\left(a^2+b^2-c^2\right)\right)\left(2ab+\left(a^2+b^2-c^2\right)\right)}{16} \\
&= \frac{\left(c^2-a^2+2ab-b^2\right)\left(a^2+2ab+b^2-c^2\right)}{16} \\
&= \frac{\left(c^2-\left(a^2-2ab+b^2\right)\right)\left(\left(a^2+2ab+b^2\right)-c^2\right)}{16} \\
&= \frac{\left(c^2-(a-b)^2\right)\left((a+b)^2-c^2\right)}{16} && \text{factor perfect square trinomials} \\
&= \frac{\left(c-(a-b)\right)\left(c+(a-b)\right)\left((a+b)-c\right)\left((a+b)+c\right)}{16} && \text{factor differences of squares} \\
&= \frac{(b+c-a)(a+c-b)(a+b-c)(a+b+c)}{16} \\
&= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2}
\end{aligned}$$

At this stage, we recognize the last factor as the semiperimeter, $s=\frac{1}{2}(a+b+c)=\frac{a+b+c}{2}$. To complete the proof, we note that

$$\begin{aligned}
(s-a) &= \frac{a+b+c}{2}-a \\
&= \frac{a+b+c-2a}{2} \\
&= \frac{b+c-a}{2}
\end{aligned}$$

Similarly, we find $(s-b)=\frac{a+c-b}{2}$ and $(s-c)=\frac{a+b-c}{2}$. Hence, we get

$$\begin{aligned}
A^2 &= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2} \\
&= (s-a)(s-b)(s-c)s
\end{aligned}$$

so that $A=\sqrt{s(s-a)(s-b)(s-c)}$ as required.

We close with an example of Heron's Formula.

Example 5.2.4. Find the area of the triangle in **Example 5.2.2**; that is, the triangle with sides of lengths $a=4$ units, $b=7$ units, and $c=5$ units.

Solution. Using the side lengths $a=4$ units, $b=7$ units, and $c=5$ units, we find the semiperimeter is $s=\frac{1}{2}(4+7+5)=8$ units. Then, applying Heron's Formula, results in

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{8(8-4)(8-7)(8-5)} \\ &= \sqrt{8(4)(1)(3)} \\ &= \sqrt{96} \\ &= 4\sqrt{6} \approx 9.80 \text{ square units} \end{aligned}$$

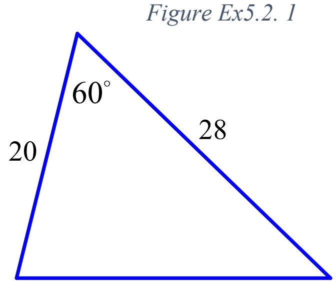
□

5.2 Exercises

In Exercises 1 – 4, solve for the length of unknown side. Round final answers to the nearest tenth.

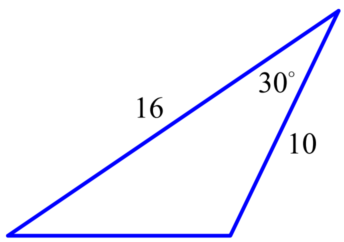
1.

Figure Ex5.2. 1



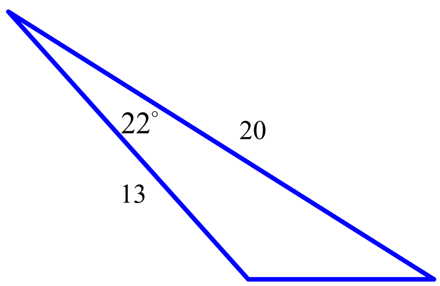
2.

Figure Ex5.2. 2



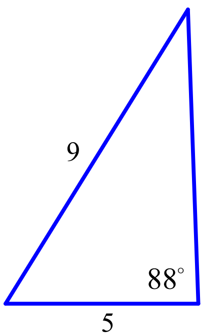
3.

Figure Ex5.2. 3



4.

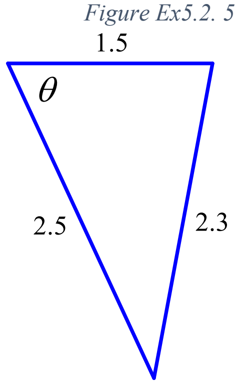
Figure Ex5.2. 4



In Exercises 5 – 8, find the measure of the angle θ. Round final answers to the nearest tenth.

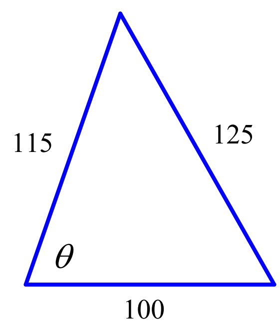
5.

Figure Ex5.2. 5



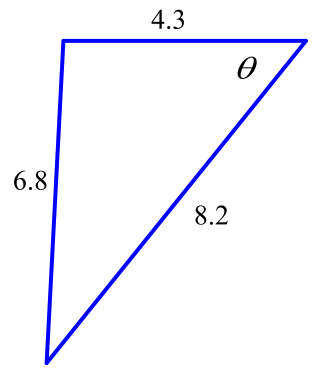
6.

Figure Ex5.2. 6



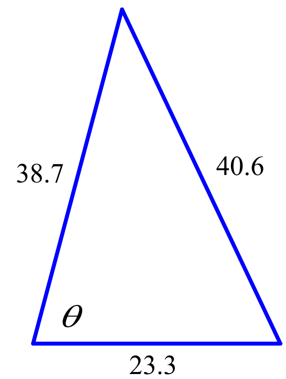
7.

Figure Ex5.2. 7



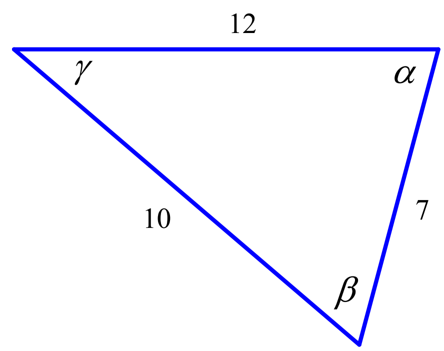
8.

Figure Ex5.2. 8



9. Find the measure of each angle in the triangle. Round final answers to the nearest tenth.

Figure Ex5.2. 9



In Exercises 10 – 19, use the Law of Cosines to find the remaining side(s) and angle(s), if possible. Round final answers to the nearest hundredth.

10. $a=7,\ b=12,\ \gamma=59.3^\circ$

11. $b=25,\ c=37,\ \alpha=104^\circ$

12. $a=153,\ c=153,\ \beta=8.2^\circ$

13. $a=3,\ b=4,\ \gamma=90^\circ$

14. $b=3,\ c=4,\ \alpha=120^\circ$

15. $a=7,\ b=10,\ c=13$

16. $a=1,\ b=2,\ c=5$

17. $a=300,\ b=302,\ c=48$

18. $a=5,\ b=5,\ c=5$

19. $a=5,\ b=12,\ c=13$

In Exercises 20 – 25, solve for the remaining side(s) and angle(s), if possible, using any appropriate technique. Round final answers to the nearest hundredth.

20. $a = 18,\ b = 20,\ \alpha = 63^\circ$

21. $a = 37,\ b = 45,\ c = 26$

22. $a = 16,\ b = 20,\ \alpha = 63^\circ$

23. $a = 22,\ b = 20,\ \alpha = 63^\circ$

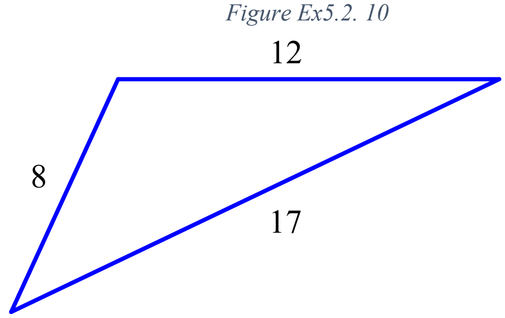
24. $b = 117,\ c = 88,\ \alpha = 42^\circ$

25. $c = 98.6,\ \beta = 7^\circ,\ \gamma = 170^\circ$

In Exercises 26 – 29, find the area of the triangle. Round final answers to the nearest hundredth.

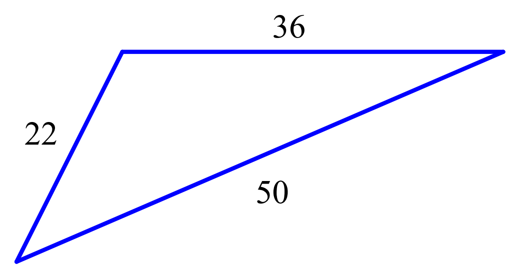
26.

Figure Ex5.2. 10



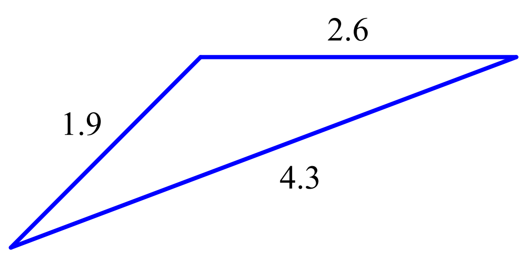
27.

Figure Ex5.2. 11



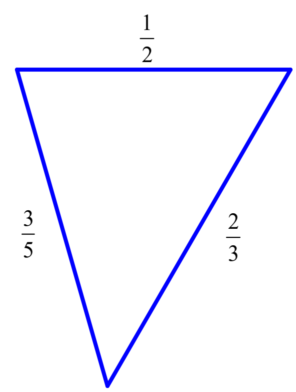
28.

Figure Ex5.2. 12



29.

Figure Ex5.2. 13



30. Find the area of the triangles. Round final answers to the nearest hundredth.

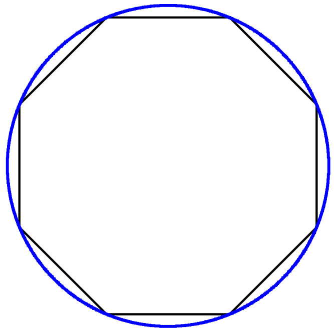
a) $a = 7,\ b = 10,\ c = 13$

b) $a = 300,\ b = 302,\ c = 48$

c) $a = 5,\ b = 12,\ c = 13$

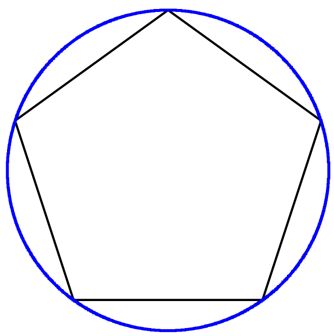
31. A regular octagon is inscribed in a circle with a radius of 8 inches. Find the perimeter of the octagon. Round your answer to the nearest hundredth.

Figure Ex5.2. 14



32. A regular pentagon is inscribed in a circle of radius 12 cm. Find the perimeter of the pentagon. Round your answer to the nearest tenth.

Figure Ex5.2. 15



33. The hour hand on an antique Seth Thomas schoolhouse clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

34. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.

35. From the Pedimaxus International Airport, a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of N8.2°E for 192 miles or it can fly to Bigfoot Falls by following a bearing of S68.5°E for 207 miles. Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.

36. Cliffs of Insanity Point and Bigfoot Falls, from **Exercise 35**, both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.

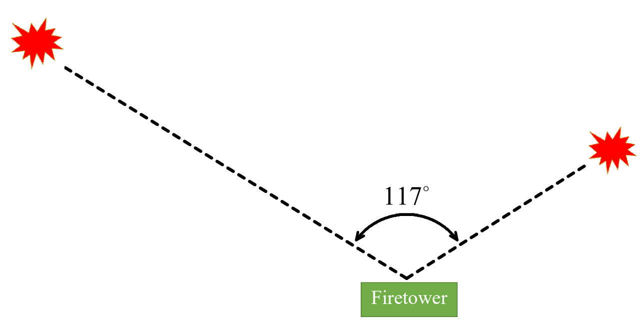
37. Sarah sets off on her hike from the lodge at a bearing of S80°W. After 1.5 miles, she changes her bearing to S17°W and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

38. The HMS Sasquatch leaves port on a bearing of N23°E and travels for 5 miles. It then changes course and follows a bearing of S41°E for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

39. The SS Bigfoot leaves harbor bound for Nessie Island, which is 300 miles away, at a bearing of N32°E. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now S70°W, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

40. From a point 300 feet above ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by the line of sight from the ranger to the second fire is 1.3°. The angle formed by the two lines of sight is 117°. Find the distance between the two fires. Round your answer to the nearest foot. (Hint: In order to use the 117° angle between the lines of sight, you will first need to use right triangle Trigonometry to find the lengths of the lines of sight. This will give you an SAS case in which to apply the Law of Cosines.)

Figure Ex5.2. 16



CHAPTER 6
POLAR COORDINATES AND APPLICATIONS

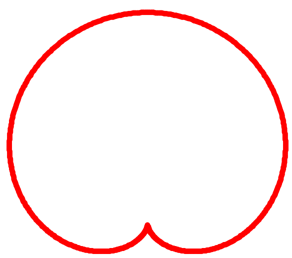


Figure 6.0. 1

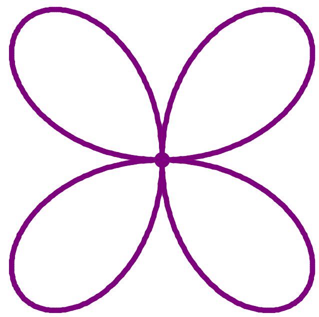


Figure 6.0. 2

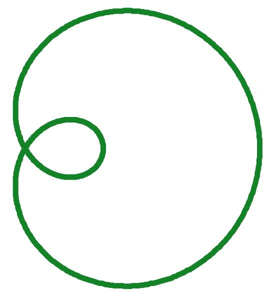


Figure 6.0. 3

Chapter Outline

Introduction

You have likely become familiar with the Cartesian coordinate system and its use of a rectangular grid for locating points. The polar coordinate system is another coordinate system that relies only on direction and distance. Making the transition from Cartesian to polar coordinates may seem awkward and strange at first. However, we will find that it becomes a more natural way of describing certain curves, like circles. In polar coordinates, some seemingly complicated curves can be represented by relatively simple polar equations. We will also describe complex numbers using polar coordinates in order to perform certain complex number calculations more simply.

Section 6.1 introduces polar coordinates, along with conversions between polar and rectangular coordinates and equations. In Section 6.2, various polar equations are discussed and graphed. Section 6.3 discusses complex numbers, their graphs in the complex plane, and properties of complex numbers. Finally, Section 6.4 is all about computing products, powers, quotients, and roots of complex numbers.

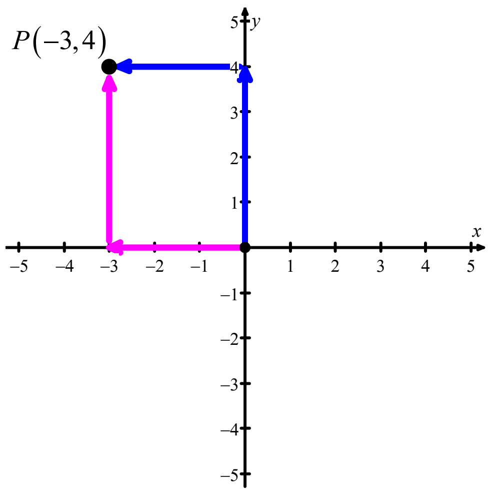
6.1 Polar Coordinates and Equations

Learning Objectives

- **Graph points in polar coordinates.**
- **Convert points in polar coordinates to rectangular coordinates and vice versa.**
- **Convert an equation from rectangular coordinates into polar coordinates.**
- **Convert an equation from polar coordinates into rectangular coordinates.**

Up to this point, we have graphed points in the Cartesian coordinate plane by assigning ordered pairs of numbers to points in the plane. We defined the Cartesian coordinate plane using two number lines, one horizontal and one vertical, which intersect at right angles at a point called the origin. To plot a point, say $P(-3,4)$, we start at the origin, travel horizontally to the left 3 units, then up 4 units.

Figure 6.1. 1



Alternatively, we could start at the origin, travel up 4 units, then to the left 3 units and arrive at the same location. For the most part, the motions of the Cartesian system (such as 'over and up') describe a rectangle, and most points can be thought of as the corner diagonally across the rectangle from the origin.[1] For this reason, the Cartesian coordinates of a point are often called rectangular coordinates.

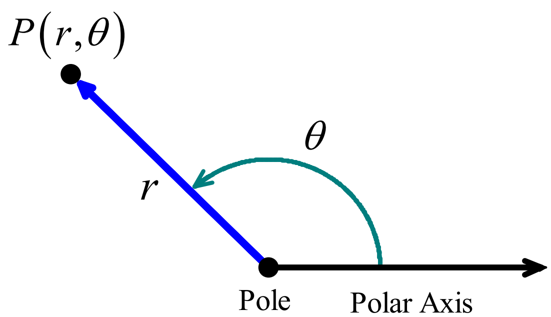
In this section, we introduce **polar coordinates**, a new system for assigning coordinates to points in the plane.

[1] Excluding, of course, the points in which one or both coordinates are 0.

Plotting Polar Coordinates

We start with a point of origin, called the **pole**, and a ray called the **polar axis** with its initial point at the pole. In Cartesian coordinates, we can interpret the pole as the origin and the polar axis as the positive side of the *x*-axis. We then locate a point P using two coordinates (r,θ), where r represents the directed distance from the pole (the distance of P from the pole is $|r|$) and θ is the measure of rotation about the pole from the polar axis. (We will discuss and demonstrate what it means for r to be negative.)

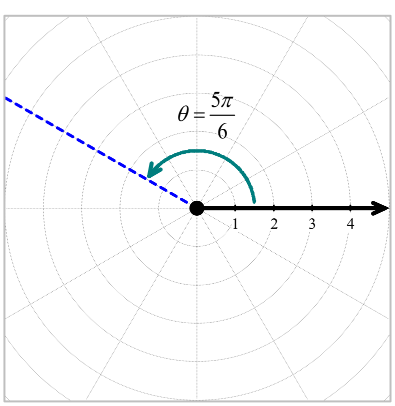
Figure 6.1. 2



Example 6.1.1. Plot the point P with polar coordinates $\left(4,\frac{5\pi}{6}\right)$.

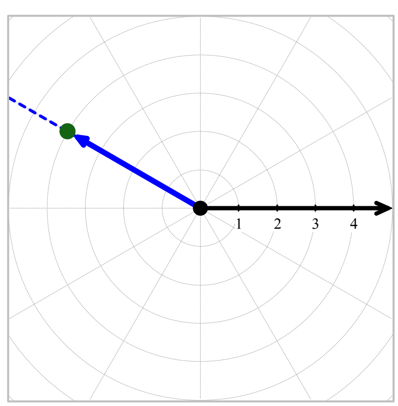
Solution. To plot the polar point $P\left(4,\frac{5\pi}{6}\right)$, we start with the rotation $\theta=\frac{5\pi}{6}$. From the polar axis, we rotate $\frac{5\pi}{6}$ radians counter-clockwise about the pole. We then move outward from the pole 4 units along the resulting ray. Essentially, we are locating a point on the terminal side of the standard position of the angle $\frac{5\pi}{6}$ that is 4 units away from the pole. Note that adding concentric circles to the sketch makes it easier to plot points.

Figure 6.1. 3



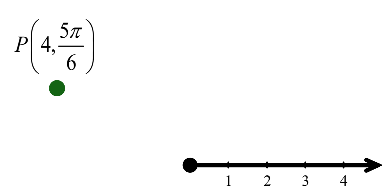
First

Figure 6.1. 4



Second

Figure 6.1. 5



The Resulting Point

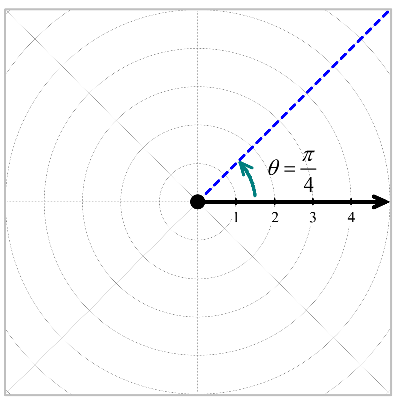
□

If $r < 0$, we first determine the ray that results from the rotation of θ, and then move $|r|$ units from the pole in the direction opposite to that ray.

Example 6.1.2. Plot the point Q with polar coordinates $\left(-3.5, \frac{\pi}{4}\right)$.

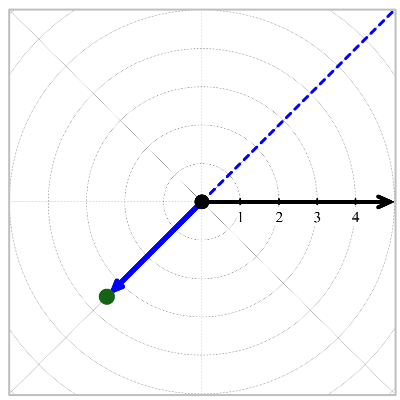
Solution. We start with a rotation of $\frac{\pi}{4}$ radians counter-clockwise from the polar axis, and then move 3.5 units in the opposite direction from the pole. Note that we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.

Figure 6.1. 6



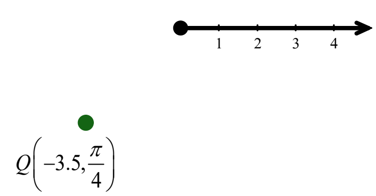
First

Figure 6.1. 7



Second

Figure 6.1. 8



The Resulting Point

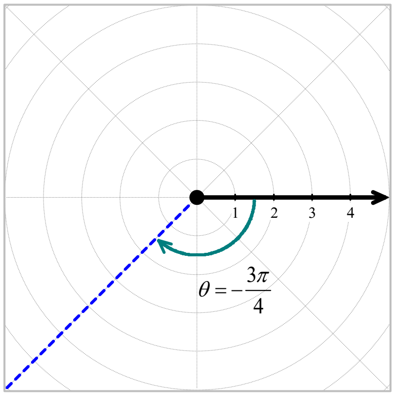
□

As you may have guessed, $\theta < 0$ means that the rotation away from the polar axis is clockwise instead of counter-clockwise.

Example 6.1.3. Plot the point R with polar coordinates $\left(3.5, -\frac{3\pi}{4}\right)$.

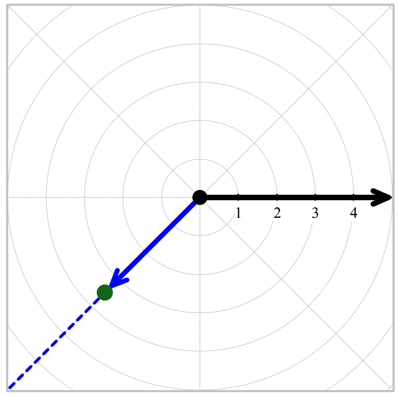
Solution. To plot $R\left(3.5, -\frac{3\pi}{4}\right)$, we first rotate $-\frac{3\pi}{4}$ radians from the polar axis, and then move out 3.5 units from the pole.

Figure 6.1. 9



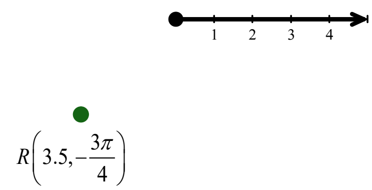
First

Figure 6.1. 10



Second

Figure 6.1. 11



The Resulting Point

□

Multiple Representations for Polar Coordinates

The points Q and R in the above examples are the same point despite the fact that their polar coordinate representations are different. Unlike Cartesian coordinates where (a,b) and (c,d) represent the same point if and only if $a=c$ and $b=d$, a point in polar coordinates has infinitely many representations. We explore this notion in the following examples.

Example 6.1.4. Plot the point $P\left(2,240^\circ\right)$, given in polar coordinates, and then find two additional representations for the point, one of which has $r>0$ and the other with $r<0$.

Solution. We rotate 240º before moving out 2 units from the pole to plot $P\left(2,240^\circ\right)$.

Figure 6.1. 12

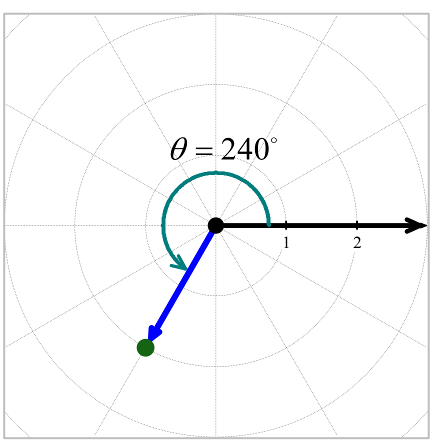
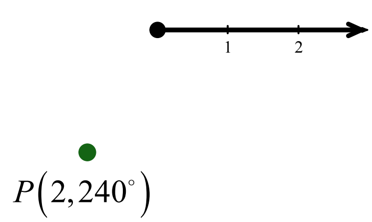


Figure 6.1. 13



We now set about finding alternate descriptions (r,θ) for the point P. Since P is 2 units from the pole, $|r|=2$, from which $r=\pm 2$. We can find appropriate θ values, different from 240°, so that each of the polar coordinates $(r,\theta)=(2,\theta)$ and $(r,\theta)=(-2,\theta)$ represent the point P.

- To have coordinates of the form $(r,\theta)=(2,\theta)$ for the point P, the angle θ must be coterminal with 240°. As seen on the graph, one such angle is $\theta=-120^{\circ}$, resulting in the representation $(2,-120^{\circ})$.

  Figure 6.1. 14

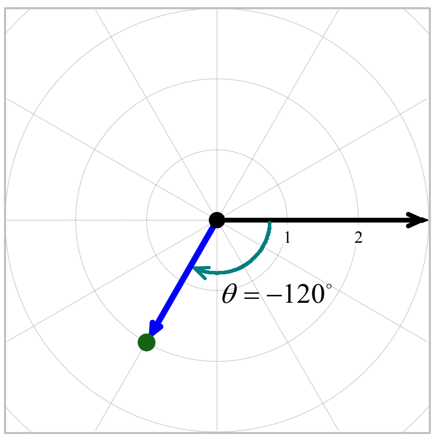
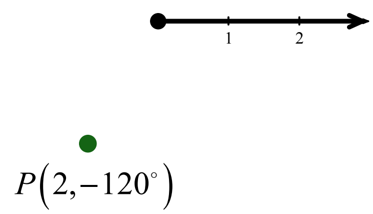
  

  Figure 6.1. 15

  Since infinitely many angles have the same terminal side, there are many other correct answers such as $(2,240^{\circ}+360^{\circ})=(2,600^{\circ})$ or $(2,-120^{\circ}-360^{\circ})=(2,-480^{\circ})$.

- To have coordinates of the form $(r,\theta)=(-2,\theta)$, the point P must be on the opposite side of the terminal side of the angle θ. As seen below, one such angle is $\theta=60^{\circ}$, resulting in the representation $(-2,60^{\circ})$.

  Figure 6.1. 16

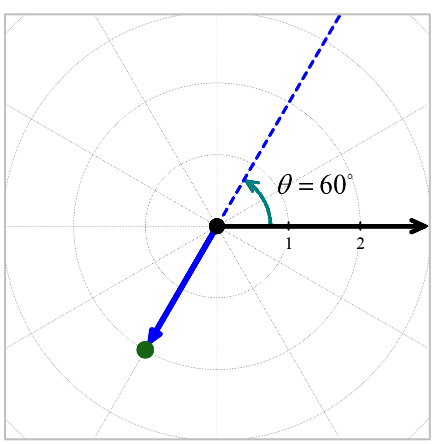
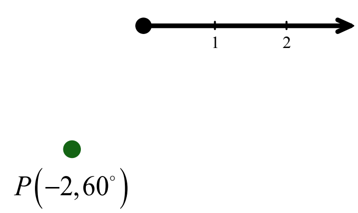
  

  Figure 6.1. 17

The infinitely many representations for the point P include $(-2, 60^\circ + 360^\circ) = (-2, 420^\circ)$ and $(-2, 60^\circ - 360^\circ) = (-2, -300^\circ)$.

□

Example 6.1.5. Plot the point $P\left(-4, \frac{7\pi}{6}\right)$, given in polar coordinates, and then find two additional representations for the point, one with $r > 0$ and one with $r < 0$.

Solution. We plot $\left(-4, \frac{7\pi}{6}\right)$ by first rotating $\frac{7\pi}{6}$ radians counter-clockwise from the polar axis, and then moving 4 units in the opposite direction from the pole.

Figure 6.1. 18

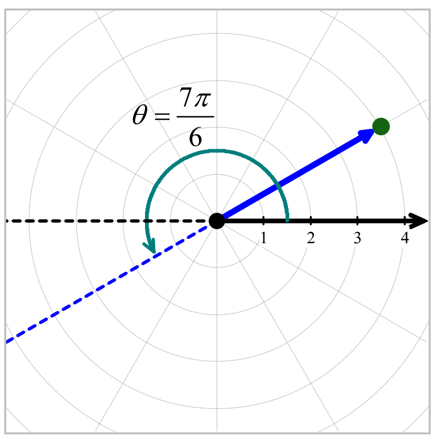
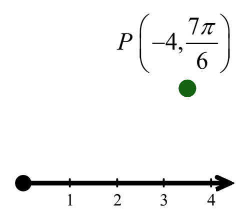


Figure 6.1. 19



To find alternate descriptions for P, we note that the distance from P to the pole is 4 units, so any representation (r, θ) for P must have $r = \pm 4$. Such representations are of the form $(r, \theta) = (4, \theta)$ or $(r, \theta) = (-4, \theta)$.

- To have coordinates of the form $(r, \theta) = (4, \theta)$ for point P, the terminal side of angle θ must lie on the line that contains the terminal side of $\frac{7\pi}{6}$. One such angle is $\frac{\pi}{6}$, as seen below in **Figure 6.1.20**. Coupled with $r = 4$, this gives $\left(4, \frac{\pi}{6}\right)$ as one solution. Noting that there are infinitely many correct answers, another solution is $\left(4, \frac{\pi}{6} + 2\pi\right) = \left(4, \frac{13\pi}{6}\right)$.
- For different coordinates of the form $(r, \theta) = (-4, \theta)$ for P, the angle θ must be coterminal with the angle $\frac{7\pi}{6}$. As seen in **Figure 6.1.21**, one such angle is $\theta = -\frac{5\pi}{6}$, resulting in the

representation $\left(-4,-\frac{5\pi}{6}\right)$. Another of the infinitely many correct answers here is

$\left(-4,-\frac{5\pi}{6}-2\pi\right)=\left(-4,-\frac{17\pi}{6}\right)$.

Figure 6.1. 20

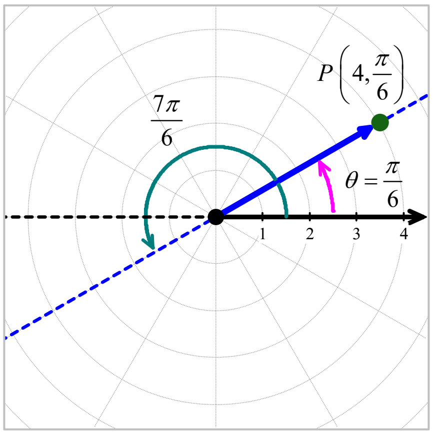
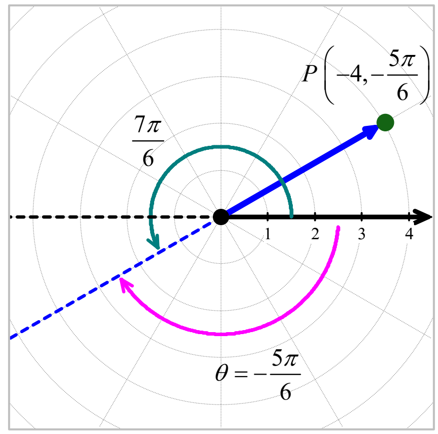


Figure 6.1. 21



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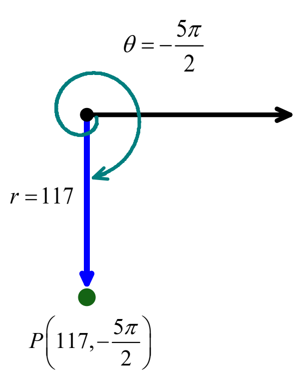
Example 6.1.6. Plot the point $P\left(117,-\frac{5\pi}{2}\right)$, given in polar coordinates, and then find two additional representations for the point, one with $r>0$ and one with $r<0$.

Solution. To plot $P\left(117,-\frac{5\pi}{2}\right)$, we rotate $\frac{5\pi}{2}$ radians clockwise from the polar axis, and then move 117 units outward from the pole.

Figure 6.1. 22



Since P is 117 units from the pole, any representation (r,θ) for P must have $r=117$ or $r=-117$.

- For the $r=117$ case, we can take θ to be any angle coterminal with $-\frac{5\pi}{2}$. We choose $\theta=\frac{3\pi}{2}$ to get $\left(117,\frac{3\pi}{2}\right)$ as one representation for P.
- For $r=-117$, the terminal side of the angle θ must lie on the line that contains the terminal side of the angle $-\frac{5\pi}{2}$. One such angle is $\theta=\frac{\pi}{2}$, so $\left(-117,\frac{\pi}{2}\right)$ is a representation for P.

Figure 6.1. 23

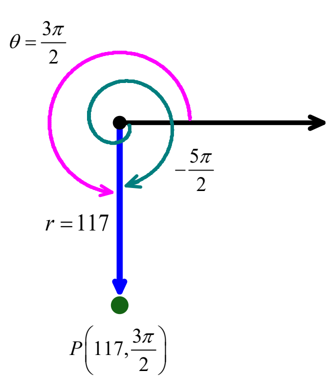
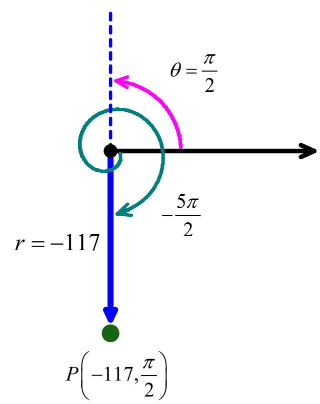


Figure 6.1. 24

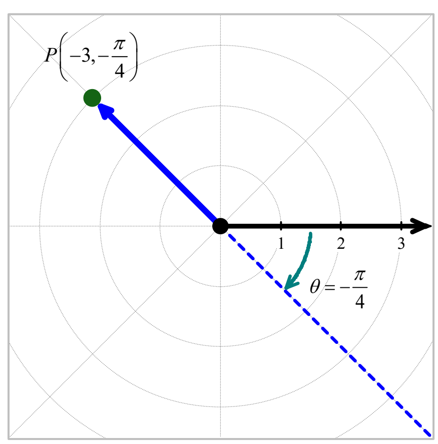


□

Example 6.1.7. Plot the point $P\left(-3,-\frac{\pi}{4}\right)$, given in polar coordinates, and then find two additional representations for the point, one with $r>0$ and one with $r<0$.

Solution. To plot $P\left(-3,-\frac{\pi}{4}\right)$, we start with a clockwise rotation of $\frac{\pi}{4}$ radians from the polar axis and follow up by moving 3 units in the opposite direction from the pole.

Figure 6.1. 25



- Since P lies on the terminal side of $\frac{3\pi}{4}$, one alternate representation for P is $\left(3,\frac{3\pi}{4}\right)$.
- To find a different representation for P with $r=-3$, we may choose any angle coterminal with $-\frac{\pi}{4}$. We select $\theta=\frac{7\pi}{4}$, for a representation of P as $\left(-3,\frac{7\pi}{4}\right)$.

Figure 6.1. 26

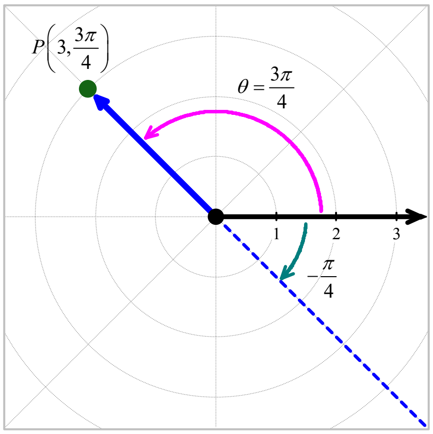
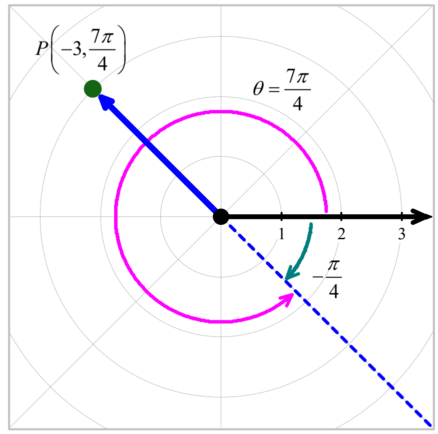


Figure 6.1. 27



□

Now that we have some practice plotting points in polar coordinates, it should come as no surprise that any given point expressed in polar coordinates has infinitely many other representations. The following property summarizes characteristics of different polar coordinates that determine the same point in the plane.

Equivalent Representations of a Point P in Polar Coordinates (r,θ)

- The polar coordinates $(r,\theta+2\pi k)$, for any integer k, represent the same point P.
- The polar coordinates $(-r,\theta+(2k+1)\pi)$, for any integer k, represent the same point P.

As verification of the above characteristics, note that the polar coordinates (r,θ) and $(r,\theta+2\pi k)$ represent the same point since θ and $\theta+2\pi k$ have the same terminal side. To have an equivalent representation with $-r$ in place of r, the terminal side of the corresponding angle must be on the opposite side of the terminal side of angle θ. Since the terminal sides of the angles θ and $\theta+\pi$ are opposite one another, the polar coordinates (r,θ) and $(-r,\theta+(2k+1)\pi)$ represent the same point.

In the special case where $r=0$, all polar coordinates $(0,\theta)$ represent the same point (the pole) regardless of the value of θ.

Converting Between Rectangular and Polar Coordinates

Next, we connect the polar coordinate system with the Cartesian (rectangular) coordinate system so that we can convert coordinates from one system to the other. We relate the pole and polar axis in the polar system to the origin and positive x-axis, respectively, in the rectangular system with the following result.

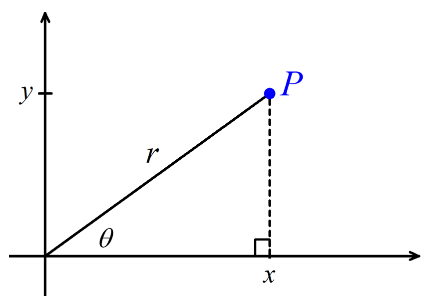
Theorem 6.1. Conversion Between Rectangular and Polar Coordinates: Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates as (r, θ). Then

- $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$

To verify this result, we check out the three cases: $r > 0$, $r < 0$ and $r = 0$.

1. In the case $r > 0$, the theorem is an immediate consequence of **Theorem 1.4**. Recall that

Figure 6.1. 28



$$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta)$$

$$\sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$$

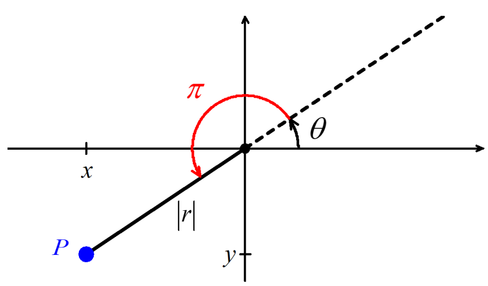
Additionally, the theorem states that

$$x^2 + y^2 = r^2$$

We apply the quotient identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ to verify that $\tan(\theta) = \dfrac{y}{x}$.

2. If $r < 0$, then an alternate representation for (r, θ) is $(-r, \theta + \pi)$.

Figure 6.1. 29



Using $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$, we apply **Theorem 1.4** as follows:

$$x = (-r)\cos(\theta + \pi) \qquad y = (-r)\sin(\theta+\pi)$$
$$= (-r)(-\cos(\theta)) \qquad = (-r)(-\sin(\theta))$$
$$= r\cos(\theta) \qquad = r\sin(\theta)$$

Moreover, $x^2 + y^2 = (-r)^2 = r^2$ and $\frac{y}{x} = \tan(\theta+\pi) = \tan(\theta)$, so the theorem is true in this case.

3. The remaining case is $r = 0$, where $(r,\theta) = (0,\theta)$ is the pole. Since the pole is identified with the origin, $(0,0)$ in rectangular coordinates, proving the theorem in this case amounts to checking '$0 = 0$'.

The following examples put **Theorem 6.1** to good use.

Example 6.1.8. Convert the point $\left(4, \frac{5\pi}{3}\right)$, in polar coordinates, to rectangular coordinates.

Solution. We convert $(r,\theta) = \left(4, \frac{5\pi}{3}\right)$ to rectangular coordinates as follows.

$$x = r\cos(\theta) = 4\cos\left(\frac{5\pi}{3}\right) = 4\left(\frac{1}{2}\right) = 2$$

$$y = r\sin(\theta) = 4\sin\left(\frac{5\pi}{3}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

The rectangular coordinates of the polar point $\left(4, \frac{5\pi}{3}\right)$ are $\left(2, -2\sqrt{3}\right)$.

□

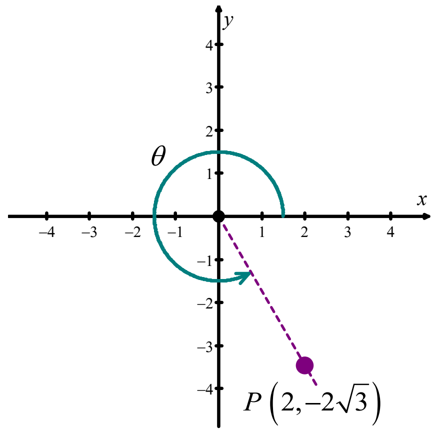
Example 6.1.9. Convert each point in rectangular coordinates, given below, to polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$. Check each answer by converting back to rectangular coordinates.

1. $P\left(2, -2\sqrt{3}\right)$
2. $Q(-3,-3)$
3. $R(0,-3)$
4. $S(-3,4)$

Solution.

1. Even though we are not explicitly told to do so, we can avoid many common mistakes by determining which quadrant $P\left(2, -2\sqrt{3}\right)$ is in, which we will do by plotting it.

Figure 6.1. 30



Plotting $P\left(2,-2\sqrt{3}\right)$ shows that P lies in Quadrant IV. With $x=2$ and $y=-2\sqrt{3}$, we use $r^2=x^2+y^2$ to get

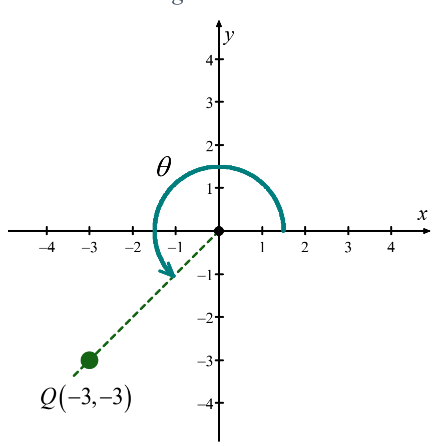
$$\begin{aligned} r^2 &= (2)^2+\left(-2\sqrt{3}\right)^2 \\ &= 4+(4)(3) \\ &= 16 \end{aligned}$$

So $r=\pm 4$ and, since we want $r \geq 0$, we choose $r=4$. To find θ, we use $\tan(\theta)=\frac{y}{x}$, with $x=2$ and $y=-2\sqrt{3}$, to get $\tan(\theta)=\frac{-2\sqrt{3}}{2}=-\sqrt{3}$. This tells us that θ has a reference angle of $\frac{\pi}{3}$ and, since θ is a Quadrant IV angle, $\theta=\frac{5\pi}{3}$. Note that this value for θ also meets the requirement that $0 \leq \theta < 2\pi$. Thus, we can write P in polar coordinates as $\left(4,\frac{5\pi}{3}\right)$.

To check our answer, we revisit **Example 6.1.8** where this result is confirmed.

2. The point $Q\left(-3,-3\right)$ is in Quadrant III.

Figure 6.1. 31



Using $x = y = -3$, we get $r^2 = (-3)^2 + (-3)^2 = 18$, so $r = \pm\sqrt{18} = \pm 3\sqrt{2}$. We are asked for $r \geq 0$ so we choose $r = 3\sqrt{2}$. To determine θ, with $x = y = -3$, we start with $\tan(\theta) = \frac{-3}{-3} = 1$. Then θ has a reference angle of $\frac{\pi}{4}$ and, since Q lies in Quadrant III, we choose $\theta = \frac{5\pi}{4}$, which satisfies the requirement that $0 \leq \theta < 2\pi$. The point Q in polar coordinates is $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$.

To check, we find

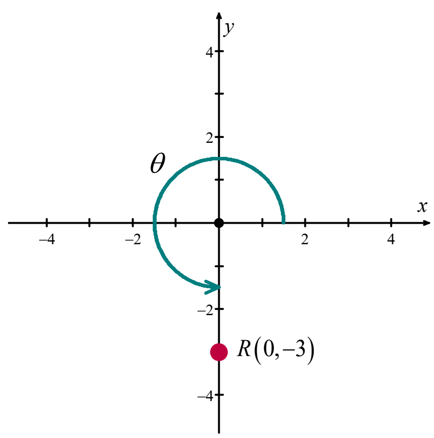
$$x = r\cos(\theta) = \left(3\sqrt{2}\right)\cos\left(\frac{5\pi}{4}\right) = \left(3\sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -3$$

$$y = r\sin(\theta) = \left(3\sqrt{2}\right)\sin\left(\frac{5\pi}{4}\right) = \left(3\sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -3$$

The resulting point $(-3,-3)$ verifies our solution.

3. The point $R(0,-3)$ lies along the negative *y*-axis.

Figure 6.1. 32



In this case, since the pole is identified with the origin, we can easily tell that the point R is 3 units from the pole, which means that $r = \pm 3$ in the polar representation, (r,θ), for R. Since we require $r \geq 0$, we choose $r = 3$. Then the angle $\theta = \frac{3\pi}{2}$, with its terminal side along the negative *y*-axis, satisfies $0 \leq \theta < 2\pi$. So R in polar coordinates is $\left(3, \frac{3\pi}{2}\right)$.

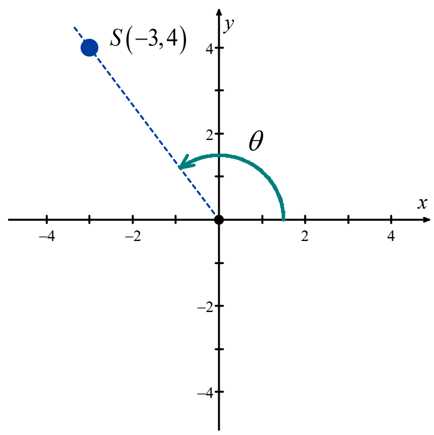
To check, we note

$$x = r\cos(\theta) = 3\cos\left(\frac{3\pi}{2}\right) = (3)(0) = 0 \qquad y = r\sin(\theta) = 3\sin\left(\frac{3\pi}{2}\right) = (3)(-1) = -3$$

The point $(0,-3)$ verifies our solution.

4. The point $S(-3,4)$ lies in Quadrant II.

Figure 6.1. 33



With $x=-3$ and $y=4$, we find $r^2 = (-3)^2 + (4)^2 = 25$, so that $r = \pm 5$. We choose $r = 5 \geq 0$ and proceed to determine θ. From $x=-3$ and $y=4$, we have $\tan(\theta) = \frac{4}{-3}$. Noting that $-\frac{4}{3}$ is not the tangent of any angle having a standard reference angle, we resort to using the arctangent function. Now, θ lies in Quadrant II and must satisfy $0 \leq \theta < 2\pi$ so we choose $\theta = \pi - \arctan\left(\frac{4}{3}\right)$ radians. So S in polar coordinates is $\left(5, \pi - \arctan\left(\frac{4}{3}\right)\right)$. Using a calculator, an approximate value for θ is 2.21 radians, from which S is approximately $(5, 2.21)$.

Checking our answer requires a bit of tenacity since we need to simplify expressions of the form $\cos\left(\pi - \arctan\left(\frac{4}{3}\right)\right)$ and $\sin\left(\pi - \arctan\left(\frac{4}{3}\right)\right)$. These are good review exercises and are hence left to the reader. We find $\cos\left(\pi - \arctan\left(\frac{4}{3}\right)\right) = -\frac{3}{5}$ and $\sin\left(\pi - \arctan\left(\frac{4}{3}\right)\right) = \frac{4}{5}$, so that

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$
$$= (5)\left(-\frac{3}{5}\right) \qquad\qquad = (5)\left(\frac{4}{5}\right)$$
$$= -3 \qquad\qquad = 4$$

This gives us the original rectangular coordinates, $(-3,4)$.

□

Now that we've converted representations of points between the rectangular and polar coordinate systems, we move on to converting equations from one system to the other. Just as we have used equations in x and y to represent relations in rectangular coordinates, equations in the variables r and θ represent relations in polar coordinates.

Converting Equations from Rectangular to Polar Coordinates

One strategy to convert an equation from rectangular to polar coordinates is to replace every occurrence of x with $r\cos(\theta)$ and every occurrence of y with $r\sin(\theta)$, and to then use identities to simplify. This is the technique we employ in the following three examples.

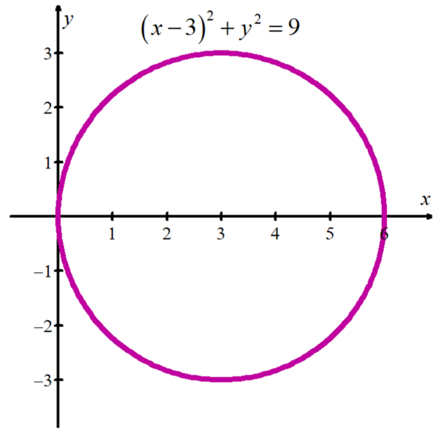
Example 6.1.10. Convert $(x-3)^2 + y^2 = 9$ from an equation in rectangular coordinates to an equation in polar coordinates.

Solution. We substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into $(x-3)^2 + y^2 = 9$ and then simplify.

$$\begin{aligned}
(x-3)^2 + y^2 &= 9 \\
\left(r\cos(\theta)-3\right)^2 + \left(r\sin(\theta)\right)^2 &= 9 \\
r^2\cos^2(\theta) - 6r\cos(\theta) + 9 + r^2\sin^2(\theta) &= 9 \\
r^2\cos^2(\theta) - 6r\cos(\theta) + r^2\sin^2(\theta) &= 0 \quad \text{subtract 9 from both sides} \\
r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) - 6r\cos(\theta) &= 0 \\
r^2(1) - 6r\cos(\theta) &= 0 \quad \text{Pythagorean identity} \\
r\left(r - 6\cos(\theta)\right) &= 0
\end{aligned}$$

Then $r = 0$ or $r = 6\cos(\theta)$. We note that the equation $(x-3)^2 + y^2 = 9$ describes a circle, as shown in the following graph.

Figure 6.1. 34



The solution $r=0$ describes only one point (namely the pole/origin), so is not the polar equation of the circle. The solution $r=6\cos(\theta)$ includes the pole and is the polar equation of the circle with rectangular form $(x-3)^2+y^2=9$.

□

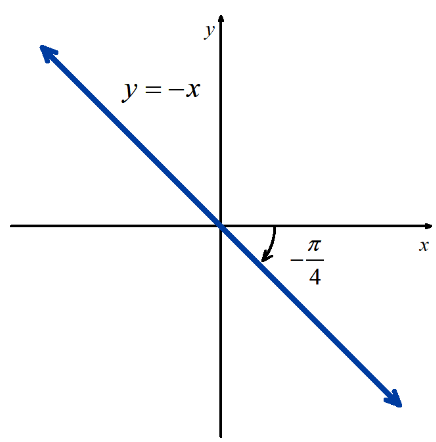
Example 6.1.11. Convert $y=-x$ from an equation in rectangular coordinates to an equation in polar coordinates.

Solution. We substitute $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $y=-x$.

$$\begin{aligned} y&=-x \\ r\sin(\theta)&=-r\cos(\theta) \\ r\sin(\theta)+r\cos(\theta)&=0 \\ r\left(\sin(\theta)+\cos(\theta)\right)&=0 \end{aligned}$$

This gives $r=0$ or $\sin(\theta)+\cos(\theta)=0$. We note that the equation $y=-x$ describes a line, which is graphed below.

Figure 6.1. 35



The solution $r=0$ is not the polar equation of this line. Now, $\theta=-\frac{\pi}{4}$ satisfies the equation $\sin(\theta)+\cos(\theta)=0$. The polar equation $\theta=-\frac{\pi}{4}$ describes all points $\left(r,-\frac{\pi}{4}\right)$, for any value of r (positive, negative, or zero). Graphically, this is the line containing the terminal side of the angle $\theta=-\frac{\pi}{4}$, whose rectangular equation is $y=-x$. Hence, our solution is $\theta=-\frac{\pi}{4}$.[2]

□

Example 6.1.12. Convert $y=x^2$ from an equation in rectangular coordinates to an equation in polar coordinates.

Solution. We substitute $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $y=x^2$.

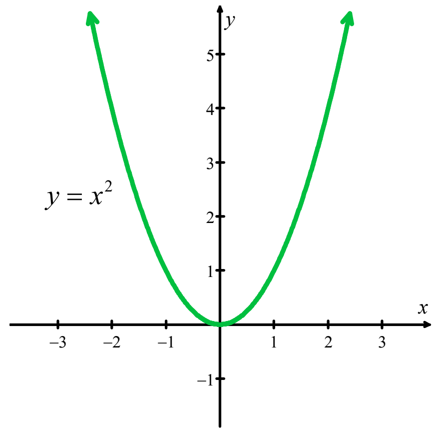
$$\begin{aligned}
y&=x^2\\
r\sin(\theta)&=\left(r\cos(\theta)\right)^2\\
r\sin(\theta)&=r^2\cos^2(\theta)\\
0&=r^2\cos^2(\theta)-r\sin(\theta)\\
0&=r\left(r\cos^2(\theta)-\sin(\theta)\right)
\end{aligned}$$

Either $r=0$ or $r\cos^2(\theta)=\sin(\theta)$. Since $r=0$ is not the polar equation of this parabola, we now can solve the latter equation for r by dividing both sides of the equation by $\cos^2(\theta)$. As a rule, we never divide through by a quantity that may be equal to 0. In this particular case, we are safe since if $\cos^2(\theta)=0$ then $\cos(\theta)=0$ and, for the equation $r\cos^2(\theta)=\sin(\theta)$ to hold, then $\sin(\theta)$ would also have to be 0. There are no angles with both $\cos(\theta)=0$ and $\sin(\theta)=0$. Thus, we are not losing any information by dividing both sides of $r\cos^2(\theta)=\sin(\theta)$ by $\cos^2(\theta)$. Doing so, we get

$$\begin{aligned}
r&=\frac{\sin(\theta)}{\cos^2(\theta)}\\
&=\frac{1}{\cos(\theta)}\cdot\frac{\sin(\theta)}{\cos(\theta)}\\
&=\sec(\theta)\tan(\theta)
\end{aligned}$$

[2] Or we could take it to be $\theta=-\frac{\pi}{4}+\pi k$ for any integer k.

Figure 6.1. 36



The solution $r = \sec(\theta)\tan(\theta)$ includes the pole and is the polar equation of the parabola $y = x^2$.

□

Converting Equations from Polar to Rectangular Coordinates

Converting equations from polar to rectangular coordinates is generally not as straightforward as the reverse process. We begin with the strategy of writing the polar equation in an equivalent form so that the left hand side of one of the following presents itself: $r^2 = x^2 + y^2$, $r\cos(\theta) = x$, $r\sin(\theta) = y$, or $\tan(\theta) = \frac{y}{x}$.

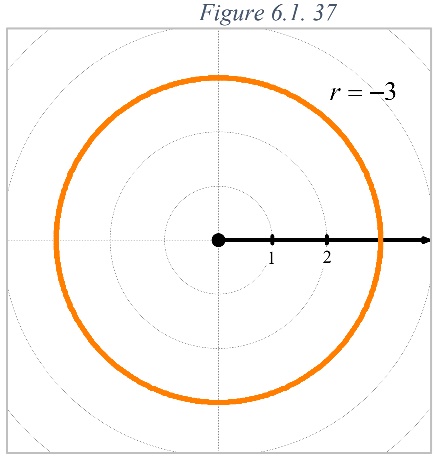
Example 6.1.13. Convert $r = -3$ from an equation in polar coordinates to an equation in rectangular coordinates.

Solution. We can start by squaring both sides of $r = -3$.

$$\begin{aligned} r &= -3 \\ r^2 &= (-3)^2 \\ r^2 &= 9 \end{aligned}$$

We may now substitute $r^2 = x^2 + y^2$ to get the equation $x^2 + y^2 = 9$.

Figure 6.1. 37



As we have seen, squaring an equation does not, in general, produce an equivalent equation. The concern here is that $r^2 = 9$ might be satisfied by more points than $r = -3$. On the surface, this appears to be the case since $r^2 = 9$ is equivalent to $r = \pm 3$, not just $r = -3$. However, any point with polar coordinates $(3, \theta)$ can be represented as $(-3, \theta + \pi)$. This means that any point (r, θ) whose polar coordinates satisfy the relation $r = \pm 3$ has an equivalent[3] representation that satisfies $r = -3$. Thus, we state our final solution as $x^2 + y^2 = 9$.

□

Example 6.1.14. Convert $\theta = \frac{4\pi}{3}$ from an equation in polar coordinates to an equation in rectangular coordinates.

Solution. We begin by taking the tangent of both sides of the equation.

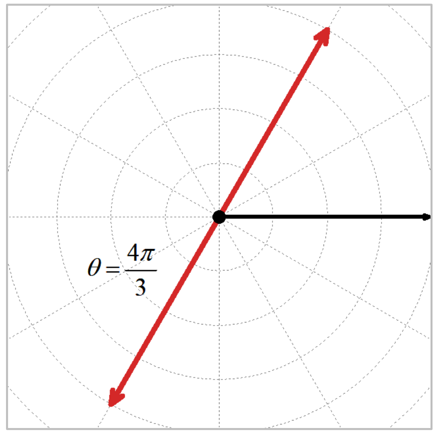
$$\theta = \frac{4\pi}{3}$$
$$\tan(\theta) = \tan\left(\frac{4\pi}{3}\right)$$
$$\tan(\theta) = \sqrt{3}$$

Since $\tan(\theta) = \frac{y}{x}$, we get $\frac{y}{x} = \sqrt{3}$, from which $y = x\sqrt{3}$.

---

[3] Here, 'equivalent' means they represent the same point in the plane. As ordered pairs, (3,0) and (−3, π) are different, but when interpreted as polar coordinates, they correspond to the same point in the plane. Mathematically speaking, relations are sets of ordered pairs, so the equations $r^2 = 9$ and $r = -3$ represent different relations since they correspond to different sets of ordered pairs. Since polar coordinates were defined geometrically to describe the location of points in the plane, however, we concern ourselves only with ensuring that the sets of points in the plane generated by two equations are the same. This was not an issue, by the way, in algebra when we first defined relations as sets of points in the plane. Back then, a point in the plane was identified with a unique ordered pair given by its Cartesian coordinates.

Figure 6.1. 38



Since all solutions to $\tan(\theta) = \sqrt{3}$, $\theta = \frac{\pi}{3} + \pi k$ for integers k, represent the same line as $\theta = \frac{4\pi}{3}$, the equation $\tan(\theta) = \sqrt{3}$ is equivalent to $\theta = \frac{4\pi}{3}$. Thus, we conclude that our answer of $y = x\sqrt{3}$ is correct.

□

Example 6.1.15. Convert $r = 1 - \cos(\theta)$ from an equation in polar coordinates to an equation in rectangular coordinates.

Solution. Once again, we need to manipulate $r = 1 - \cos(\theta)$ a bit before using the conversion formulas given in **Theorem 6.1**. We could square both sides of this equation like we did in **Example 6.1.13** to obtain an r^2 on the left side, but that does nothing helpful for the right side. Instead, we begin by multiplying both sides by r.

$$r = 1 - \cos(\theta)$$

$$r^2 = r - r\cos(\theta) \quad \text{multiply through by } r$$

$$r^2 + r\cos(\theta) = r$$

$$\left(r^2 + r\cos(\theta)\right)^2 = r^2 \quad \text{square both sides}$$

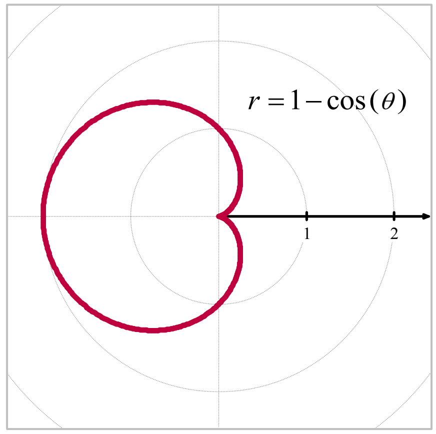
$$\left(x^2 + y^2 + x\right)^2 = x^2 + y^2 \quad \text{convert from polar to rectangular coordinates}$$

We have the equation $\left(x^2 + y^2 + x\right)^2 = x^2 + y^2$ as a solution.

□

Following is a graph of this polar equation, $r = 1 - \cos(\theta)$.

Figure 6.1. 39



It is easier to express certain relations in polar form than Cartesian form; for example, the circle $r = 4$, the line $\theta = \frac{\pi}{3}$, or the curve in the previous example. The curve in the previous example is referred to as a cardioid. We will graph cardioids, along with other polar equations, in **Section 6.2**.

6.1 Exercises

In Exercises 1 – 16, each point is given in polar coordinates. Plot the point and then give three additional representations for the point such that

(a) $r<0$ and $0\le\theta<2\pi$ (b) $r>0$ and $-2\pi\le\theta<0$ (c) $r>0$ and $2\pi\le\theta<4\pi$

1. $\left(2,\frac{\pi}{3}\right)$
2. $\left(5,\frac{7\pi}{4}\right)$
3. $\left(\frac{1}{3},\frac{3\pi}{2}\right)$
4. $\left(\frac{5}{2},\frac{5\pi}{6}\right)$
5. $\left(12,-\frac{7\pi}{6}\right)$
6. $\left(3,-\frac{5\pi}{4}\right)$
7. $\left(2\sqrt{2},-\pi\right)$
8. $\left(\frac{7}{2},-\frac{13\pi}{6}\right)$
9. $\left(-20,3\pi\right)$
10. $\left(-4,\frac{5\pi}{4}\right)$
11. $\left(-1,\frac{2\pi}{3}\right)$
12. $\left(-3,\frac{\pi}{2}\right)$
13. $\left(-3,-\frac{11\pi}{6}\right)$
14. $\left(-2.5,-\frac{\pi}{4}\right)$
15. $\left(-\sqrt{5},-\frac{4\pi}{3}\right)$
16. $\left(-\pi,-\pi\right)$

In Exercises 17 – 36, convert the point from polar coordinates to rectangular coordinates.

17. $\left(5,\frac{7\pi}{4}\right)$
18. $\left(2,\frac{\pi}{3}\right)$
19. $\left(11,-\frac{7\pi}{6}\right)$
20. $\left(-20,3\pi\right)$
21. $\left(\frac{3}{5},\frac{\pi}{2}\right)$
22. $\left(-4,\frac{5\pi}{6}\right)$
23. $\left(9,\frac{7\pi}{2}\right)$
24. $\left(-5,-\frac{9\pi}{4}\right)$
25. $\left(42,\frac{13\pi}{6}\right)$
26. $\left(-117,117\pi\right)$
27. $\left(6,\arctan(2)\right)$
28. $\left(10,\arctan(3)\right)$
29. $\left(-3,\arctan\left(\frac{4}{3}\right)\right)$
30. $\left(5,\arctan\left(-\frac{4}{3}\right)\right)$
31. $\left(\pi,\arctan(\pi)\right)$
32. $\left(13,\arctan\left(\frac{12}{5}\right)\right)$
33. $\left(2,\pi-\arctan\left(\frac{1}{2}\right)\right)$
34. $\left(-\frac{1}{2},\pi-\arctan(5)\right)$
35. $\left(-1,\pi+\arctan\left(\frac{3}{4}\right)\right)$
36. $\left(\frac{2}{3},\pi+\arctan\left(2\sqrt{2}\right)\right)$

In Exercises 37 – 56, convert the point from rectangular coordinates to polar coordinates with $r\ge 0$ and $0\le\theta<2\pi$.

37. $(0,5)$
38. $\left(3,\sqrt{3}\right)$
39. $(7,-7)$
40. $\left(-3,-\sqrt{3}\right)$

41. $(-3,0)$
42. $(-\sqrt{2},\sqrt{2})$
43. $(-4,-4\sqrt{3})$
44. $\left(\frac{\sqrt{3}}{4},-\frac{1}{4}\right)$
45. $\left(-\frac{3}{10},-\frac{3\sqrt{3}}{10}\right)$
46. $(-\sqrt{5},-\sqrt{5})$
47. $(6,8)$
48. $(\sqrt{5},2\sqrt{5})$
49. $(-8,1)$
50. $(-2\sqrt{10},6\sqrt{10})$
51. $(-5,-12)$
52. $\left(-\frac{\sqrt{5}}{15},-\frac{2\sqrt{5}}{15}\right)$
53. $(24,-7)$
54. $(12,-9)$
55. $\left(\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4}\right)$
56. $\left(-\frac{\sqrt{65}}{5},\frac{2\sqrt{65}}{5}\right)$

In Exercises 57 – 76, convert the equation from rectangular coordinates to polar coordinates. Solve for r in all but Exercises 60 – 63. In Exercises 60 – 63, solve for θ, assuming $0 \le \theta < \pi$.

57. $x=6$
58. $x=-3$
59. $y=7$
60. $y=0$
61. $y=-x$
62. $y=x\sqrt{3}$
63. $y=2x$
64. $x^2+y^2=25$
65. $x^2+y^2=117$
66. $y=4x-19$
67. $x=3y+1$
68. $y=-3x^2$
69. $4x=y^2$
70. $x^2+y^2-2y=0$
71. $x^2-4x+y^2=0$
72. $x^2+y^2=x$
73. $y^2=7y-x^2$
74. $(x+2)^2+y^2=4$
75. $x^2+(y-3)^2=9$
76. $4x^2+4\left(y-\frac{1}{2}\right)^2=1$

In Exercises 77 – 96, convert the equation from polar coordinates to rectangular coordinates.

77. $r=7$
78. $r=-3$
79. $r=\sqrt{2}$
80. $\theta=\frac{\pi}{4}$
81. $\theta=\frac{2\pi}{3}$
82. $\theta=\pi$
83. $\theta=\frac{3\pi}{2}$
84. $r=4\cos(\theta)$
85. $5r=\cos(\theta)$
86. $r=3\sin(\theta)$
87. $r=-2\sin(\theta)$
88. $r=7\sec(\theta)$
89. $12r=\csc(\theta)$
90. $r=-2\sec(\theta)$
91. $r=-\sqrt{5}\csc(\theta)$
92. $r=2\sec(\theta)\tan(\theta)$
93. $r^2=\sin(2\theta)$
94. $r=1-2\cos(\theta)$
95. $r=1+\sin(\theta)$
96. $r=-\csc(\theta)\cot(\theta)$

97. Convert the origin $(0,0)$ to polar coordinates in four different ways.

98. Use the Law of Cosines to develop a formula for the distance between two points in polar coordinates.

6.2 Graphing Polar Equations

Learning Objectives

- **Graph polar equations.**
- **From their equations, identify the shape and location of polar graphs in the polar plane.**

In this section, we discuss how to graph polar equations. A **polar equation** is an equation involving polar coordinates (r,θ). Some examples of polar equations are $r=4$, $\theta=-\frac{3\pi}{2}$, and $r=4-2\sin(\theta)$. Since any point in the plane has infinitely many different representations in polar coordinates, practice with graphing polar equations will be an essential part of the learning process. We begin with a definition.

Definition 6.1. The **graph of a polar equation** is the set of all points $P(r,\theta)$ that satisfy the equation.

Plotting Polar Equations of the Form $r=a$ and $\theta=\alpha$, for constants a and α

Our first example focuses on some of the more structurally simple polar equations.

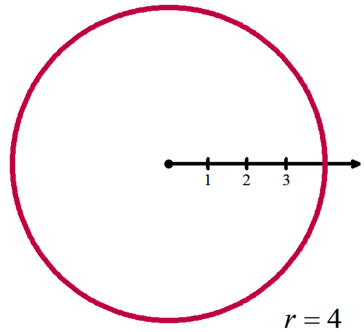
Example 6.2.1. Graph the following polar equations.

1. $r=4$
2. $r=-3\sqrt{2}$
3. $\theta=\frac{5\pi}{4}$
4. $\theta=-\frac{3\pi}{2}$

Solution. In each of these equations, only r or θ is present, resulting in the missing variable taking on all values without restriction. This makes these graphs easier to visualize than graphs of other polar equations.

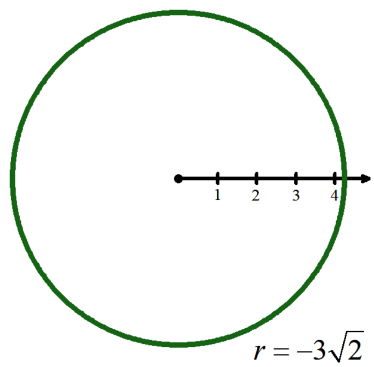
1. The variable θ does not appear in the equation $r=4$. Therefore, the graph of this equation is all points that have a polar coordinate representation $(4,\theta)$, for any choice of θ. Graphically, this translates into tracing out all points that are 4 units away from the pole. This is exactly the definition of a circle centered at the pole with a radius of 4.

Figure 6.2. 1



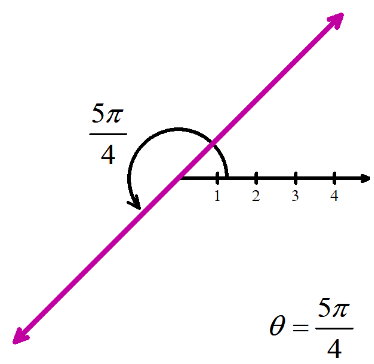
2. Once again, the variable θ does not appear in the equation, $r = -3\sqrt{2}$. Plotting all points of the form $\left(-3\sqrt{2}, \theta\right)$ gives us a circle of radius $3\sqrt{2} \approx 4.24$ centered at the pole.

Figure 6.2. 2



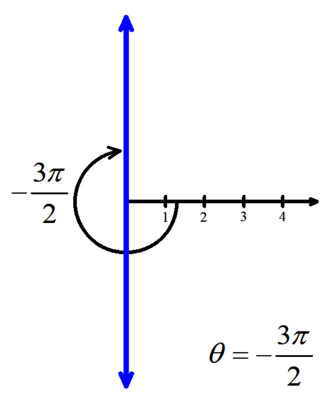
3. The variable r does not appear in the equation $\theta = \dfrac{5\pi}{4}$, so we plot all points with polar representations $\left(r, \dfrac{5\pi}{4}\right)$. What we find is that we are tracing out the line that contains the terminal side of $\theta = \dfrac{5\pi}{4}$, when plotted in standard position.

Figure 6.2. 3



4. As in the previous problem, the variable r does not appear in the equation, $\theta = -\frac{3\pi}{2}$. Plotting $\left(r, -\frac{3\pi}{2}\right)$ for all values of r results in tracing out the vertical line that passes through the pole.

Figure 6.2. 4



□

Example 6.2.1 leads us to the following.

Graphs of Polar Equations $r = a$ and $\theta = \alpha$, for constants a and α

- The graph of the polar equation $r = a$ is a circle centered at the pole that has radius $|a|$. If $a = 0$, the graph is one single point (the pole).
- The graph of the polar equation $\theta = \alpha$ is the line containing the terminal side of α, when α is plotted in standard position.

Graphing Polar Equations of the Form $r = a\sin(\theta)$ or $r = a\cos(\theta)$, $a \neq 0$

In general, to graph equations of the form $r = a\sin(\theta)$ or $r = a\cos(\theta)$, we choose values for the independent variable θ, determine corresponding values of the dependent variable r, and then plot the resulting polar coordinates (r,θ).

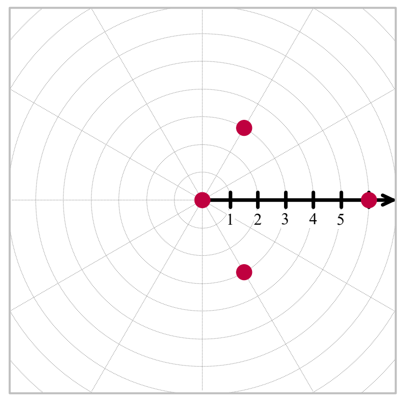
Example 6.2.2. Graph the polar equation $r = 6\cos(\theta)$.

Solution. We start by choosing some standard angle values for θ. After generating the table below, the resulting points are graphed in the plane.[4]

θ	$r = 6\cos(\theta)$	(r,θ)
0	6	$(6,0)$
$\frac{\pi}{3}$	3	$\left(3,\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	0	$\left(0,\frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	-3	$\left(-3,\frac{2\pi}{3}\right)$
π	-6	$(-6,\pi)$

θ	$r = 6\cos(\theta)$	(r,θ)
$\frac{4\pi}{3}$	-3	$\left(-3,\frac{4\pi}{3}\right)$
$\frac{3\pi}{2}$	0	$\left(0,\frac{3\pi}{2}\right)$
$\frac{5\pi}{3}$	3	$\left(3,\frac{5\pi}{3}\right)$
2π	6	$(6,2\pi)$

Figure 6.2. 5



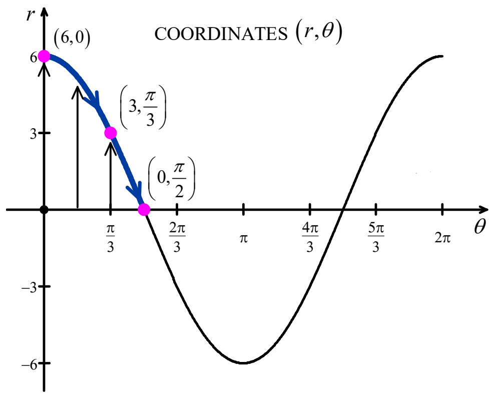
Despite having nine ordered pairs, we only get four distinct points on the graph. For this reason, we employ a slightly different strategy. We graph $r = 6\cos(\theta)$ in the θr-plane[5] and use it as a guide for graphing the equation in the polar plane.

We first see that as θ ranges from 0 to $\frac{\pi}{2}$, r ranges from 6 to 0. In the polar plane, this means the curve starts 6 units from the pole on the polar axis, when $\theta = 0$, and gradually returns to the pole, at $\theta = \frac{\pi}{2}$.

[4] Note that values for θ were chosen here to result in cosine values of 0, ± 1, or $\pm\frac{1}{2}$, allowing for easier calculations of r. Including additional values for θ would result in additional points that would make graphing easier.

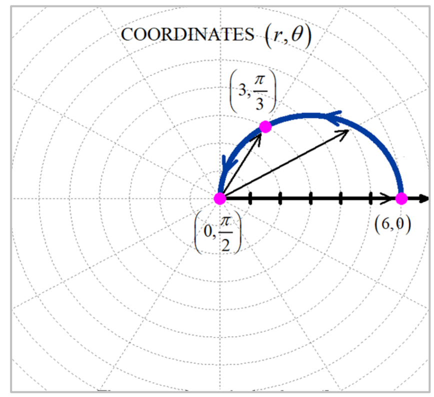
[5] This graph looks exactly like $y = 6\cos(x)$ in the xy-plane, and for good reason. At this stage, we are just graphing the relationship between r and θ before we interpret them as polar coordinates (r, θ) in the polar plane.

Figure 6.2. 6



$r = 6\cos(\theta)$ in the θr -plane

Figure 6.2. 7

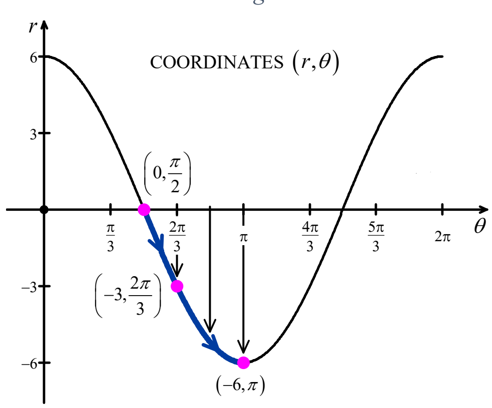


$r = 6\cos(\theta)$ in the polar plane

The arrows drawn in the above figures are meant to help you visualize this process. In the θr -plane, the vertical arrows are drawn from the θ -axis to the curve $r = 6\cos(\theta)$. In the polar plane, each of the arrows corresponds to those drawn in the θr -plane, by rotating through an angle of θ from the polar axis and then plotting a point r units from the pole.

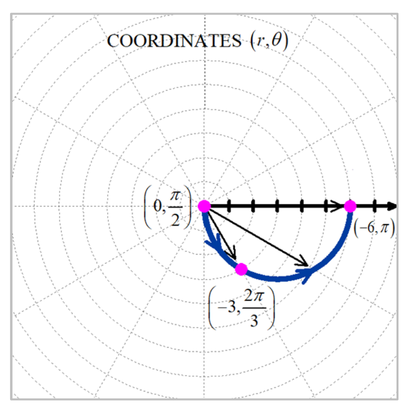
Next, we repeat the process as θ ranges from $\frac{\pi}{2}$ to π. Between $\frac{\pi}{2}$ and π, the r -values are negative. This means that in the polar plane, instead of graphing in the region corresponding to Quadrant II, we graph in the region corresponding to Quadrant IV.

Figure 6.2. 8



$r = 6\cos(\theta)$ in the θr -plane

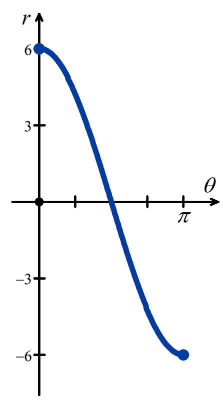
Figure 6.2. 9



$r = 6\cos(\theta)$ in the polar plane

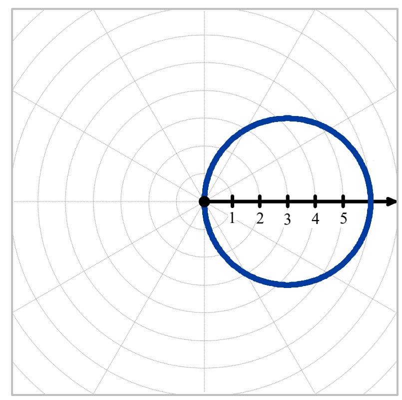
As θ ranges from π to $\frac{3\pi}{2}$, the r -values are still negative, which means the graph is traced out in the region corresponding to Quadrant I instead of the region corresponding to Quadrant III. Since $|r|$ for these values of θ matches the r values for θ in $\left[0, \frac{\pi}{2}\right]$, the curve begins to retrace itself at this point. Proceeding further, we find that when $\frac{3\pi}{2} \le \theta \le 2\pi$, we retrace the part of the curve that we first traced out as $\frac{\pi}{2} \le \theta \le \pi$. The reader is invited to verify that plotting any range of θ outside the interval $[0, \pi]$ results in retracing some portion of the curve. We present the final graph below.

Figure 6.2. 10



$r = 6\cos(\theta)$ in the θr-plane

Figure 6.2. 11



$r = 6\cos(\theta)$ in the polar plane

□

The graph of $r = 6\cos(\theta)$ looks suspiciously like a circle, for good reason. We can convert this polar equation to a rectangular equation through multiplying both sides by r, to give us $r^2 = 6r\cos(\theta)$. We substitute $r^2 = x^2 + y^2$ and $r\cos(\theta) = x$ to get

$$\begin{aligned} x^2 + y^2 &= 6x \\ x^2 - 6x + y^2 &= 0 \\ x^2 - 6x + 9 + y^2 &= 9 \\ (x-3)^2 + y^2 &= 9 \end{aligned}$$

This rectangular equation is easily recognizable as a circle in the *xy*-plane with center $(3,0)$ and radius $\sqrt{9} = 3$. Any polar equation of the form $r = a\sin(\theta)$ or $r = a\cos(\theta)$, where a is a non-zero constant, represents a circle, as summarized below. Recall that $r = a$ also represents a circle, as discussed earlier.

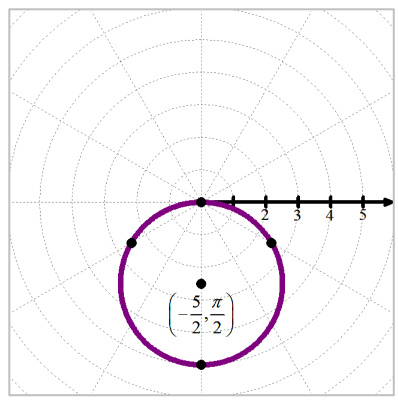
Graphs of Polar Equations $r = a\cos(\theta)$ and $r = a\sin(\theta)$, $a \neq 0$

- The graph of polar equation $r = a\cos(\theta)$, $a \neq 0$, is a circle centered at the polar point $\left(\frac{a}{2}, 0\right)$ with radius $\left|\frac{a}{2}\right|$.
- The graph of polar equation $r = a\sin(\theta)$, $a \neq 0$, is a circle centered at the polar point $\left(\frac{a}{2}, \frac{\pi}{2}\right)$ with radius $\left|\frac{a}{2}\right|$.

An example of a polar equation of the form $r = a\sin(\theta)$, $a \neq 0$, is $r = -5\sin(\theta)$. From the information in the preceding box, this is a circle centered at $\left(-\frac{5}{2}, \frac{\pi}{2}\right)$ with radius $\left|-\frac{5}{2}\right| = \frac{5}{2}$. With this information, plotting a few points is sufficient to draw the graph, as follows.

θ	$r = -5\sin(\theta)$	(r, θ)
0	0	$(0,0)$
$\frac{\pi}{6}$	$-\frac{5}{2}$	$\left(-\frac{5}{2}, \frac{\pi}{6}\right)$
$\frac{\pi}{2}$	-5	$\left(-5, \frac{\pi}{2}\right)$
$\frac{5\pi}{6}$	$-\frac{5}{2}$	$\left(-\frac{5}{2}, \frac{5\pi}{6}\right)$
π	0	$(\pi, 0)$

Figure 6.2. 12



$r = -5\sin(\theta)$

Note that the graph of $r = 6\cos(\theta)$ is symmetric about the polar axis, while the graph of $r = -5\sin(\theta)$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$. The following tests for symmetry may prove helpful in graphing polar equations.

Symmetry in Graphs of Polar Equations: If a polar equation is unchanged when

- θ is replaced by $-\theta$, the graph is symmetric about the polar axis;
- r is replaced by $-r$, the graph is symmetric about the pole;
- θ is replaced by $\pi - \theta$, the graph is symmetric about the vertical line $\theta = \frac{\pi}{2}$.

For $r = 6\cos(\theta)$, symmetry about the polar axis can be verified by noting that $6\cos(-\theta) = 6\cos(\theta)$, since cosine is an even function. To verify that the graph of $r = -5\sin(\theta)$ is symmetric about the line $\theta = \frac{\pi}{2}$, we have

$$\begin{aligned} -5\sin(\pi - \theta) &= -5\left[\sin(\pi)\cos(\theta) - \cos(\pi)\sin(\theta)\right] \\ &= -5\left[(0)\cos(\theta) - (-1)\sin(\theta)\right] \\ &= -5\sin(\theta) \end{aligned}$$

While noting that recognizing the symmetry of $r = 6\cos(\theta)$ in **Example 6.2.2** could have shortened the graphing process, we will primarily use symmetry as a check for our graphs in this section.

Graphing Polar Equations of the Form $r = a \pm b\sin(\theta)$ or $r = a \pm b\cos(\theta)$, $a > 0$ and $b > 0$

Example 6.2.3. Graph the polar equation $r = 4 - 2\sin(\theta)$.

Solution. Once again, we begin with a table of values.

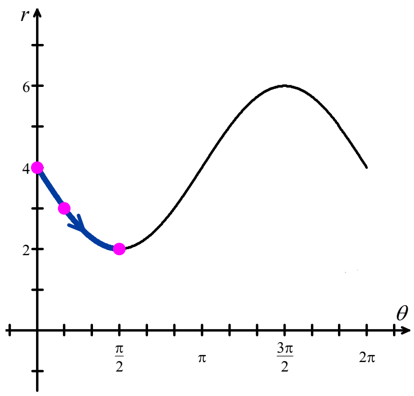
θ	$r = 4 - 2\sin(\theta)$	(r, θ)
0	4	$(4, 0)$
$\frac{\pi}{6}$	3	$\left(3, \frac{\pi}{6}\right)$
$\frac{\pi}{2}$	2	$\left(2, \frac{\pi}{2}\right)$
$\frac{5\pi}{6}$	3	$\left(3, \frac{5\pi}{6}\right)$
π	4	$(4.\pi)$

θ	$r = 4 - 2\sin(\theta)$	(r, θ)
$\frac{7\pi}{6}$	5	$\left(5, \frac{7\pi}{6}\right)$
$\frac{3\pi}{2}$	6	$\left(6, \frac{3\pi}{2}\right)$
$\frac{11\pi}{6}$	5	$\left(5, \frac{11\pi}{6}\right)$
2π	4	$(4, 2\pi)$

We first plot the fundamental cycle of $r = 4 - 2\sin(\theta)$ in the θr-plane. To help visualize what is going on graphically, we divide $[0, 2\pi]$ into the four subintervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$, and $\left[\frac{3\pi}{2}, 2\pi\right]$, then proceed as we did in **Example 6.2.2**.

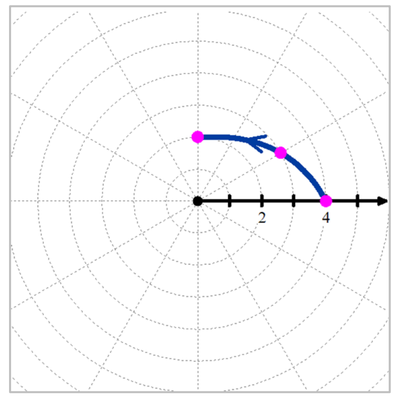
1. As θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 4 to 2. This means that in the polar plane, the curve starts 4 units from the pole on the polar axis, when $\theta = 0$, and gradually pulls in toward a point 2 units from the pole, at $\theta = \frac{\pi}{2}$.

Figure 6.2. 13



$r = 4 - 2\sin(\theta)$ in the θr-plane

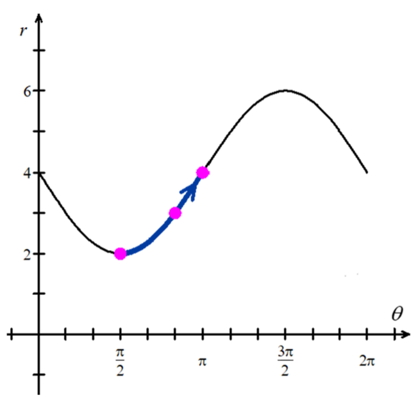
Figure 6.2. 14



$r = 4 - 2\sin(\theta)$ in the polar plane

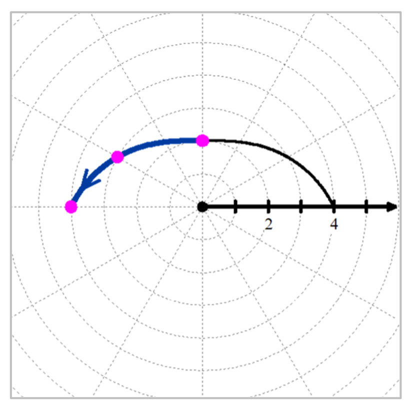
2. Next, as θ runs from $\frac{\pi}{2}$ to π, we see that r increases from 2 to 4. In the polar plane, picking up where we left off, we gradually pull the graph toward the point 4 units away from the pole, at $\theta = \pi$.

Figure 6.2. 15



$r = 4 - 2\sin(\theta)$ in the θr-plane

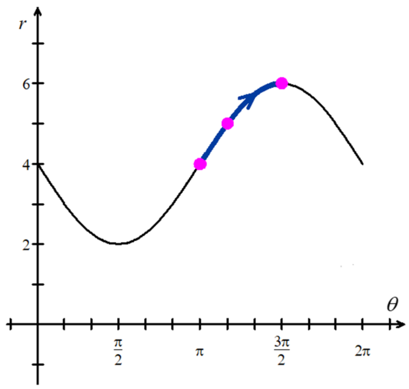
Figure 6.2. 16



$r = 4 - 2\sin(\theta)$ in the polar plane

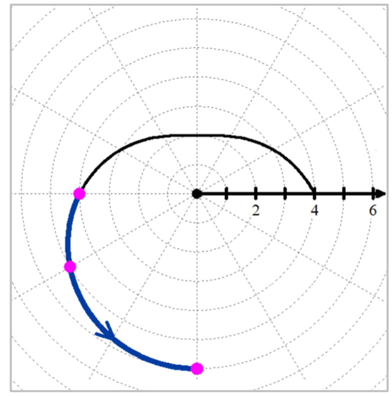
3. Over the interval $\left[\pi, \frac{3\pi}{2}\right]$, we see that r increases from 4 to 6. In the polar plane, the curve sweeps out away from the point 4 units from the pole, at $\theta = \pi$, to a point 6 units from the pole, at $\theta = \frac{3\pi}{2}$.

Figure 6.2. 17



$r = 4 - 2\sin(\theta)$ in the θr-plane

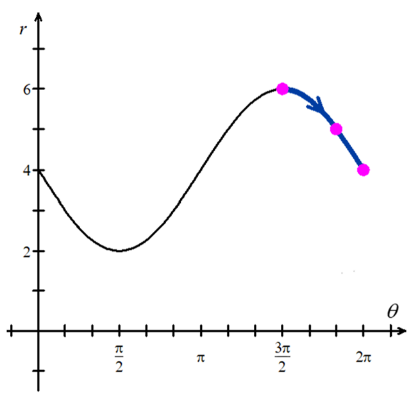
Figure 6.2. 18



$r = 4 - 2\sin(\theta)$ in the polar plane

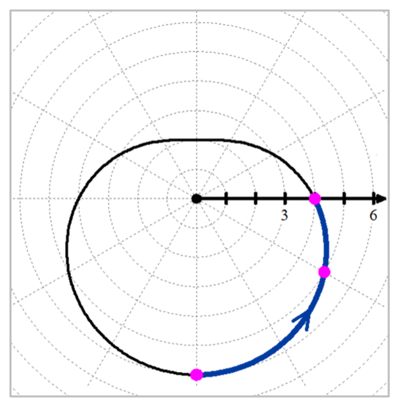
4. Finally, as θ takes on values from $\frac{3\pi}{2}$ to 2π, r decreases from 6 back to 4. The graph in the polar plane pulls in from the point 6 units from the pole, at $\theta = \frac{3\pi}{2}$, to finish where we started.

Figure 6.2. 19



$r = 4 - 2\sin(\theta)$ in the θr-plane

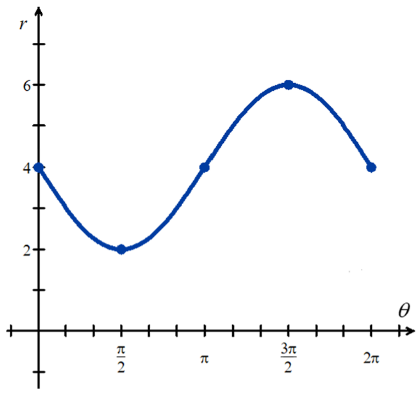
Figure 6.2. 20



$r = 4 - 2\sin(\theta)$ in the polar plane

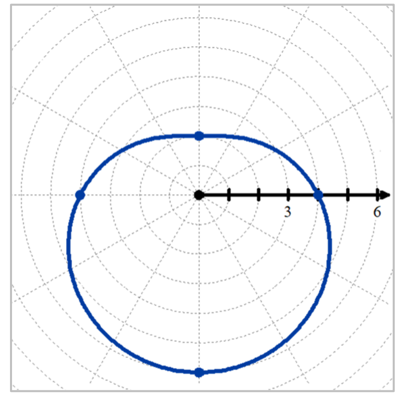
We leave it to the reader to verify that plotting points corresponding to values of θ outside the interval $[0, 2\pi]$ results in retracing portions of the curve. The final graph is symmetric about the line $\theta = \frac{\pi}{2}$, as can be verified by showing that $4 - 2\sin(\pi - \theta) = 4 - 2\sin(\theta)$.

Figure 6.2. 21



$r = 4 - 2\sin(\theta)$ in the θr-plane

Figure 6.2. 22



$r = 4 - 2\sin(\theta)$ in the polar plane

□

The following example is similar, but with an interesting 'twist' to the curve.

Example 6.2.4. Graph the polar equation $r = 2 + 4\cos(\theta)$.

Solution. Following is a table of values.

θ	$r = 2 + 4\cos(\theta)$	(r,θ)
0	6	$(6,0)$
$\frac{\pi}{3}$	4	$\left(4,\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	2	$\left(2,\frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	0	$\left(0,\frac{2\pi}{3}\right)$
π	-2	$(-2,\pi)$

θ	$r = 2 + 4\cos(\theta)$	(r,θ)
$\frac{4\pi}{3}$	0	$\left(0,\frac{4\pi}{3}\right)$
$\frac{3\pi}{2}$	2	$\left(2,\frac{3\pi}{2}\right)$
$\frac{5\pi}{3}$	4	$\left(4,\frac{5\pi}{3}\right)$
2π	6	$(6,2\pi)$

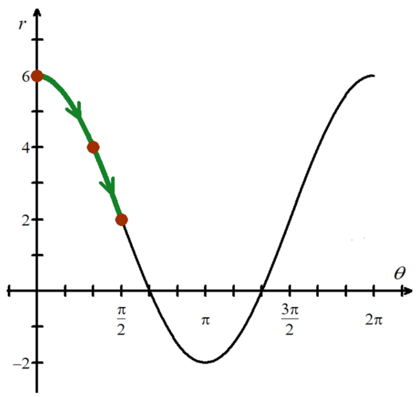
The first thing to note when graphing $r = 2 + 4\cos(\theta)$ in the θr-plane over the interval $[0,2\pi]$ is that the graph crosses through the θ-axis. This corresponds to the graph of the curve passing through the pole in the polar plane. We note from the table of values, above, that $r = 0$ when $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$.

Since these values are of significance geometrically, we break the interval $[0,2\pi]$ into six subintervals:

$\left[0,\frac{\pi}{2}\right]$, $\left[\frac{\pi}{2},\frac{2\pi}{3}\right]$, $\left[\frac{2\pi}{3},\pi\right]$, $\left[\pi,\frac{4\pi}{3}\right]$, $\left[\frac{4\pi}{3},\frac{3\pi}{2}\right]$, and $\left[\frac{3\pi}{2},2\pi\right]$.

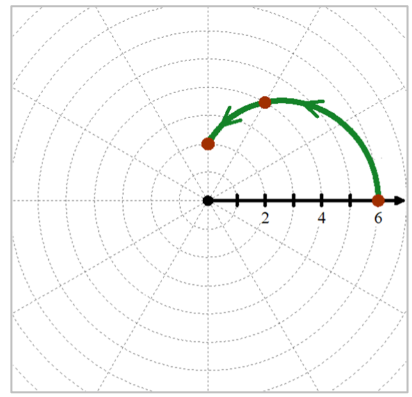
1. As θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 6 to 2. Plotting this in the polar plane, we start 6 units out from the pole on the polar axis, when $\theta = 0$, and slowly pull in toward a point 2 units from the pole, at $\theta = \frac{\pi}{2}$.

Figure 6.2. 23



$r = 2 + 4\cos(\theta)$ in the θr-plane

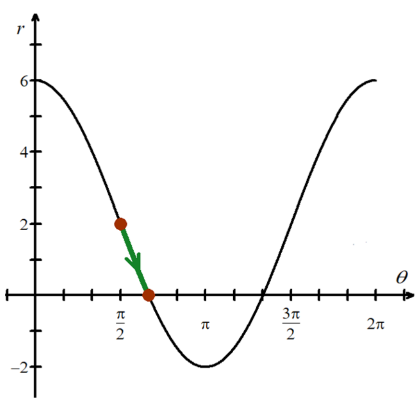
Figure 6.2. 24



$r = 2 + 4\cos(\theta)$ in the polar plane

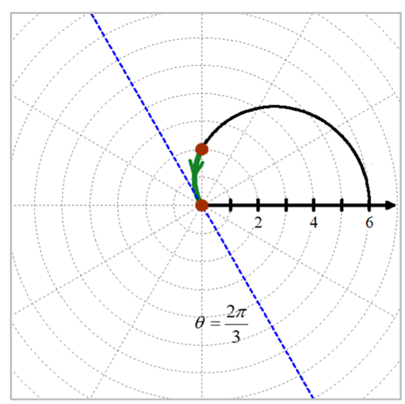
2. On the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, r decreases from 2 to 0, which means the polar graph is heading into (and will eventually cross through) the pole. Not only do we reach the pole when $\theta = \frac{2\pi}{3}$, the curve hugs the line $\theta = \frac{2\pi}{3}$ as it approaches the pole.

Figure 6.2. 25



$r = 2 + 4\cos(\theta)$ in the θr-plane

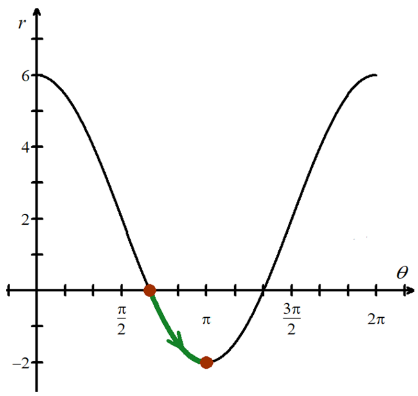
Figure 6.2. 26



$r = 2 + 4\cos(\theta)$ in the polar plane

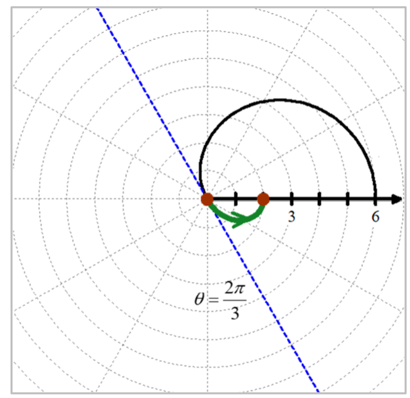
3. On the interval $\left[\frac{2\pi}{3},\pi\right]$, r ranges from 0 to −2. Since $r \le 0$, the curve passes through the pole in the polar plane, following the line $\theta = \frac{2\pi}{3}$, and continues upward toward the polar axis, stopping on the polar axis at a point 2 units to the right of the pole.

Figure 6.2. 27



$r = 2 + 4\cos(\theta)$ in the θr-plane

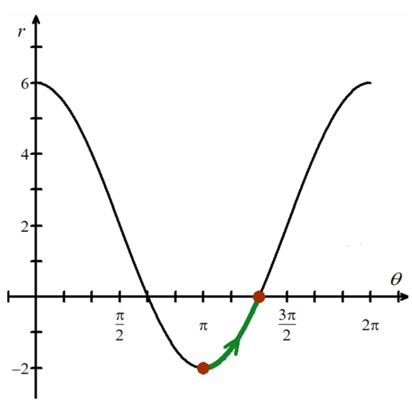
Figure 6.2. 28



$r = 2 + 4\cos(\theta)$ in the polar plane

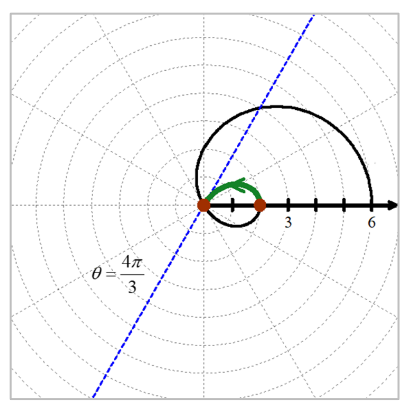
4. Next, as θ progresses from π to $\frac{4\pi}{3}$, r ranges from −2 to 0. With $r \le 0$, the polar graph continues a gradual return to the pole, following the line $\theta = \frac{4\pi}{3}$ as it gets closer.

Figure 6.2. 29



$r = 2 + 4\cos(\theta)$ in the θr-plane

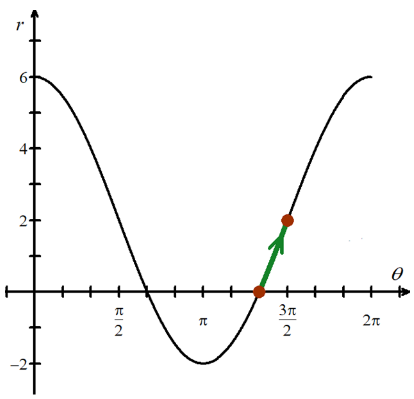
Figure 6.2. 30



$r = 2 + 4\cos(\theta)$ in the polar plane

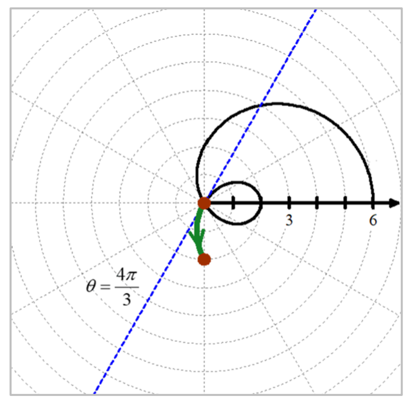
5. On the interval $\left[\frac{4\pi}{3},\frac{3\pi}{2}\right]$, r returns to positive values and increases from 0 to 2. The polar graph hugs the line $\theta=\frac{4\pi}{3}$ as it moves through the pole and heads toward a point 2 units from the pole, at $\theta=\frac{3\pi}{2}$.

Figure 6.2. 31



$r=2+4\cos(\theta)$ in the θr-plane

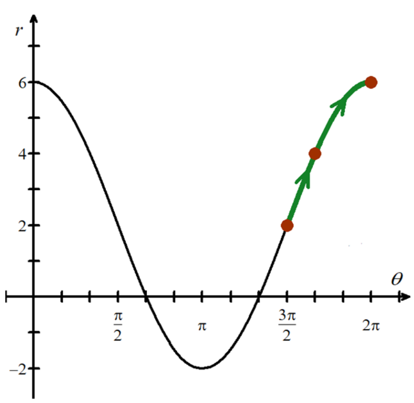
Figure 6.2. 32



$r=2+4\cos(\theta)$ in the polar plane

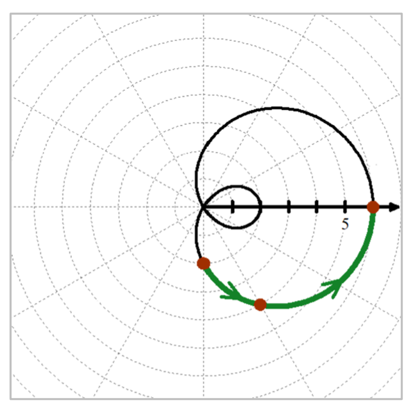
6. In the last step, as θ runs through $\frac{3\pi}{2}$ to 2π, r increases from 2 to 6, and we end up back where we started, 6 units from the pole on the polar axis.

Figure 6.2. 33



$r=2+4\cos(\theta)$ in the θr-plane

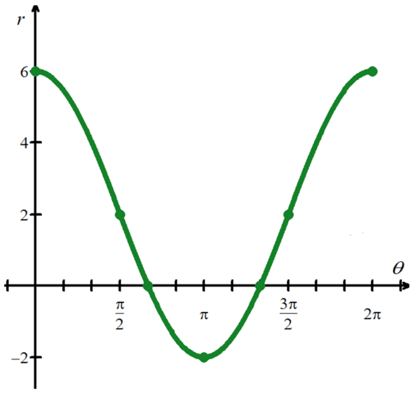
Figure 6.2. 34



$r=2+4\cos(\theta)$ in the polar plane

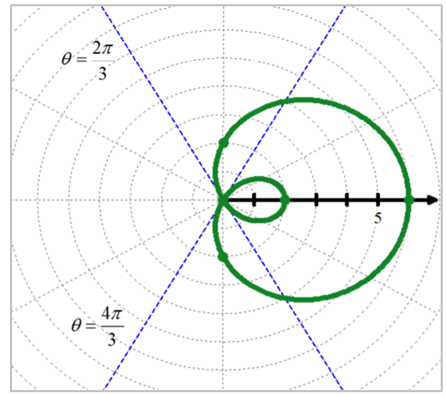
Again, the reader is invited to show that plotting the curve for values of θ outside $[0,2\pi]$ results in retracing a portion of the curve already traced. We conclude this example with a final graph. Note the symmetry about the polar axis.

Figure 6.2. 35



$r = 2 + 4\cos(\theta)$ in the θr-plane

Figure 6.2. 36

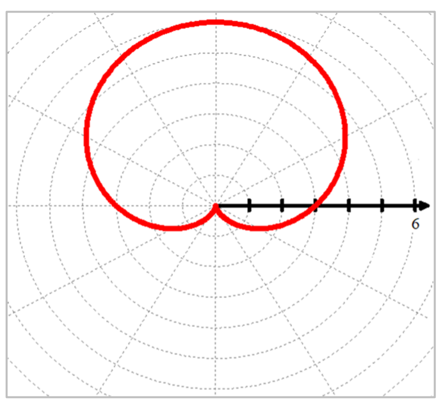


$r = 2 + 4\cos(\theta)$ in the polar plane

□

The polar curves in the previous two examples are **limaçons**, identified by a polar equation of the form $r = a \pm b\sin(\theta)$ or $r = a \pm b\cos(\theta)$, where $a > 0$ and $b > 0$. In **Example 6.2.4**, the inner loop is a property of limaçons in which $a < b$. Another case worthy of note is a limaçon in which $a = b$. This curve is referred to as a **cardioid**, as demonstrated in the following graph. Can you guess where the designation 'cardioid' comes from?

Figure 6.2. 37



$r = 3 + 3\sin(\theta)$

Graphing Polar Equations of the Form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$, $a \neq 0$

Example 6.2.5. Graph the polar equation $r = 5\sin(2\theta)$.

Solution. Once again, we begin with a table of values, being a bit creative in using input values of θ that will result in helpful output values for $\sin(2\theta)$.

θ	$r = 5\sin(2\theta)$	(r,θ)
0	0	$(0,0)$
$\frac{\pi}{12}$	$\frac{5}{2}$	$\left(\frac{5}{2},\frac{\pi}{12}\right)$
$\frac{\pi}{4}$	5	$\left(5,\frac{\pi}{4}\right)$
$\frac{5\pi}{12}$	$\frac{5}{2}$	$\left(\frac{5}{2},\frac{5\pi}{12}\right)$
$\frac{\pi}{2}$	0	$\left(0,\frac{\pi}{2}\right)$
$\frac{7\pi}{12}$	$-\frac{5}{2}$	$\left(-\frac{5}{2},\frac{7\pi}{12}\right)$

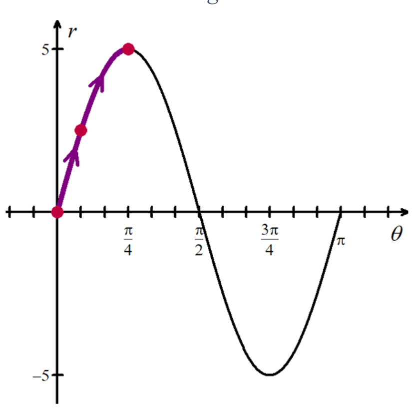
θ	$r = 5\sin(2\theta)$	(r,θ)
$\frac{3\pi}{4}$	-5	$\left(-5,\frac{3\pi}{4}\right)$
$\frac{11\pi}{12}$	$-\frac{5}{2}$	$\left(-\frac{5}{2},\frac{11\pi}{12}\right)$
π	0	$(0,\pi)$
$\frac{13\pi}{12}$	$\frac{5}{2}$	$\left(\frac{5}{2},\frac{13\pi}{12}\right)$
$\frac{5\pi}{4}$	5	$\left(5,\frac{5\pi}{4}\right)$
$\frac{17\pi}{12}$	$\frac{5}{2}$	$\left(\frac{5}{2},\frac{17\pi}{12}\right)$

θ	$r = 5\sin(2\theta)$	(r,θ)
$\frac{3\pi}{2}$	0	$\left(0,\frac{3\pi}{2}\right)$
$\frac{19\pi}{12}$	$-\frac{5}{2}$	$\left(-\frac{5}{2},\frac{19\pi}{12}\right)$
$\frac{7\pi}{4}$	-5	$\left(-5,\frac{7\pi}{4}\right)$
$\frac{23\pi}{12}$	$-\frac{5}{2}$	$\left(-\frac{5}{2},\frac{23\pi}{12}\right)$
2π	0	$(0,2\pi)$

We start by graphing the subintervals $\left[0,\frac{\pi}{4}\right]$, $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$, $\left[\frac{\pi}{2},\frac{3\pi}{4}\right]$, and $\left[\frac{3\pi}{4},\pi\right]$.

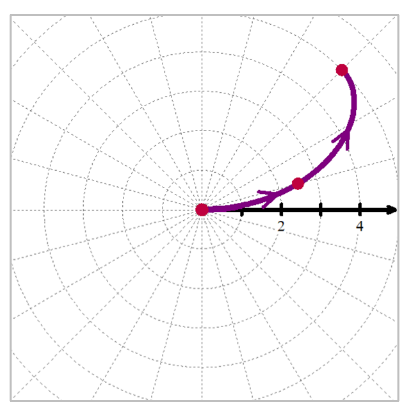
1. As θ ranges from 0 to $\frac{\pi}{4}$, r increases from 0 to 5. This means the graph of $r = 5\sin(2\theta)$ in the polar plane starts at the pole and gradually sweeps out so that it is 5 units away from the pole at $\theta = \frac{\pi}{4}$.

Figure 6.2. 38



$r = 5\sin(2\theta)$ in the θr-plane

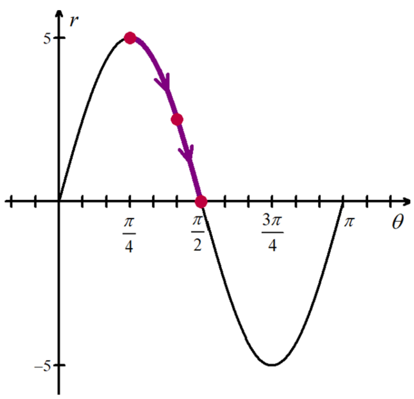
Figure 6.2. 39



$r = 5\sin(2\theta)$ in the polar plane

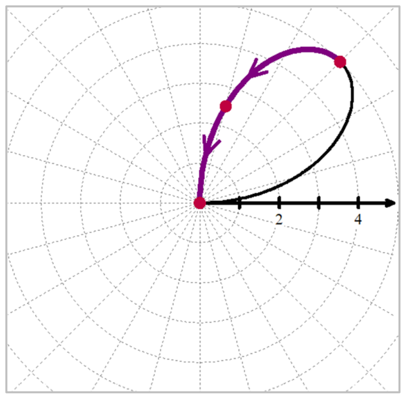
2. Next, we see that r decreases from 5 to 0 as θ runs through $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and, furthermore, r is heading toward negative values as the graph crosses the θ-axis at $\frac{\pi}{2}$ in the θr-plane. Hence, in the polar plane, we draw the curve hugging the line $\theta = \frac{\pi}{2}$ as the curve heads toward the pole.

Figure 6.2. 40



$r = 5\sin(2\theta)$ in the θr-plane

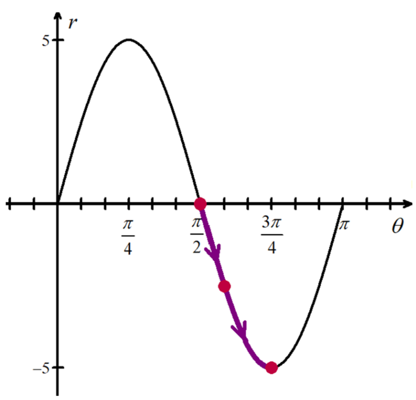
Figure 6.2. 41



$r = 5\sin(2\theta)$ in the polar plane

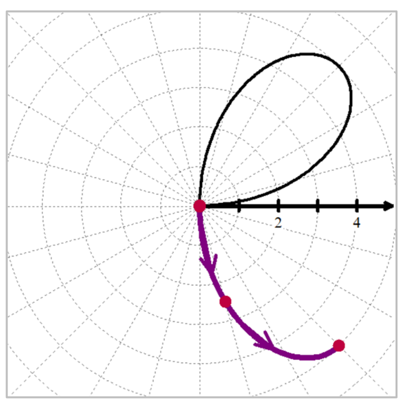
3. As θ runs from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r becomes negative and ranges from 0 to −5. The polar curve starts at the pole and stops at a distance of 5 units from the pole, at the point $\left(-5, \frac{3\pi}{4}\right)$.

Figure 6.2. 42



$r = 5\sin(2\theta)$ in the θr-plane

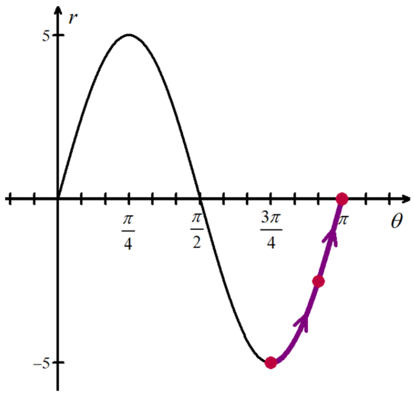
Figure 6.2. 43



$r = 5\sin(2\theta)$ in the polar plane

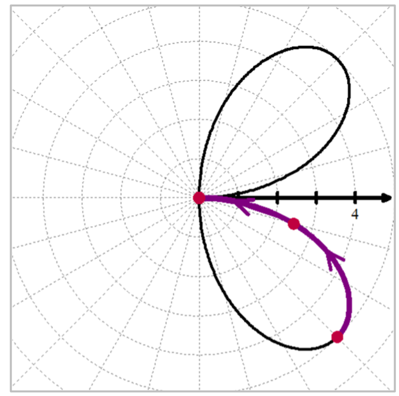
4. For $\frac{3\pi}{4} \le \theta \le \pi$, r increases from −5 to 0, so the polar curve pulls back to the pole.

Figure 6.2. 44



$r = 5\sin(2\theta)$ in the θr-plane

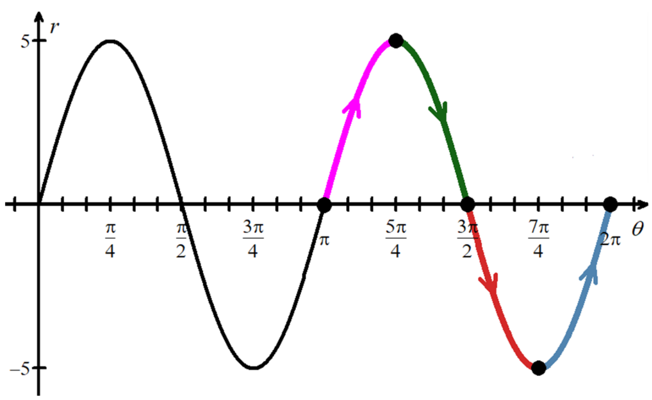
Figure 6.2. 45



$r = 5\sin(2\theta)$ in the polar plane

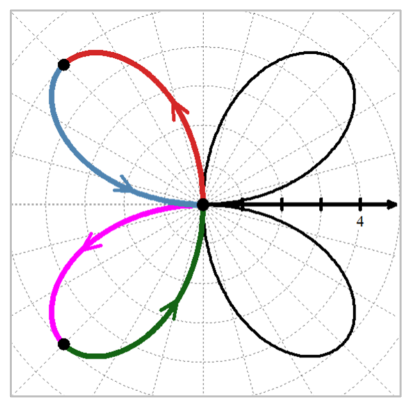
Below, we continue our sketch of the graph from $\theta = \pi$ to $\theta = 2\pi$.

Figure 6.2. 46



$r = 5\sin(2\theta)$ in the θr-plane

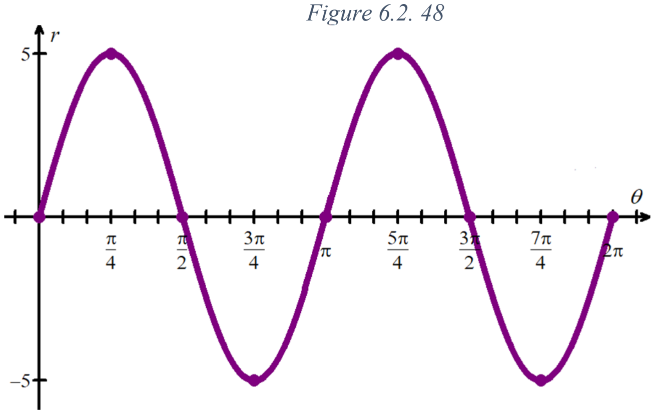
Figure 6.2. 47



$r = 5\sin(2\theta)$ in the polar plane

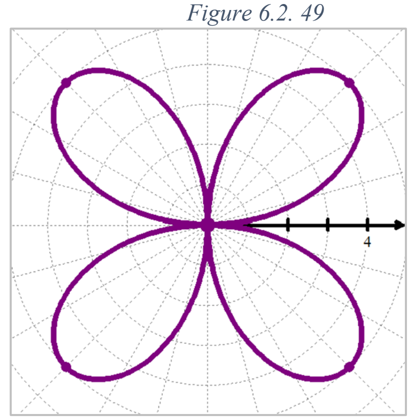
Below is the final graph.

Figure 6.2. 48



$r = 5\sin(2\theta)$ in the θr-plane

Figure 6.2. 49



$r = 5\sin(2\theta)$ in the polar plane

□

The polar curve in **Example 6.2.5** is a **rose**. A polar equation of the form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ will result in a graph having the shape of a rose, with n petals if n is odd or $2n$ petals if n is even.

Graphing Polar Equations of the Form $r^2 = a^2\sin(2\theta)$ or $r^2 = a^2\cos(2\theta)$, $a \neq 0$

Example 6.2.6. Graph $r^2 = 16\cos(2\theta)$.

Solution. We can make our task easier by first checking for symmetries. Notice that replacing θ by $-\theta$ does not change the equation:

$$16\cos(2(-\theta)) = 16\cos(-2\theta) = 16\cos(2\theta)$$

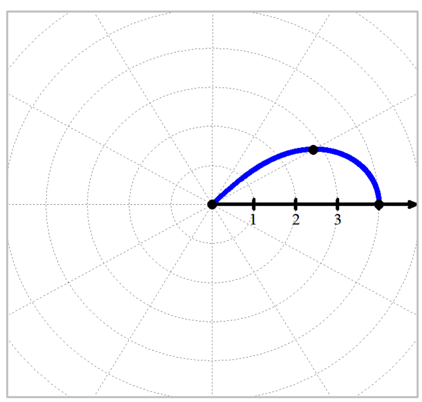
Thus, the graph is symmetric about the polar axis. Also, the graph is symmetric about the pole since replacing r by $-r$ does not change the equation:

$$(-r)^2 = 16\cos(2\theta) \Leftrightarrow r^2 = 16\cos(2\theta)$$

To find points on the graph, we solve $r^2 = 16\cos(2\theta)$ for r to get $r = \pm 4\sqrt{\cos(2\theta)}$. Since the graph is symmetric about the pole, we need only consider $r = 4\sqrt{\cos(2\theta)}$, and since $\sqrt{\cos(2\theta)}$ is undefined when $\cos(2\theta) < 0$, or $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, we start with $0 \le \theta \le \frac{\pi}{4}$.

θ	$r = 4\sqrt{\cos(2\theta)}$	(r,θ)
0	4	$(4,0)$
$\frac{\pi}{6}$	$\frac{4}{\sqrt{2}} \approx 2.8$	$\left(\frac{4}{\sqrt{2}}, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	0	$\left(0, \frac{\pi}{4}\right)$

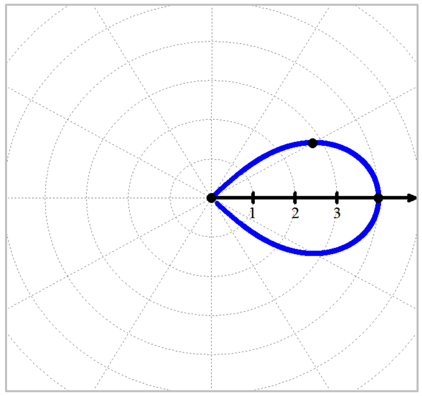
Figure 6.2. 50



$r = 4\sqrt{\cos(2\theta)}$, $0 \le \theta \le \frac{\pi}{4}$

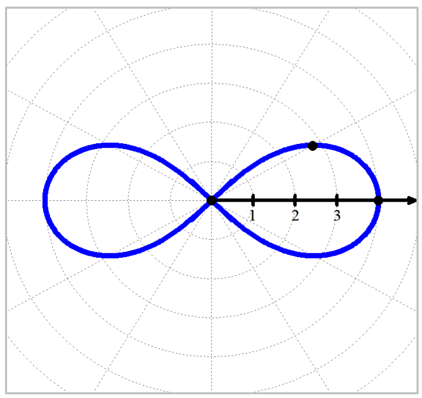
Adding the symmetry about the polar axis, we have the following.

Figure 6.2. 51



Finally, incorporating the property of symmetry about the pole, we have the graph of $r^2 = 16\cos(2\theta)$.

Figure 6.2. 52

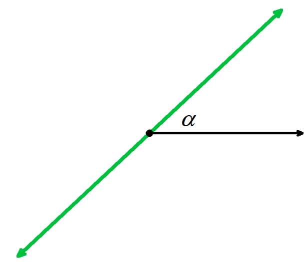
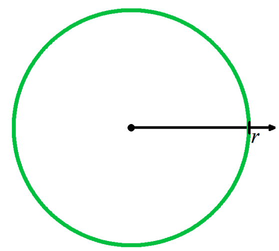
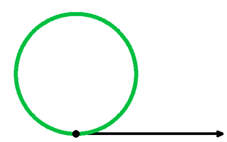
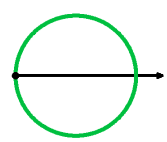
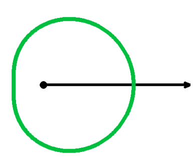


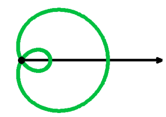
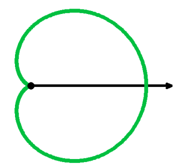
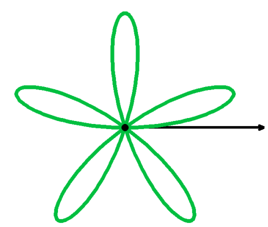
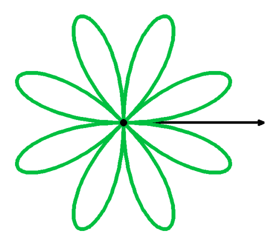
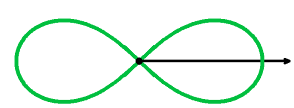
$r^2 = 16\cos(2\theta)$

□

The polar curve in **Example 6.2.6** is a **lemniscate**. Note that, instead of symmetry about the pole, we could have shown and used symmetry about the line $\theta = \frac{\pi}{2}$.

This section ends with a summary of common polar graphs and their associated equations. The exact location/orientation of the curves depends on the values of the constants and the specific function.

Polar Equation	Description	Representative Graph
$\theta = \alpha$	Line passing through the Pole	*Figure 6.2. 53* 
$r = a$	Circle centered at the Pole	*Figure 6.2. 54* 
$r = a\sin(\theta)$	Circle touching the Pole with the center on the line $\theta = \frac{\pi}{2}$	*Figure 6.2. 55* 
$r = a\cos(\theta)$	Circle touching the Pole with center on the Polar Axis	*Figure 6.2. 56* 
$r = a \pm b\sin(\theta)$ or $r = a \pm b\cos(\theta)$ $0 < b < a$	Limaçon without inner loop	*Figure 6.2. 57* 

Polar Equation	Description	Representative Graph
$r = a \pm b\sin(\theta)$ or $r = a \pm b\cos(\theta)$ $0 < a < b$	Limaçon with inner loop	*Figure 6.2. 58* 
$r = a \pm a\sin(\theta)$ or $r = a \pm a\cos(\theta)$	Cardioid	*Figure 6.2. 59* 
$r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ n odd	Rose with n petals	*Figure 6.2. 60* 
$r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ n even	Rose with $2n$ petals	*Figure 6.2. 61* 
$r^2 = a^2\sin(2\theta)$ $r^2 = a^2\cos(2\theta)$	Lemniscate	*Figure 6.2. 62* 

6.2 Exercises

In Exercises 1 – 10, plot the graph of the polar equation by hand, without the aid of a calculator. Label the polar axis and identify points that you use in plotting the graph.

1. Circle: $r = 6\sin(\theta)$
2. Rose: $r = 2\sin(2\theta)$
3. Rose: $r = 4\cos(2\theta)$
4. Rose: $r = \cos(5\theta)$
5. Cardioid: $r = 3 - 3\cos(\theta)$
6. Cardioid: $r = 2 + 2\cos(\theta)$
7. Limaçon: $r = 1 - 2\sin(\theta)$
8. Limaçon: $r = 3 - 5\cos(\theta)$
9. Limaçon: $r = 5 - 3\sin(\theta)$
10. Lemniscate: $r^2 = 4\cos(2\theta)$

In Exercises 11 – 20, plot the graph of the polar equation by hand, without the aid of a calculator. Label the polar axis and identify points that you use in plotting the graph.

11. $r = 2\cos(\theta)$
12. $r = 5 + 5\sin(\theta)$
13. $r = 5\sin(3\theta)$
14. $r = 1 - 2\cos(\theta)$
15. $r = 2 + 7\sin(\theta)$
16. $r = \sin(4\theta)$
17. $r = 3\cos(4\theta)$
18. $r = 2\sqrt{3} + 4\cos(\theta)$
19. $r = 1 - \sin(\theta)$
20. $r^2 = \sin(2\theta)$

Exercises 21 – 30 give you some curves to graph using a graphing calculator or other form of technology. Notice that some of the curves have explicit bounds on θ and others do not.

21. $r = \theta$, $0 \le \theta \le 12\pi$
22. $r = \ln(\theta)$, $1 \le \theta \le 12\pi$
23. $r = e^{0.1\theta}$, $0 \le \theta \le 12\pi$
24. $r = \theta^3 - \theta$, $-1.2 \le \theta \le 1.2$
25. $r = \sin(5\theta) - 3\cos(\theta)$
26. $r = \sin^3\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{3}\right)$
27. $r = \arctan(\theta)$, $-\pi \le \theta \le \pi$
28. $r = \dfrac{1}{1 - \cos(\theta)}$
29. $r = \dfrac{1}{2 - \cos(\theta)}$
30. $r = \dfrac{1}{2 - 3\cos(\theta)}$

31. How many petals does the polar rose $r = \sin(2\theta)$ have? What about $r = \sin(3\theta)$, $r = \sin(4\theta)$ and $r = \sin(5\theta)$? With the help of your classmates, make a conjecture as to how many petals the polar rose $r = \sin(n\,\theta)$ has for any natural number n. Replace sine with cosine and repeat the investigation. How many petals does $r = \cos(n\,\theta)$ have for each natural number n?

32. In this exercise, we have you and your classmates explore transformations of polar graphs. For both parts (a) and (b), let $f(\theta) = \cos(\theta)$ and $g(\theta) = 2 - \sin(\theta)$.

   a) Using a graphing calculator or other form of technology, compare the graph of $r = f(\theta)$ to each of the graphs of $r = f\left(\theta + \frac{\pi}{4}\right)$, $r = f\left(\theta + \frac{3\pi}{4}\right)$, $r = f\left(\theta - \frac{\pi}{4}\right)$ and $r = f\left(\theta - \frac{3\pi}{4}\right)$. Repeat the process for $g(\theta)$. In general, how do you think the graph of $r = f(\theta + \alpha)$ compares with the graph of $r = f(\theta)$?

   b) Using a graphing calculator or other form of technology, compare the graph of $r = f(\theta)$ to each of the graphs of $r = 2f(\theta)$, $r = \frac{1}{2}f(\theta)$, $r = -f(\theta)$ and $r = -3f(\theta)$. Repeat this process for $g(\theta)$. In general, how do you think the graph of $r = k\,f(\theta)$ compares with the graph of $r = f(\theta)$? (Does it matter if $k > 0$ or $k < 0$?)

33. With the help of your classmates, research cardioid microphones.

6.3 Polar Representations of Complex Numbers

Learning Objectives

- **Find the real part, the imaginary part, the modulus, and the argument of a complex number.**
- **Graph complex numbers.**
- **Know and apply properties of complex numbers.**

Complex Numbers and the Complex Plane

While the equation $x^2 = -1$ has no real solutions, it prompts us to look for a quantity x whose square is -1. We write such a quantity as i, or $\sqrt{-1}$, and refer to it as the **imaginary unit**. The imaginary unit i is a different kind of number with the property that $i^2 = -1$. The properties of i that distinguish it from the real numbers are listed below.

Properties of i: The imaginary unit i satisfies the following two properties.

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = \left(\sqrt{c}\right) \cdot i$

Property 1 establishes that i does act as a square root of -1. Property 2 establishes what we mean by the 'principal square root' of a negative real number. For Property 2, it is important to remember the restriction on c, requiring that $c \geq 0$. For example, $\sqrt{-4} = \sqrt{4}\, i = 2i$, but $\sqrt{-(-4)} \neq \sqrt{-4}\, i$.[6] We are now ready to define complex numbers.

Definition 6.2. A **complex number** is defined as $z = a + bi$ where a and b are real numbers and i is the imaginary unit. The number a is the **real part** of the complex number, denoted $\operatorname{Re}(z)$, and the number b is the **imaginary part**, denoted $\operatorname{Im}(z)$, of the complex number.

The arithmetic of complex numbers is performed by treating i as a variable, but with the additional two properties listed above.

[6] This would result in $2 = \sqrt{4} = \sqrt{-(-4)} = \left(\sqrt{-4}\right)i = (2i)i = 2i^2 = 2(-1) = -2$ which is, obviously, false.

Example 6.3.1. Perform the indicated operations. Write your answer in the form $a+bi$.

1. $\sqrt{-3}\sqrt{-12}$ 2. $(1-2i)-(3+4i)$ 3. $\left(2\sqrt{3}+2i\right)\left(-1+\sqrt{3}\,i\right)$ 4. $\left(-1+\sqrt{3}\,i\right)^3$

Solution.

1. We use Property 2 first, then apply rules of real radicals and Property 1.

$$\begin{aligned}\sqrt{-3}\sqrt{-12} &= \left(\sqrt{3}\,i\right)\left(\sqrt{12}\,i\right)\\ &= \sqrt{3}\sqrt{12}\,i^2\\ &= \sqrt{36}(-1)\\ &= -6\end{aligned}$$

2. Combining like terms, we get

$$\begin{aligned}(1-2i)-(3+4i) &= 1-2i-3-4i\\ &= -2-6i\end{aligned}$$

3. Using the distributive property results in

$$\begin{aligned}\left(2\sqrt{3}+2i\right)\left(-1+\sqrt{3}\,i\right) &= \left(2\sqrt{3}\right)(-1)+\left(2\sqrt{3}\right)\left(\sqrt{3}\,i\right)+(2i)(-1)+(2i)\left(\sqrt{3}\,i\right)\\ &= -2\sqrt{3}+2(3)i-2i+2\sqrt{3}\,i^2\\ &= -2\sqrt{3}+6i-2i+2\sqrt{3}(-1)\\ &= -4\sqrt{3}+4i\end{aligned}$$

4. Again using the distributive property,

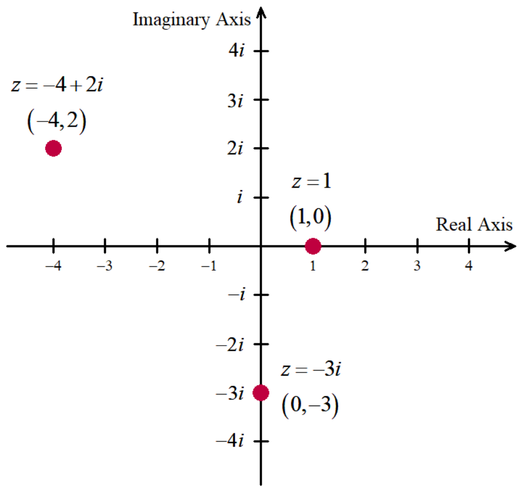
$$\begin{aligned}\left(-1+\sqrt{3}\,i\right)^3 &= \left(-1+\sqrt{3}\,i\right)\left(-1+\sqrt{3}\,i\right)\left(-1+\sqrt{3}\,i\right)\\ &= \left(-1+\sqrt{3}\,i\right)\left(1-\sqrt{3}\,i-\sqrt{3}\,i+3i^2\right)\\ &= \left(-1+\sqrt{3}\,i\right)\left(-2-2\sqrt{3}\,i\right)\\ &= 2+2\sqrt{3}\,i-2\sqrt{3}\,i-2\left(\sqrt{3}\right)^2 i^2\\ &= 2-(-6)\\ &= 8\end{aligned}$$

□

The last problem implies that a solution of the equation $w^3=8$ is $w=-1+\sqrt{3}\,i$, or that a third root of the number 8 is $-1+\sqrt{3}\,i$. Of course, a real valued solution to $w^3=8$, or a third root of the number 8, is the number 2. We will discuss finding all roots of a number in **Section 6.4**.

We can visualize complex numbers as follows. Associate each complex number $z = a + bi$ with the point (a,b) on the coordinate plane. In this case, the *x*-axis is relabeled as the **real axis**, which corresponds to the real number line, and the *y*-axis is relabeled as the **imaginary axis**, which is demarcated in increments of the imaginary unit i. The plane determined by these two axes is called the **complex plane**. Several complex numbers are plotted in the following complex plane.

Figure 6.3. 1



Complex Plane

Since the ordered pair (a,b) gives the rectangular coordinates associated with the complex number $z = a + bi$, the expression $z = a + bi$ is called the **rectangular form** of z. Of course, we could just as easily associate z with a pair of polar coordinates (r,θ). Although it is not as straightforward as the definitions of $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$, we can still give r and θ names in relation to z.

Definition 6.3. The Modulus and Argument of Complex Numbers: Suppose $z = a + bi$ is a complex number. Let (r,θ) be the polar representation of the point having rectangular coordinates (a,b), with $r \geq 0$ and $0 \leq \theta < 2\pi$.

- The **modulus** of z, denoted $|z|$, is defined by $|z| = r = \sqrt{a^2 + b^2}$.
- For $z \neq 0$, the angle θ is called the **argument** of z.[7] For $z = 0$, the argument is not defined.

Note that if $b = 0$ then the modulus of z is equal to the square root of a^2, and $\sqrt{a^2} = |a|$, which explains the use of the absolute value notation.

[7] The argument may be restricted to other intervals such as $(-\pi,\pi]$, in which case $-\pi < \theta \leq \pi$.

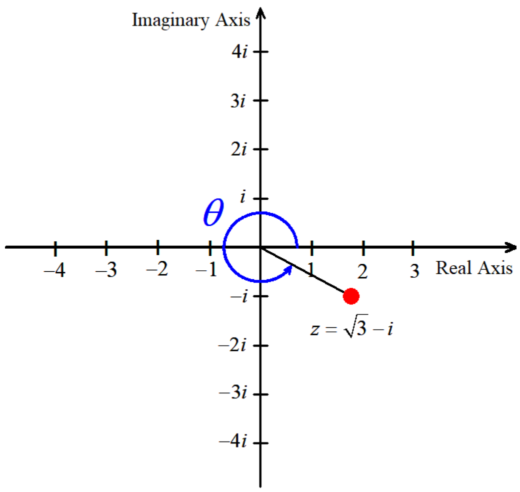
Example 6.3.2. For each of the following complex numbers, find $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, and the argument θ, $0 \le \theta < 2\pi$.

1. $z = \sqrt{3} - i$ 2. $z = -2 + 4i$ 3. $z = 3i$ 4. $z = -117$

Solution.

1. For $z = \sqrt{3} - i = \sqrt{3} + (-1)i$, we have $\text{Re}(z) = \sqrt{3}$ and $\text{Im}(z) = -1$. We plot this complex number, with rectangular coordinates $(\sqrt{3}, -1)$, in the complex plane.

Figure 6.3. 2



To find $|z|$ and θ, we need a polar representation (r, θ) for $(a, b) = (\sqrt{3}, -1)$, with $r \ge 0$ and $0 \le \theta < 2\pi$. Then, by **Definition 6.3**,

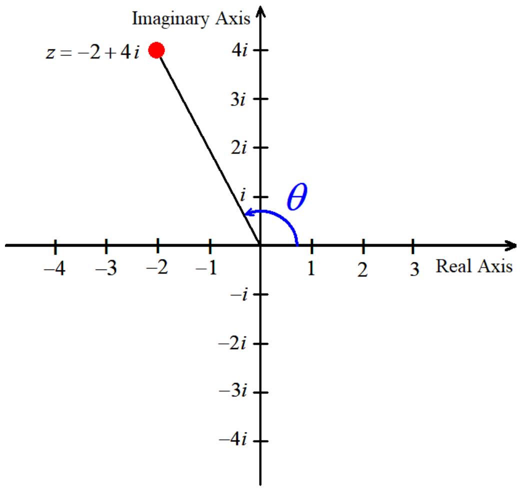
$$\begin{aligned} r &= \sqrt{(\sqrt{3})^2 + (-1)^2} \quad \text{from } r = \sqrt{a^2 + b^2} \\ &= \sqrt{3+1} \\ &= 2 \end{aligned}$$

Thus, $|z| = r = 2$.

To determine θ, we see that θ is in Quadrant IV and that $\tan(\theta) = \frac{b}{a} = \frac{-1}{\sqrt{3}}$. Note that this is equivalent to $\mathbf{tan}(\boldsymbol{\theta}) = \frac{\mathbf{Im(z)}}{\mathbf{Re(z)}}$. To meet the requirement that $0 \le \theta < 2\pi$, we find $\theta = \frac{11\pi}{6}$.

2. The complex number $z = -2 + 4i$ has $\text{Re}(z) = -2$ and $\text{Im}(z) = 4$. We plot the number $z = -2 + 4i$, with rectangular coordinates $(-2, 4)$, in the complex plane.

Figure 6.3. 3



We look for a polar representation (r,θ) where $r \geq 0$.

$$r = \sqrt{(-2)^2 + (4)^2}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

Then $|z| = r = 2\sqrt{5}$. To find θ, we use $\tan(\theta) = \dfrac{4}{-2} = -2$. We will need to resort to using the arctangent since this is not a standard angle.

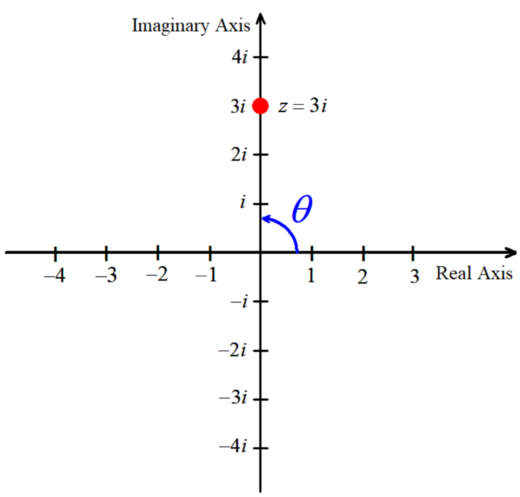
With $(-2,4)$ in Quadrant II and $\arctan(-2)$ in Quadrant IV,

$$\theta = \pi + \arctan(-2)$$
$$\theta = \pi - \arctan(2) \quad \text{since arctangent is an odd function}$$

Note that $\theta = \pi - \arctan(2)$ meets the requirement that $0 \leq \theta < 2\pi$.

3. We rewrite $z = 3i$ as $z = 0 + 3i$ to find $\text{Re}(z) = 0$ and $\text{Im}(z) = 3$. We plot the number $z = 3i$ in the complex plane, using rectangular coordinates $(0,3)$.

Figure 6.3. 4

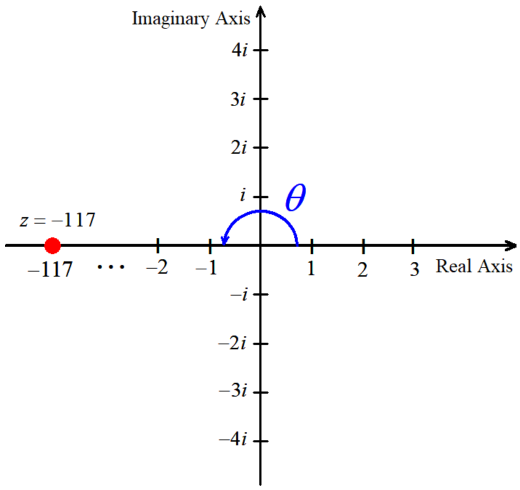


While we could go through the usual calculations to find the required polar form of this point, we can almost 'see' the answer. The point $(0,3)$ lies 3 units away from the origin on the positive imaginary axis.

Hence, $|z| = r = 3$ and $\theta = \dfrac{\pi}{2}$.

4. As in the previous problem, we write $z = -117$ as $z = -117 + 0i$, so $\text{Re}(z) = -117$ and $\text{Im}(z) = 0$. The number $z = -117$ is the rectangular point $(-117,0)$ in the complex plane.

Figure 6.3. 5



This is another instance where we can determine the polar form 'by eye'. The point $(-117,0)$ is 117 units away from the origin along the negative real axis.

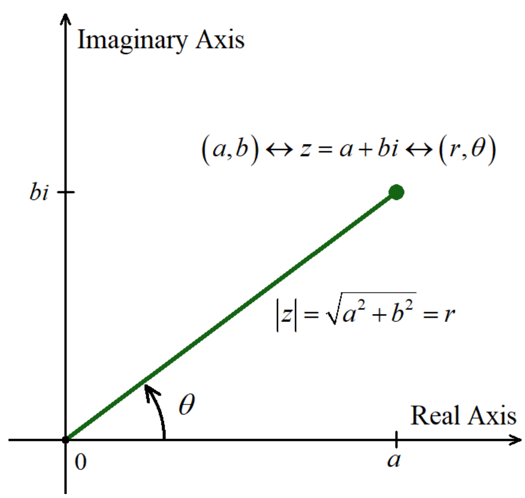
Hence, $|z| = r = 117$ and $\theta = \pi$.

□

Polar Form of Complex Numbers

Our next goal is to link the geometry and algebra of the complex numbers. To that end, consider the following figure.

Figure 6.3. 6



Polar coordinate (r,θ) associated with $z = a+bi$, $r \geq 0$ and $0 \leq \theta < 2\pi$

We know from **Theorem 6.1** that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Making these substitutions for a and b gives

$$\begin{aligned} z &= a + bi \\ &= r\cos(\theta) + r\sin(\theta)\, i \\ &= r\left(\cos(\theta) + i\,\sin(\theta)\right) \end{aligned}$$

The expression $\cos(\theta) + i\,\sin(\theta)$ can be abbreviated $\text{cis}(\theta)$, so that $z = r\,\text{cis}(\theta)$. From $r = |z|$, we have the following definition.

Definition 6.4. Polar Form of a Complex Number: Suppose z is a complex number and θ is the argument of z. A polar form for z is

$$|z|\operatorname{cis}(\theta) = |z|\left(\cos(\theta) + i\,\sin(\theta)\right)$$

Other polar forms of z are $|z|\operatorname{cis}(\theta + 2\pi k) = |z|\left(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)\right)$ for integers k.

Example 6.3.3. Find the rectangular form of the following complex numbers. Identify $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$.

1. $z = 4\operatorname{cis}\left(\frac{2\pi}{3}\right)$ 2. $z = 2\operatorname{cis}\left(\frac{5\pi}{4}\right)$ 3. $z = 3\operatorname{cis}(0)$ 4. $z = \operatorname{cis}\left(\frac{\pi}{2}\right)$

Solution. The key to finding the rectangular form of these complex numbers is to write $\operatorname{cis}(\theta)$ as $\cos(\theta) + i\,\sin(\theta)$.

1. By definition,

$$\begin{aligned} z &= 4\operatorname{cis}\left(\frac{2\pi}{3}\right) \\ &= 4\left(\cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 4\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right) \end{aligned}$$

   After simplifying, $z = -2 + 2\sqrt{3}\,i$ so that $\operatorname{Re}(z) = -2$ and $\operatorname{Im}(z) = 2\sqrt{3}$.

2. Expanding, we get

$$\begin{aligned} z &= 2\operatorname{cis}\left(\frac{5\pi}{4}\right) \\ &= 2\left(\cos\left(\frac{5\pi}{4}\right) + i\,\sin\left(\frac{5\pi}{4}\right)\right) \\ &= 2\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right) \end{aligned}$$

   Then $z = -\sqrt{2} - \sqrt{2}\,i$, so $\operatorname{Re}(z) = -\sqrt{2} = \operatorname{Im}(z)$.

3. We have

$$\begin{aligned} z &= 3\operatorname{cis}(0) \\ &= 3\left(\cos(0) + i\,\sin(0)\right) \\ &= 3\left(1 + i\,(0)\right) \end{aligned}$$

Simplifying, $z=3$, from which $\operatorname{Re}(z)=3$ and $\operatorname{Im}(z)=0$. Note that 3 is a real number, so it makes sense to have an imaginary part of 0.

4. Lastly,

$$\begin{aligned} z &= \operatorname{cis}\left(\frac{\pi}{2}\right) \\ &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \\ &= 0 + i\,(1) \end{aligned}$$

We find $z=i$, $\operatorname{Re}(z)=0$, and $\operatorname{Im}(z)=1$.

□

Example 6.3.4. Use the results from **Example 6.3.2** to find a polar form of the following complex numbers.

1. $z=\sqrt{3}-i$ 2. $z=-2+4i$ 3. $z=3i$ 4. $z=-117$

Solution. To write a polar form of a complex number z, we need two pieces of information: the modulus $|z|$ and the argument θ for z. This information, for each of these complex numbers, was included in the solution to **Example 6.3.2**.

1. For $z=\sqrt{3}-i$, $|z|=2$ and $\theta=\frac{11\pi}{6}$, so $z=2\operatorname{cis}\left(\frac{11\pi}{6}\right)$. We can check our answer by converting it back to rectangular form to see that it simplifies to $z=\sqrt{3}-i$.
2. For $z=-2+4i$, $|z|=2\sqrt{5}$ and $\theta=\pi-\arctan(2)$. Hence, $z=2\sqrt{5}\operatorname{cis}(\pi-\arctan(2))$. It is a good exercise to show that this polar form reduces to $z=-2+4i$.
3. Next, $z=3i$ has $|z|=3$ and $\theta=\frac{\pi}{2}$. In this case, $z=3\operatorname{cis}\left(\frac{\pi}{2}\right)$. This can be checked geometrically; rotate $\frac{\pi}{2}$ radians counter-clockwise from the polar axis, about the pole, then move 3 units from the pole along the resulting ray. This positions you exactly 3 units above 0 on the imaginary axis at $z=3i$.
4. Last but not least, for $z=-117$, $|z|=117$ and $\theta=\pi$. So $z=117\operatorname{cis}(\pi)$. As in the previous problem, the answer is easily checked geometrically.

□

6.3 Exercises

In Exercises 1 – 20, find a polar representation for the complex number z. Identify $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, and the argument θ, $0 \le \theta < 2\pi$. These exercises should be worked without the aid of a calculator.

1. $z = 9 + 9\,i$
2. $z = 5 + 5\,i\sqrt{3}$
3. $z = 6\,i$
4. $z = -3\sqrt{2} + 3\,i\sqrt{2}$
5. $z = -6\sqrt{3} + 6\,i$
6. $z = -2$
7. $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}\,i$
8. $z = -3 - 3\,i$
9. $z = -5\,i$
10. $z = 6$
11. $z = i\sqrt[3]{7}$
12. $z = 2\sqrt{2} - 2\,i\sqrt{2}$
13. $z = 3 + 4\,i$
14. $z = \sqrt{2} + i$
15. $z = -7 + 24\,i$
16. $z = -2 + 6\,i$
17. $z = -12 - 5\,i$
18. $z = -5 - 2\,i$
19. $z = 4 - 2\,i$
20. $z = 1 - 3\,i$

In Exercises 21 – 40, find the rectangular form of the given complex number. Use whatever identities are necessary to find the exact values. These exercises should be worked without the aid of a calculator.

21. $z = 6\,\text{cis}(0)$
22. $z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$
23. $z = 7\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$
24. $z = 3\,\text{cis}\left(\frac{\pi}{2}\right)$
25. $z = 4\,\text{cis}\left(\frac{2\pi}{3}\right)$
26. $z = \sqrt{6}\,\text{cis}\left(\frac{3\pi}{4}\right)$
27. $z = 9\,\text{cis}(\pi)$
28. $z = 3\,\text{cis}\left(\frac{4\pi}{3}\right)$
29. $z = 7\,\text{cis}\left(\frac{5\pi}{4}\right)$
30. $z = \sqrt{13}\,\text{cis}\left(\frac{3\pi}{2}\right)$
31. $z = \frac{1}{2}\,\text{cis}\left(\frac{7\pi}{4}\right)$
32. $z = 12\,\text{cis}\left(\frac{5\pi}{3}\right)$
33. $z = 8\,\text{cis}\left(\frac{\pi}{12}\right)$
34. $z = 2\,\text{cis}\left(\frac{7\pi}{8}\right)$
35. $z = 5\,\text{cis}\left(\arctan\left(\frac{4}{3}\right)\right)$
36. $z = \sqrt{10}\,\text{cis}\left(\arctan\left(\frac{1}{3}\right)\right)$
37. $z = 15\,\text{cis}(\pi + \arctan(-2))$
38. $z = \sqrt{3}\,\text{cis}\left(\pi + \arctan\left(-\sqrt{2}\right)\right)$
39. $z = 50\,\text{cis}\left(\pi - \arctan\left(\frac{7}{24}\right)\right)$
40. $z = \frac{1}{2}\,\text{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$

41. The complex conjugate of a complex number $z = a + b\,i$ is denoted $\overline{z}$ and is given by $\overline{z} = a - b\,i$.

a) Prove that $|\overline{z}| = |z|$.

b) Prove that $|z| = \sqrt{z \cdot \overline{z}}$.

c) Show that $\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\operatorname{Im}(z) = \dfrac{z - \overline{z}}{2\,i}$.

6.4 Complex Products, Powers, Quotients, and Roots

Learning Objectives

- **Find the product and quotient of complex numbers.**
- **Find the power and roots of complex numbers.**

Products and Quotients of Complex Numbers

The following theorem requires that complex numbers be written in polar form before making use of the formula.

Theorem 6.2. Products and Quotients of Complex Numbers: Suppose z and w are complex numbers with polar forms $z=|z|\operatorname{cis}(\alpha)$ and $w=|w|\operatorname{cis}(\beta)$.

- **Product Rule:** $zw=|z||w|\operatorname{cis}(\alpha+\beta)$

  (To multiply two complex numbers, multiply their moduli and add their arguments.)

- **Quotient Rule:** $\dfrac{z}{w}=\dfrac{|z|}{|w|}\operatorname{cis}(\alpha-\beta)$, provided $|w|\neq 0$

  (To divide one complex number by another complex number, divide the modulus of the first by the modulus of the second and subtract the second argument from the first argument.)

The proof of **Theorem 6.2** requires a mix of definitions, arithmetic, and identities.

- We start with the product rule.

$$\begin{aligned} zw &= \left[|z|\operatorname{cis}(\alpha)\right]\left[|w|\operatorname{cis}(\beta)\right] \\ &= |z||w|\operatorname{cis}(\alpha)\operatorname{cis}(\beta) \\ &= |z||w|\left[\cos(\alpha)+i\sin(\alpha)\right]\left[\cos(\beta)+i\sin(\beta)\right] && \text{definition of cis} \end{aligned}$$

  We now focus on the quantities in brackets on the right side of the equation.

$$\begin{aligned} &\left[\cos(\alpha)+i\sin(\alpha)\right]\left[\cos(\beta)+i\sin(\beta)\right] \\ &= \cos(\alpha)\cos(\beta)+i\cos(\alpha)\sin(\beta)+i\sin(\alpha)\cos(\beta)+i^2\sin(\alpha)\sin(\beta) \\ &= \cos(\alpha)\cos(\beta)+i^2\sin(\alpha)\sin(\beta)+i\sin(\alpha)\cos(\beta)+i\cos(\alpha)\sin(\beta) && \text{rearrange terms} \\ &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)+i\left[\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\right] && i^2=-1;\text{ factor out } i \\ &= \cos(\alpha+\beta)+i\sin(\alpha+\beta) && \text{sum identities} \\ &= \operatorname{cis}(\alpha+\beta) \end{aligned}$$

  Putting this result together with our earlier work, we have $zw=|z||w|\operatorname{cis}(\alpha+\beta)$.

- To prove the quotient rule, assuming $|w| \neq 0$,

$$\begin{aligned}\frac{z}{w} &= \frac{|z|\operatorname{cis}(\alpha)}{|w|\operatorname{cis}(\beta)} \\ &= \frac{|z|}{|w|}\cdot\frac{\cos(\alpha)+i\sin(\alpha)}{\cos(\beta)+i\sin(\beta)}\end{aligned}$$

Multiplying by $\dfrac{\cos(\beta)-i\sin(\beta)}{\cos(\beta)-i\sin(\beta)}$,

$$\begin{aligned}\frac{z}{w} &= \frac{|z|}{|w|}\cdot\frac{\cos(\alpha)+i\sin(\alpha)}{\cos(\beta)+i\sin(\beta)}\cdot\frac{\cos(\beta)-i\sin(\beta)}{\cos(\beta)-i\sin(\beta)} \\ &= \frac{|z|}{|w|}\cdot\frac{\left[\cos(\alpha)+i\sin(\alpha)\right]\left[\cos(\beta)-i\sin(\beta)\right]}{\left[\cos(\beta)+i\sin(\beta)\right]\left[\cos(\beta)-i\sin(\beta)\right]} \\ &= \frac{|z|}{|w|}\cdot\frac{\cos(\alpha)\cos(\beta)-i\cos(\alpha)\sin(\beta)+i\sin(\alpha)\cos(\beta)-i^2\sin(\alpha)\sin(\beta)}{\cos^2(\beta)-i\cos(\beta)\sin(\beta)+i\sin(\beta)\cos(\beta)-i^2\sin^2(\beta)} \\ &= \frac{|z|}{|w|}\cdot\frac{\left[\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\right]+i\left[\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\right]}{\cos^2(\beta)-i^2\sin^2(\beta)} \\ &= \frac{|z|}{|w|}\cdot\frac{\cos(\alpha-\beta)+i\sin(\alpha-\beta)}{1}\end{aligned}$$

Finally, we have $\dfrac{z}{w} = \dfrac{|z|}{|w|}\operatorname{cis}(\alpha-\beta)$.

Example 6.4.1. Let $z = 2\sqrt{3}+2i$ and $w = -1+\sqrt{3}\,i$. Use **Theorem 6.2** to find the following, with the final answer expressed in rectangular form.

1. zw
2. $\dfrac{z}{w}$

Solution. In order to use **Theorem 6.2**, we first write z and w in polar form.

For $z = 2\sqrt{3}+2i$,

$$\begin{aligned}|z| &= \sqrt{\left(2\sqrt{3}\right)^2+(2)^2} \\ &= \sqrt{16} \\ &= 4\end{aligned}$$

To determine the argument α,

$$\begin{aligned}\tan(\alpha) &= \frac{\text{Im}(z)}{\text{Re}(z)} \\ &= \frac{2}{2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}\end{aligned}$$

Noting that z lies in Quadrant I, we find $\alpha = \frac{\pi}{6}$.[8] Hence, $z = 4\text{cis}\left(\frac{\pi}{6}\right)$.

For $w = -1 + \sqrt{3}\,i$,

$$\begin{aligned}|w| &= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \\ &= 2\end{aligned}$$

The argument β has

$$\begin{aligned}\tan(\beta) &= \frac{\sqrt{3}}{-1} \\ &= -\sqrt{3}\end{aligned}$$

Since w lies in Quadrant II, it follows that $\beta = \frac{2\pi}{3}$. So, $w = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$.

We can now proceed with the solution.

1. To determine $z\,w$, we use the product rule.

$$\begin{aligned}z\,w &= \left[4\,\text{cis}\left(\frac{\pi}{6}\right)\right]\left[2\,\text{cis}\left(\frac{2\pi}{3}\right)\right] \\ &= 8\,\text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) \\ &= 8\,\text{cis}\left(\frac{5\pi}{6}\right) \\ &= 8\left[\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right]\end{aligned}$$

After converting to rectangular form and simplifying, $z\,w = -4\sqrt{3} + 4i$.

2. Using the quotient rule,

[8] Unless otherwise specified, we will select arguments between 0 (inclusive) and 2π.

$$\begin{aligned}\frac{z}{w} &= \frac{4\operatorname{cis}\left(\frac{\pi}{6}\right)}{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}\\ &= \frac{4}{2}\operatorname{cis}\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)\\ &= 2\operatorname{cis}\left(-\frac{\pi}{2}\right)\end{aligned}$$

Since $-\frac{\pi}{2}$ is a quadrantal angle, we can 'see' the rectangular form by rotating $\frac{\pi}{2}$ radians clockwise from the positive real axis, and then moving out 2 units along the negative imaginary axis. We find that $\frac{z}{w} = -2i$.

□

Powers of Complex Numbers

The following theorem requires that a complex number first be written in polar form.

Theorem 6.3. Power Rule (DeMoivre's Theorem): Suppose z is a complex number with polar form $z = |z|\operatorname{cis}(\alpha)$. Then

$$z^n = |z|^n \operatorname{cis}(n\alpha) \text{ for every natural number } n$$

(To raise a complex number to the power n, raise its modulus to the power n and multiply its argument by n.)

We prove **Theorem 6.3** by induction. Let $P(n)$ denote the statement $z^n = |z|^n \operatorname{cis}(n\alpha)$.

- Then $P(1)$ is true since

$$\begin{aligned}z^1 &= z\\ &= |z|\operatorname{cis}(\alpha)\\ &= |z|^1\operatorname{cis}(1\cdot\alpha)\end{aligned}$$

- We now assume that $P(k)$ is true; that is, we assume $z^k = |z|^k\operatorname{cis}(k\alpha)$ for some $k \geq 1$. Our goal is to show that $P(k+1)$ is true, or that $z^{k+1} = |z|^{k+1}\operatorname{cis}((k+1)\alpha)$. We have

$$
\begin{aligned}
z^{k+1} &= z^k z && \text{property of exponents} \\
&= \left(|z|^k \operatorname{cis}(k\alpha)\right)\left(|z|\operatorname{cis}(\alpha)\right) && \text{induction hypothesis} \\
&= \left(|z|^k |z|\right)\operatorname{cis}(k\alpha+\alpha) && \text{product rule} \\
&= |z|^{k+1}\operatorname{cis}((k+1)\alpha)
\end{aligned}
$$

Hence, assuming $P(k)$ is true, then $P(k+1)$ is true so, by the Principle of Mathematical Induction[9], $z^n = |z|^n \operatorname{cis}(n\alpha)$ for all natural numbers n.

Example 6.4.2. Let $w = -1 + \sqrt{3}\, i$. Use **Theorem 6.3** to find w^5, with the final answer expressed in rectangular form.

Solution. From **Example 6.4.1**, the polar form of w is $w = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$. To find w^5, we use DeMoivre's Theorem as follows.

$$
\begin{aligned}
w^5 &= \left[2\operatorname{cis}\left(\frac{2\pi}{3}\right)\right]^5 \\
&= 2^5 \operatorname{cis}\left(5\cdot\frac{2\pi}{3}\right) \\
&= 32\operatorname{cis}\left(\frac{10\pi}{3}\right)
\end{aligned}
$$

Since $\frac{10\pi}{3}$ is coterminal with $\frac{4\pi}{3}$, we get

$$
\begin{aligned}
w^5 &= 32\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] \\
&= 32\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) \\
&= -16 - 16\sqrt{3}\, i
\end{aligned}
$$

□

Some remarks are in order.

- The reader may not be sold on using the polar form of complex numbers to find their product, especially if the numbers are not given in polar form to begin with. Indeed, a lot of work was needed to convert the numbers z and w in **Example 6.4.1** into polar form, compute their

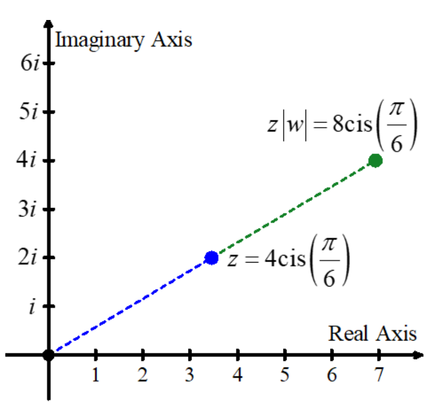
[9] The reader is encouraged to look up the Principle of Mathematical Induction.

product, and then convert the product back to rectangular form; certainly more work than is required to multiply out $zw = \left(2\sqrt{3}+2i\right)\left(-1+\sqrt{3}i\right)$ as we did in **Example 6.3.1**.

- In **Example 6.4.1**, we may (or may not) have saved time using **Theorem 6.2** to find and simplify $\dfrac{z}{w}$ as opposed to starting with $\dfrac{2\sqrt{3}+2i}{-1+\sqrt{3}i}$, rationalizing the denominator, and simplifying. (Try it!)
- **Theorem 6.3** pays huge dividends when computing large powers of complex numbers. Consider how we computed w^5 in **Example 6.4.2** and compare that to accomplishing the same feat by expanding $\left(-1+\sqrt{3}i\right)^5$.
- There is a geometric reason for studying these polar forms. Take the product rule, for instance. If $z = |z|\operatorname{cis}(\alpha)$ and $w = |w|\operatorname{cis}(\beta)$, the formula $zw = |z||w|\operatorname{cis}(\alpha+\beta)$ can be viewed geometrically as a two-step process. The multiplication of $|z|$ by $|w|$, $|w|>1$, can be interpreted as magnifying $|z|$, the distance from 0 to z, by $|w|$. Adding the argument of w to the argument of z can be geometrically interpreted as a rotation of β radians counter-clockwise about the pole.

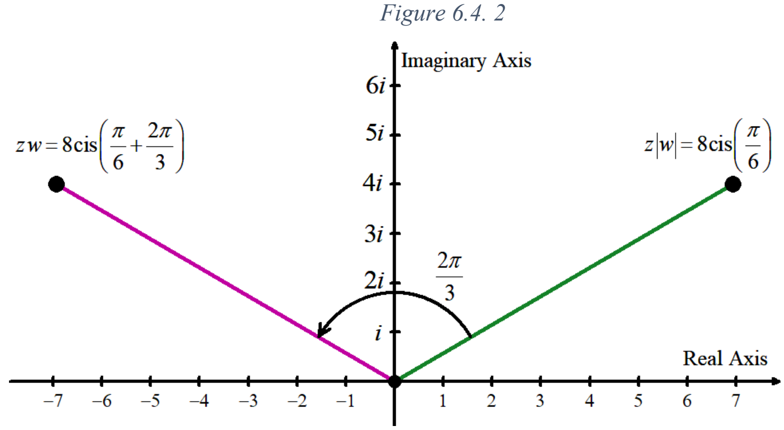
Focusing on $z = 4\operatorname{cis}\left(\dfrac{\pi}{6}\right)$ and $w = 2\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$, we can arrive at the product zw from **Example 6.4.1** by plotting z, doubling its distance from 0, since $|w| = 2$, and rotating $\dfrac{2\pi}{3}$ radians counter-clockwise. The following sequence of diagrams attempts to describe this process geometrically.

Figure 6.4. 1



Multiplying z by $|w| = 2$

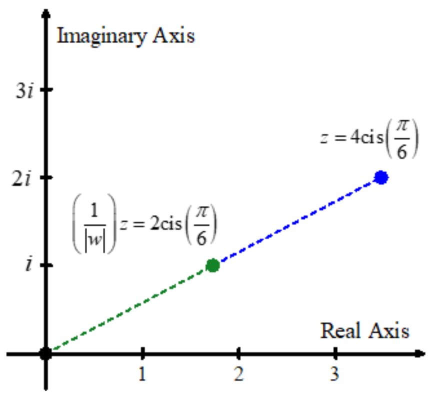
Figure 6.4. 2



Rotating counter-clockwise by $\beta = \dfrac{2\pi}{3}$, the argument of w

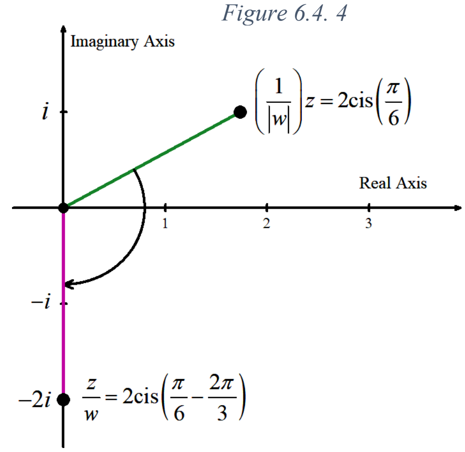
We may visualize division similarly. For $|w|>1$, the formula $\frac{z}{w}=\frac{|z|}{|w|}\text{cis}(\alpha-\beta)$ can be interpreted as shrinking $|z|$, the distance from 0 to z, by the factor $|w|$, and rotating β radians. In the case of $z=4\text{cis}\left(\frac{\pi}{6}\right)$ and $w=2\text{cis}\left(\frac{2\pi}{3}\right)$ from **Example 6.4.1**, we arrive at $\frac{z}{w}$ by halving the distance from 0 to z, and rotating clockwise $\frac{2\pi}{3}$ radians, as visualized below.

Figure 6.4. 3



Dividing z by $|w|=2$

Figure 6.4. 4



Rotating clockwise by $\beta=\frac{2\pi}{3}$, the argument of w

Roots of Complex Numbers

Our last goal of the section is to reverse DeMoivre's Theorem to extract roots of complex numbers.

> **Definition 6.5.** Let n be a positive integer. An ***n*th root** of a complex number z is a complex number w such that $w^n=z$.

Here, we do not specify one particular *principal* n^{th} root, hence the use of the indefinite article in defining w as 'an' n^{th} root of z. Using this definition, both 4 and -4 are square roots of 16, while $\sqrt{16}$ means the principal square root of 16, as in $\sqrt{16}=4$.

Suppose we wish to find the complex third roots of 8. Algebraically, we are trying to solve $w^3=8$. This is equivalent to solving $w^3-8=0$ which, after factoring, becomes $(w-2)(w^2+2w+4)=0$. Now, we need to solve $w-2=0$ and $w^2+2w+4=0$. One solution is $w=2$ and the other two solutions, by the quadratic formula, are $w=\frac{-2\pm\sqrt{4-16}}{2}=-1\pm\sqrt{3}\,i$. So, the complex third roots of 8 are 2 and $-1\pm\sqrt{3}\,i$, for a total of three third roots of 8.

To solve the same problem using techniques developed in this section, we express $z = 8$ in polar form. Noting that $z = 8$ lies 8 units away from the origin on the positive real axis, we get $z = 8\text{cis}(0)$. If we let $w = |w|\text{cis}(\alpha)$ represent the polar form of w, the equation $w^3 = 8$ becomes

$$w^3 = 8$$
$$\left(|w|\text{cis}(\alpha)\right)^3 = 8\text{cis}(0)$$
$$|w|^3 \text{cis}(3\alpha) = 8\text{cis}(0) \quad \text{DeMoivre's Theorem}$$

The complex number on the left side of the equation corresponds to the point with polar coordinates $\left(|w|^3, 3\alpha\right)$ while the complex number on the right side corresponds to the point with polar coordinates $(8,0)$. The two polar representations $\left(|w|^3, 3\alpha\right)$ and $(8,0)$ correspond to the same complex number, and we use this correspondence to determine three distinct points w such that $w^3 = 8$.

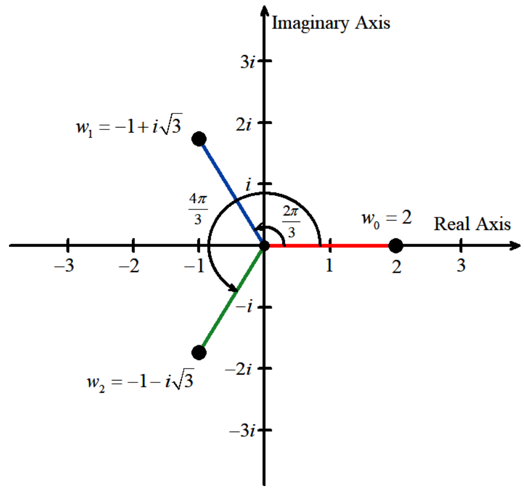
- By definition, $|w| \geq 0$, so it follows that $|w|^3 \geq 0$. Since $|w|$ is a real number, we solve $|w|^3 = 8$ by extracting the principal cube root to get $|w| = \sqrt[3]{8} = 2$.
- Noting that the angle 0 is coterminal with any angle $0 + 2\pi k$, where k is an integer, we use $3\alpha = 0 + 2\pi k$ to determine the angle α needed in the polar representation for each of the three distinct cube roots. Then $\alpha = \frac{2\pi k}{3}$ and setting k equal to 0, 1, and 2, the resulting angles are 0 radians, $\frac{2\pi}{3}$ radians, and $\frac{4\pi}{3}$ radians, respectively.

The resulting polar coordinates for the three roots of 8 and their corresponding numbers (w_0, w_1, and w_2) in both complex and polar form are shown in the following table. For extra practice, try deriving the rectangular forms on your own.

Polar Coordinate	$(2,0)$	$\left(2, \frac{2\pi}{3}\right)$	$\left(2, \frac{4\pi}{3}\right)$
Complex Number	$w_0 = 2\text{cis}(0)$	$w_1 = 2\text{cis}\left(\frac{2\pi}{3}\right)$	$w_2 = 2\text{cis}\left(\frac{4\pi}{3}\right)$
Rectangular Form	$w_0 = 2$	$w_1 = -1 + i\sqrt{3}$	$w_2 = -1 - i\sqrt{3}$

The cube roots of 8 can be visualized geometrically in the complex plane, as follows.

Figure 6.4. 5



Keeping the geometric picture in mind throughout the remainder of this section will lead to an interesting observation regarding geometric properties of complex numbers.

While the process for finding cube roots of 8 seems more involved than our previous factoring approach, this procedure can be generalized to find, for example, all of the fifth roots of 32. (Try using factoring techniques on that!) If we start with a generic complex number in polar form, $z = |z|\operatorname{cis}(\theta)$, and solve $w^n = z$ in the same manner as above, we arrive at the following theorem.

Theorem 6.4. The n^{th} Roots of a Complex Number: Let $z \neq 0$ be a complex number with polar form $z = |z|\operatorname{cis}(\theta)$. For each natural number n, z has n distinct n^{th} roots, which we denote w_0, $w_1, \ldots,$ w_{n-1}, and they are given by the formula

$$w_k = |z|^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)$$

The proof of **Theorem 6.4** breaks into two parts: first, showing that each w_k is an n^{th} root; second, showing that the set $\{w_k \mid k = 0, 1, \ldots, (n-1)\}$ consists of n different complex numbers.

- To show w_k is an n^{th} root of z, we use DeMoivre's Theorem to show $(w_k)^n = z$.

$$\begin{aligned}(w_k)^n &= \left(\sqrt[n]{|z|}\operatorname{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)\right)^n \\ &= \left(\sqrt[n]{|z|}\right)^n \operatorname{cis}\left(n \cdot \left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)\right) \quad \text{DeMoivre's Theorem} \\ &= |z|\operatorname{cis}(\theta + 2\pi k)\end{aligned}$$

Since k is an integer, $\cos(\theta+2\pi k)=\cos(\theta)$ and $\sin(\theta+2\pi k)=\sin(\theta)$. It follows that $\text{cis}(\theta+2\pi k)=\text{cis}(\theta)$, so $(w_k)^n=|z|\text{cis}(\theta)=z$, as required.

- To show that the formula in **Theorem 6.4** generates n distinct numbers, we assume $n\geq 2$ (or else there is nothing to prove) and note that the modulus of each w_k is the same, namely $\sqrt[n]{|z|}$. Therefore, the only way any two of these polar forms correspond to the same number is if their arguments are coterminal; that is, if the arguments differ by an integer multiple of 2π.

  Suppose k and j are integers between 0 and $(n-1)$, inclusive, with $k>j$. Then $\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)-\left(\frac{\theta}{n}+\frac{2\pi}{n}j\right)=2\pi\left(\frac{k-j}{n}\right)$. For this to be an integer multiple of 2π, $(k-j)$ must be a multiple of n. But because of the restrictions on k and j, $0<k-j\leq n-1$. (Think this through.) Hence, $(k-j)$ is a positive number less than n, so it cannot be a multiple of n. As a result, w_k and w_j are different complex numbers, and we are done.

From College Algebra, we know there are at most n distinct solutions to $w^n=z$, and we have just found all of them. We illustrate **Theorem 6.4** in the following examples.

Example 6.4.3. Find the two square roots of $z=-2+2\sqrt{3}\,i$. Express your answers in rectangular form.

Solution. To find both square roots of $z=-2+2\sqrt{3}\,i$, we start by writing z in its polar form, $z=4\,\text{cis}\left(\frac{2\pi}{3}\right)$. We identify $|z|=4$, $\theta=\frac{2\pi}{3}$, and $n=2$. We are looking for two roots, and in keeping with the notation in **Theorem 6.4**, we will call them w_0 and w_1. We get

$$\begin{aligned} w_0 &= \sqrt{4}\,\text{cis}\left(\frac{2\pi/3}{2}+\frac{2\pi}{2}\cdot 0\right) && \text{Theorem 6.4 with } k=0 \\ &= 2\,\text{cis}\left(\frac{\pi}{3}\right) \\ &= 1+\sqrt{3}\,i && \text{rectangular form} \end{aligned}$$

and

$$\begin{aligned} w_1 &= \sqrt{4}\,\text{cis}\left(\frac{2\pi/3}{2}+\frac{2\pi}{2}\cdot 1\right) && \text{Theorem 6.4 with } k=1 \\ &= 2\,\text{cis}\left(\frac{4\pi}{3}\right) \\ &= 1-\sqrt{3}\,i && \text{rectangular form} \end{aligned}$$

We can check each of these roots by squaring to get $z=-2+2\sqrt{3}\,i$.

□

Example 6.4.4. Find the four fourth roots of $z=-16$. Express your answers in rectangular form.

Solution. To find the fourth roots of $z=-16$, proceeding as above, we write z in its polar form as $z=16\,\text{cis}(\pi)$. With $|z|=16$, $\theta=\pi$, and $n=4$, we get the four fourth roots:

$$\begin{aligned} w_0 &= \sqrt[4]{16}\,\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}\cdot 0\right)=2\,\text{cis}\left(\frac{\pi}{4}\right) \\ w_1 &= \sqrt[4]{16}\,\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}\cdot 1\right)=2\,\text{cis}\left(\frac{3\pi}{4}\right) \\ w_2 &= \sqrt[4]{16}\,\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}\cdot 2\right)=2\,\text{cis}\left(\frac{5\pi}{4}\right) \\ w_3 &= \sqrt[4]{16}\,\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}\cdot 3\right)=2\,\text{cis}\left(\frac{7\pi}{4}\right) \end{aligned}$$

Converting these to rectangular form gives $w_0=\sqrt{2}+\sqrt{2}\,i$, $w_1=-\sqrt{2}+\sqrt{2}\,i$, $w_2=-\sqrt{2}-\sqrt{2}\,i$, and $w_3=\sqrt{2}-\sqrt{2}\,i$.

□

Example 6.4.5. Find the three cube roots of $z=\sqrt{2}+\sqrt{2}\,i$.

Solution. For finding the cube roots of $z=\sqrt{2}+\sqrt{2}\,i$, we have $z=2\,\text{cis}\left(\frac{\pi}{4}\right)$. With $|z|=2$, $\theta=\frac{\pi}{4}$, and $n=3$, our computations yield

$$\begin{aligned} w_0 &= \sqrt[3]{2}\,\text{cis}\left(\frac{\pi}{12}\right) \\ w_1 &= \sqrt[3]{2}\,\text{cis}\left(\frac{9\pi}{12}\right)=\sqrt[3]{2}\,\text{cis}\left(\frac{3\pi}{4}\right) \\ w_2 &= \sqrt[3]{2}\,\text{cis}\left(\frac{17\pi}{12}\right) \end{aligned}$$

If we were to convert these to rectangular form, we would need to use either sum and difference identities or half-angle identities to evaluate w_0 and w_2. Since we are not explicitly told to do so, we leave this as a good, but messy, exercise.

□

Example 6.4.6. Find the five fifth roots of $z = 1$.

Solution. To find the five fifth roots of 1, we write z in the polar form $z = 1\operatorname{cis}(0)$. Then $|z| = 1$, $\theta = 0$, and $n = 5$. Since $\sqrt[5]{1} = 1$, the roots are

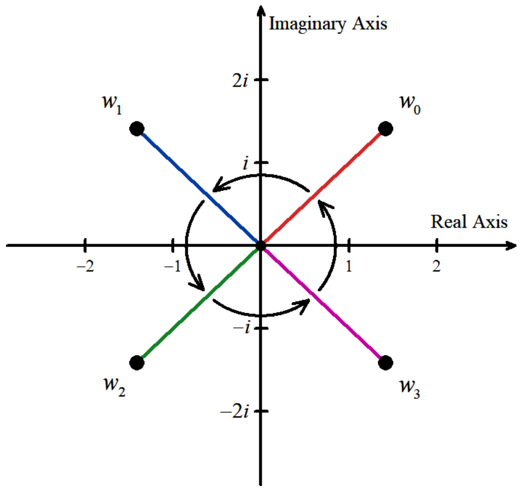
$$\begin{aligned} w_0 &= \operatorname{cis}(0) = 1 \\ w_1 &= \operatorname{cis}\left(\frac{2\pi}{5}\right) \\ w_2 &= \operatorname{cis}\left(\frac{4\pi}{5}\right) \\ w_3 &= \operatorname{cis}\left(\frac{6\pi}{5}\right) \\ w_4 &= \operatorname{cis}\left(\frac{8\pi}{5}\right) \end{aligned}$$

The situation here is even graver than in the previous example, since we have not developed any identities to help us determine the cosine or sine of $\frac{2\pi}{5}$. At this stage, we could approximate our answers using a calculator, and we leave this as an exercise.

□

Now that we have done some computations using **Theorem 6.4**, we take a step back to look at things geometrically. Essentially, **Theorem 6.4** says that to find the n^{th} roots of a complex number, we take the n^{th} root of the modulus and divide the argument by n. This gives the first root w_0. Each successive root is found by adding $\frac{2\pi}{n}$ to the argument, which amounts to rotating w_0 by $\frac{2\pi}{n}$ radians. This results in n roots, spaced equally around the complex plane. As an example of this, the answers to **Example 6.4.4** are plotted below.

Figure 6.4. 6



The four fourth roots of $z = -16$ equally spaced $\frac{2\pi}{4} = \frac{\pi}{2}$ radians around the plane

We have only glimpsed at the beauty of the complex numbers in this section. The complex plane is without a doubt one of the most important mathematical constructs ever devised. Coupled with Calculus, it is the venue for incredibly important science and engineering applications.

6.4 Exercises

In Exercises 1 – 12, use $z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}\,i$ and $w = 3\sqrt{2} - 3\sqrt{2}\,i$ to compute the quantity. Express your answers in polar form using the argument θ, with $0 \le \theta < 2\pi$. These exercises should be worked without the aid of a calculator.

1. zw
2. $\frac{z}{w}$
3. $\frac{w}{z}$
4. z^4
5. w^3
6. $z^5 w^2$
7. $z^3 w^2$
8. $\frac{z^2}{w}$
9. $\frac{w}{z^2}$
10. $\frac{z^3}{w^2}$
11. $\frac{w^2}{z^3}$
12. $\left(\frac{w}{z}\right)^6$

In Exercises 13 – 24, use the power rule (DeMoivre's Theorem) to find the indicated power of the given complex number. Express your final answers in rectangular form. These exercises should be worked without the aid of a calculator.

13. $\left(-2 + 2\sqrt{3}\,i\right)^3$
14. $\left(-\sqrt{3} - i\right)^3$
15. $\left(-3 + 3\,i\right)^4$
16. $\left(\sqrt{3} + i\right)^4$
17. $\left(\frac{5}{2} + \frac{5}{2}\,i\right)^3$
18. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)^6$
19. $\left(\frac{3}{2} - \frac{3}{2}\,i\right)^3$
20. $\left(\frac{\sqrt{3}}{3} - \frac{1}{3}\,i\right)^4$
21. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right)^4$
22. $\left(2 + 2\,i\right)^5$
23. $\left(\sqrt{3} - i\right)^5$
24. $\left(1 - i\right)^8$

In Exercises 25 – 36, find the indicated complex roots. Express your answers in polar form and then convert them into rectangular form. These exercises should be worked without the aid of a calculator.

25. The two square roots of $z = 4\,i$
26. The two square roots of $z = -25\,i$
27. The two square roots of $z = 1 + \sqrt{3}\,i$
28. The two square roots of $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$
29. The three cube (third) roots of $z = 64$
30. The three cube roots of $z = -125$
31. The three cube roots of $z = i$
32. The three cube roots of $z = -8\,i$
33. The four fourth roots of $z = 16$
34. The four fourth roots of $z = -81$
35. The six sixth roots of $z = 64$
36. The six sixth roots of $z = -729$

37. Find the four complex fourth roots of −4. Use them to factor $p(x)=x^4+4$. Multiply appropriate factors to show that $p(x)=(x^2-2x+2)(x^2+2x+2)$ is the factorization of p over the real numbers.

38. Use the 12 complex 12th roots of 4096 to factor $p(x)=x^{12}-4096$ over the real numbers. The result will be a product of linear and irreducible quadratic factors.

39. Given any natural number $n\geq 2$, the complex n^{th} roots of the number $z=1$ are called the **n^{th} Roots of Unity**. In the following exercises, assume that n is a fixed, but arbitrary, natural number such that $n\geq 2$.

a) Show that $w=1$ is an n^{th} root of unity.

b) Show that if both w_j and w_k are n^{th} roots of unity then so is their product $w_j w_k$.

c) Show that if w_j is an n^{th} root of unity then there exists another n^{th} root of unity w_j' such that $w_j w_j'=1$. Hint: If $w_j=\text{cis}(\theta)$ let $w_j'=\text{cis}(2\pi-\theta)$. You'll need to verify that $w_j'=\text{cis}(2\pi-\theta)$ is indeed an n^{th} root of unity.

40. Another way to express the polar form of a complex number is to use the exponential function. For real numbers t, Euler's Formula defines $e^{it}=\cos(t)+i\sin(t)$.

a) Use the definition $e^{it}=\cos(t)+i\sin(t)$ to show that $e^{ix}e^{iy}=e^{i(x+y)}$ for all real numbers x and y.

b) Use the definition $e^{it}=\cos(t)+i\sin(t)$ to show that $\left(e^{ix}\right)^n=e^{i(nx)}$ for any real number x and any natural number n.

c) Use the definition $e^{it}=\cos(t)+i\sin(t)$ to show that $\dfrac{e^{ix}}{e^{iy}}=e^{i(x-y)}$ for all real numbers x and y.

d) If $z=r\ \text{cis}(\theta)$ is the polar form of z, show that $z=re^{it}$ where $\theta=t$ radians.

e) Show that $e^{i\pi}+1=0$. (This famous equation relates the five most important constants in all of Mathematics with the three most fundamental operations in Mathematics.)

f) Show that $\cos(t)=\dfrac{e^{it}+e^{-it}}{2}$ and that $\sin(t)=\dfrac{e^{it}-e^{-it}}{2i}$ for all real numbers t.

CHAPTER 7
VECTORS

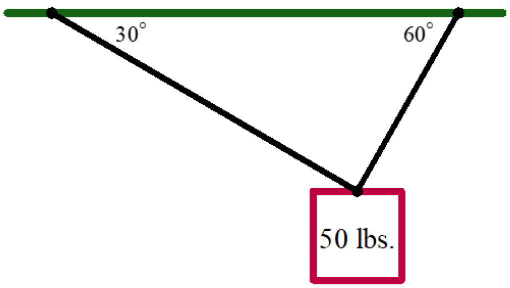


Figure 7.0. 1

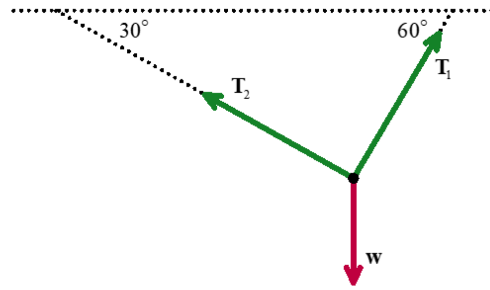


Figure 7.0. 2

Chapter Outline

Introduction

As we have seen numerous times in this book, Trigonometry is used to model and solve real-world problems. In some cases, the solution to a real-world problem is just a number or the size or magnitude of a quantity. However, there are instances when both magnitude and direction are required to describe the answer. To describe such quantities we use vectors. For example, to describe vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ above, we need both their sizes and their directions.

In this chapter, we introduce vectors and many of their applications. We begin in Section 7.1 with the geometric representation of vectors along with vector arithmetic, properties, and applications involving bearings. Section 7.2 continues with applications by focusing on component forms of vectors. The unit vector is introduced, which allows alternate representations of vectors. Also in this section, vectors are used to model forces. The focus of Section 7.3 is the dot product, its properties, and its applications.

7.1 Vector Properties and Operations

Learning Objectives

- **Interpret vectors geometrically.**
- **Write vectors in component form.**
- **Perform vector addition, subtraction, and scalar multiplication.**
- **Determine the magnitude and direction of vectors.**

In using Trigonometry to model and solve real-world problems, the solution may only require a magnitude. For example, when asked how close the nearest Sasquatch nest is, the answer might be three miles. There are other instances, however, when more information is required. For example, it may be important to know how close the nearest Sasquatch nest is as well as the direction in which it lies. To answer questions like these, which involve both a magnitude and a direction, we use **vectors**.[1]

The Geometry of Vectors

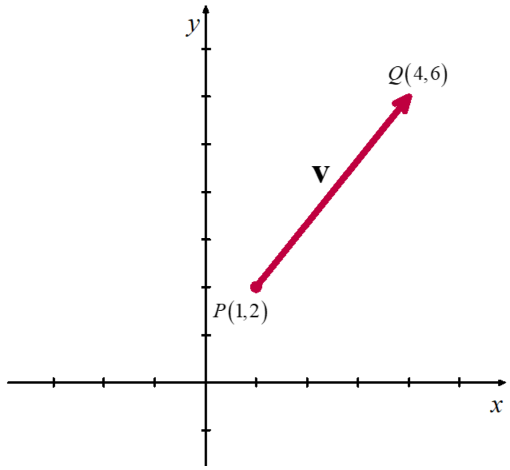
A **vector** is represented geometrically as a directed line segment where the **magnitude** of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrowhead at one endpoint of the segment. A vector has an **initial point**, where it begins, and a **terminal point**, indicated by an arrowhead, where it ends. There are various symbols that distinguish vectors from other quantities:

- Lower case type, boldfaced or with an arrow on top, such as $\mathbf{v}$ or $\vec{v}$.[2]
- Given an initial point P and a terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrow on top indicates its direction. This is different from $\overline{PQ}$, which represents the line segment between P and Q.

The following diagram shows a typical vector $\mathbf{v}$. The point $P(1,2)$ is the initial point of $\mathbf{v}$ and the point $Q(4,6)$ is the terminal point of $\mathbf{v}$. Since $\mathbf{v}$ is a vector from the point P to the point Q, we write $\mathbf{v} = \overrightarrow{PQ}$, where the order of points P (the initial point) and Q (the terminal point) is important. (Think about this before moving on.)

[1] The word 'vector' comes from the Latin 'vehere', meaning to convey or carry.
[2] In this textbook, we will usually adopt the boldfaced type notation for vectors, without the arrow. In the classroom, your instructor will likely use arrow notation, and arrow notation should be used whenever vectors are written by hand.

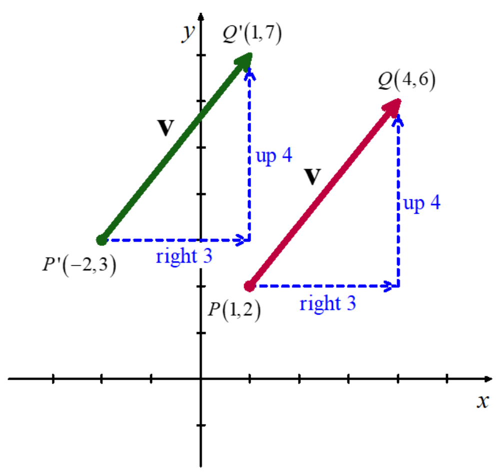
Figure 7.1. 1



$\mathbf{v} = \overrightarrow{PQ}$

While it is true that P and Q completely determine $\mathbf{v}$, it is important to note that since vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\mathbf{v}$ is considered to be the same vector as $\mathbf{v}$, regardless of its initial point. In the case of our vector $\mathbf{v}$ above, any vector that moves three units to the right and four units up from its initial point to arrive at its terminal point is considered the same vector as $\mathbf{v}$. For example, the vector from the initial point $P'(-2,3)$ to the terminal point $Q'(1,7)$ is the same vector $\mathbf{v}$, as shown below.

Figure 7.1. 2



The Component Form of a Vector

The notation we use to capture that the vector $\mathbf{v}$ moves three units to the right and four units up is $\mathbf{v} = \langle 3,4 \rangle$. This is called the **component form** of the vector $\mathbf{v}$, where the first number, 3, is the **horizontal component** (or the *x*-component) of $\mathbf{v}$ and the second number, 4, is the **vertical component** (or the *y*-component) of $\mathbf{v}$. If we wanted to reconstruct $\mathbf{v} = \langle 3,4 \rangle$ with initial point $P'(-2,3)$ then we would find the terminal point of $\mathbf{v}$ by adding 3 to the *x*-coordinate and adding 4 to the *y*-coordinate to obtain the terminal point $Q'(1,7)$, as demonstrated in the previous figure. The component form of a vector ties these geometric objects back to algebra, and ultimately trigonometry. A definition follows.

Definition 7.1. Suppose a vector $\mathbf{v}$ is represented by the directed line segment with an initial point $P(x_0, y_0)$ and a terminal point $Q(x_1, y_1)$. Then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle.$$

Example 7.1.1. Consider the vector whose initial point is $P(2,3)$ and whose terminal point is $Q(6,4)$. Write $\mathbf{v} = \overrightarrow{PQ}$ in component form.

Solution. Using the definition of component form, we get

$$\begin{aligned}\mathbf{v} &= \langle 6-2, 4-3 \rangle \\ &= \langle 4,1 \rangle\end{aligned}$$

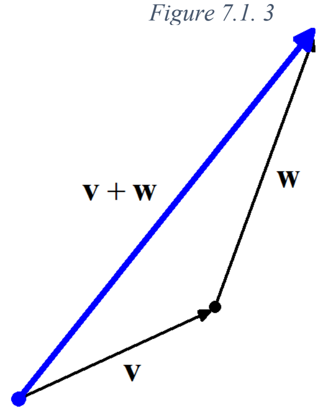
□

Using the language of components, two vectors are **equal** if and only if their corresponding components are equal. That is, $\langle v_1, v_2 \rangle = \langle v_3, v_4 \rangle$ if and only if $v_1 = v_3$ and $v_2 = v_4$. (Think about this before reading on.) We now define operations on vectors.

Vector Addition

Suppose we are given two vectors $\mathbf{v}$ and $\mathbf{w}$. The sum, or **resultant vector**, $\mathbf{v} + \mathbf{w}$ is obtained geometrically as follows. First, plot $\mathbf{v}$. Next, plot $\mathbf{w}$ so that its initial point is the terminal point of $\mathbf{v}$. To plot the vector $\mathbf{v} + \mathbf{w}$, we begin at the initial point of $\mathbf{v}$ and end at the terminal point of $\mathbf{w}$. It is helpful to think of the vector $\mathbf{v} + \mathbf{w}$ as the 'net result' of moving along $\mathbf{v}$ and then moving along $\mathbf{w}$.

Figure 7.1. 3



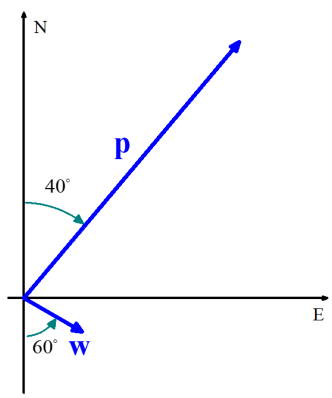
$\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$

Our next example demonstrates the use of resultant vectors to solve real-world problems, while reviewing bearings and the laws of sines and cosines.

Example 7.1.2. A plane leaves the local airport with an airspeed of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

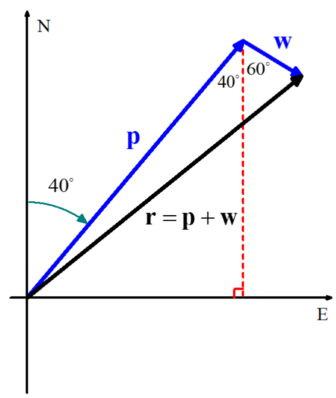
Solution. For both the plane and the wind, we are given their speeds and directions. Coupling speed (as a magnitude) with direction is the concept of velocity, which we have seen a few times before in this textbook. We let $\mathbf{p}$ denote the plane's velocity and $\mathbf{w}$ denote the wind's velocity in the diagrams below. The true speed and bearing are found by analyzing the resultant vector, $\mathbf{p}+\mathbf{w}$, which we refer to as $\mathbf{r}$.

Figure 7.1. 4



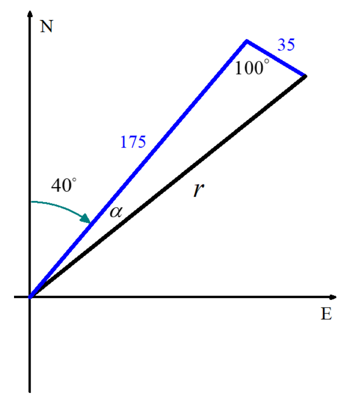
Vectors $\mathbf{p}$ and $\mathbf{w}$

Figure 7.1. 5



Vector $\mathbf{r}=\mathbf{p}+\mathbf{w}$

Figure 7.1. 6



Magnitudes of vectors

From the vector diagrams, we get a triangle, the lengths of whose sides are the magnitude of $\mathbf{p}$, which is 175 mph, the magnitude of $\mathbf{w}$, which is 35 mph, and the magnitude of the resultant vector $\mathbf{r}=\mathbf{p}+\mathbf{w}$, which we refer to as r. From the given bearing information, we use geometric properties of angles to determine that the angle between the sides of lengths 175 and 35 measures 100°. The Law of Cosines can then be used to determine r.

$$r^2 = 175^2 + 35^2 - 2(175)(35)\cos(100^\circ)$$

$$r = \sqrt{31850 - 12250\cos(100^\circ)} \approx 184.33$$

The true speed of the plane is approximately 184 miles per hour.

To find the true bearing of the plane, we use the Law of Sines to determine the angle α.[3]

[3] Since the angle 100° is obtuse, the Law of Sines can be used without any ambiguity here.

$$\frac{\sin(\alpha)}{35} = \frac{\sin(100^\circ)}{r}$$
$$r\sin(\alpha) = 35\sin(100^\circ)$$
$$\sin(\alpha) = \frac{35\sin(100^\circ)}{r}$$

Using the inverse sine, along with the value of r from our prior calculation, we find $\alpha \approx 10.78^\circ$. Given the geometry of the situation, we add α to the given 40° to find that the true bearing of the plane is approximately N51°E.

□

We next define the addition of vectors component-wise to match the geometry.[4]

Definition 7.2. Vector Addition: Suppose $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$. Then

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

Example 7.1.3. Let $\mathbf{v} = \langle 3,4 \rangle$ and $\mathbf{w} = \overrightarrow{PQ}$, for points $P(-3,7)$ and $Q(-2,5)$. Find $\mathbf{v} + \mathbf{w}$ and interpret this sum geometrically.

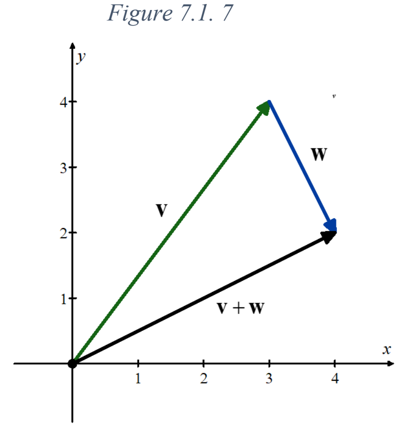
Solution. We begin by writing $\mathbf{w}$ in component form.

$$\begin{aligned} \mathbf{w} &= \overrightarrow{PQ} && \text{for } P(-3,7) \text{ and } Q(-2,5) \\ &= \langle -2-(-3), 5-7 \rangle \\ &= \langle 1,-2 \rangle \end{aligned}$$

Using the definition, $\mathbf{v} + \mathbf{w} = \langle 3,4 \rangle + \langle 1,-2 \rangle = \langle 3+1, 4+(-2) \rangle$, from which $\mathbf{v} + \mathbf{w} = \langle 4,2 \rangle$.

To visualize this sum, we draw $\mathbf{v}$ with its initial point at $(0,0)$, for convenience, so that its terminal point is $(3,4)$. Next, we graph $\mathbf{w}$ with its initial point at $(3,4)$. Moving one unit to the right and two units down, we find the terminal point of $\mathbf{w}$ to be $(4,2)$. Then, the vector $\mathbf{v} + \mathbf{w}$ has initial point $(0,0)$ and terminal point $(4,2)$, so its component form is $\langle 4,2 \rangle$, as required.

Figure 7.1. 7



□

[4] Adding vectors component-wise should look familiar. Compare this with matrix addition. In fact, in more advanced courses such as Linear Algebra, vectors are defined as '1 by n' or 'n by 1' matrices, depending on the situation.

In order for vector addition to have properties similar to real number addition, it is necessary to extend the definition of vectors to include a **zero vector**, $\mathbf{0} = \langle 0, 0 \rangle$. Geometrically, $\mathbf{0}$ represents a point, which can be thought of as a directed line segment having the same initial and terminal points. While it seems clear that the magnitude of $\mathbf{0}$ should be 0, it is not clear what the direction is. The direction of $\mathbf{0}$ is, in fact, undefined. We have the following theorem.

Theorem 7.1. Properties of Vector Addition.

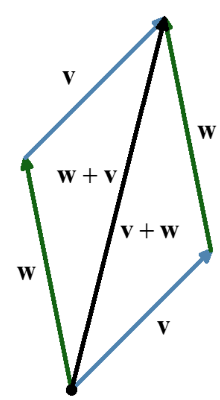
- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- **Associative Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- **Identity Property:** The vector $\mathbf{0}$ acts as the additive identity for vectors. That is, for all vectors $\mathbf{v}$, $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$.
- **Inverse Property:** Every vector $\mathbf{v}$ has a unique additive inverse, denoted $-\mathbf{v}$. That is, for every vector $\mathbf{v}$, there is a vector $-\mathbf{v}$ so that $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$.

The properties in **Theorem 7.1** are easily verified using the definition of vector addition. For the commutative property, we note that if $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$, then

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle \\
&= \langle v_1 + w_1, v_2 + w_2 \rangle && \text{definition of vector addition} \\
&= \langle w_1 + v_1, w_2 + v_2 \rangle && \text{commutative property of real number addition} \\
&= \langle w_1, w_2 \rangle + \langle v_1, v_2 \rangle && \text{definition of vector addition} \\
&= \mathbf{w} + \mathbf{v}
\end{aligned}$$

Geometrically, we can 'see' the commutative property by realizing that the sums $\mathbf{v} + \mathbf{w}$ and $\mathbf{w} + \mathbf{v}$ are the same directed diagonal determined by the parallelogram.

Figure 7.1. 8



The proofs of the associative and identity properties proceed similarly, and the reader is encouraged to verify these properties, and provide accompanying diagrams.

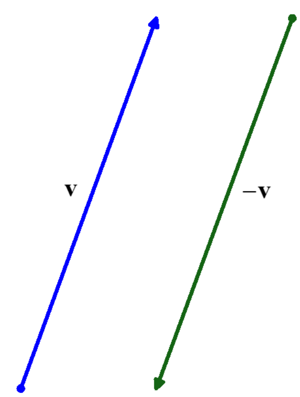
The Additive Inverse

The existence and uniqueness of the additive inverse of a vector is yet another property inherited from the real numbers. Given a vector $\mathbf{v} = \langle v_1, v_2 \rangle$, suppose we wish to find a vector $\mathbf{w} = \langle w_1, w_2 \rangle$ so that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. By the definition of vector addition, we have

$$\mathbf{v} + \mathbf{w} = \mathbf{0}$$
$$\langle v_1 + w_1, v_2 + w_2 \rangle = \langle 0, 0 \rangle$$

Hence, $v_1 + w_1 = 0$ and $v_2 + w_2 = 0$, from which $w_1 = -v_1$ and $w_2 = -v_2$, resulting in $\mathbf{w} = \langle -v_1, -v_2 \rangle$. So $\mathbf{v}$ has an additive inverse; it is unique and can be obtained by the formula $-\mathbf{v} = \langle -v_1, -v_2 \rangle$.

Figure 7.1. 9



Geometrically, the vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ and $-\mathbf{v} = \langle -v_1, -v_2 \rangle$ have the same length but opposite directions. As a result, when adding the vectors geometrically, the sum $\mathbf{v} + (-\mathbf{v})$ starts at the initial point of $\mathbf{v}$ and ends back at the initial point of $\mathbf{v}$. Or, in other words, the net result of moving along $\mathbf{v}$ and then along $-\mathbf{v}$ is returning to the original position.[5]

Using the additive inverse of a vector, we can define **vector subtraction**, or the difference of two vectors, as $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$. If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$, then

$$\begin{aligned} \mathbf{v} - \mathbf{w} &= \mathbf{v} + (-\mathbf{w}) \\ &= \langle v_1, v_2 \rangle + \langle -w_1, -w_2 \rangle \\ &= \langle v_1 + (-w_1), v_2 + (-w_2) \rangle \\ &= \langle v_1 - w_1, v_2 - w_2 \rangle \end{aligned}$$

In other words, like vector addition, vector subtraction is performed component-wise. To interpret the vector $\mathbf{v} - \mathbf{w}$ geometrically, we note

[5] An interesting property of a vector and its additive inverse is that the two vectors are 'parallel'. In fact, we say two non-zero vectors are **parallel** when they have the same or opposite directions. That is, $\mathbf{v}$ is parallel to $\mathbf{w}$ if $\mathbf{v} = k\,\mathbf{w}$ for some non-zero scalar *k*. (Scalar multiplication will be defined shortly.)

$$\begin{aligned} \mathbf{w}+(\mathbf{v}-\mathbf{w}) &= \mathbf{w}+(\mathbf{v}+(-\mathbf{w})) && \text{definition of vector subtraction} \\ &= \mathbf{w}+((-\mathbf{w})+\mathbf{v}) && \text{commutativity of vector addition} \\ &= (\mathbf{w}+(-\mathbf{w}))+\mathbf{v} && \text{associativity of vector addition} \\ &= \mathbf{0}+\mathbf{v} && \text{definition of additive inverse} \\ &= \mathbf{v} && \text{definition of additive identity} \end{aligned}$$

This means that the net result of moving along $\mathbf{w}$, then moving along $\mathbf{v}-\mathbf{w}$, is just $\mathbf{v}$ itself. From the diagram below, to the left, we see that $\mathbf{v}-\mathbf{w}$ may be interpreted as the vector whose initial point is the terminal point of $\mathbf{w}$ and whose terminal point is the terminal point of $\mathbf{v}$. It is also worth mentioning that in the parallelogram, to the right, determined by the vectors $\mathbf{v}$ and $\mathbf{w}$, the vector $\mathbf{v}-\mathbf{w}$ is one of the diagonals, the other being $\mathbf{v}+\mathbf{w}$.[6]

Figure 7.1. 10

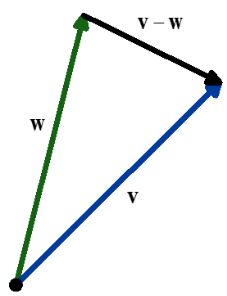
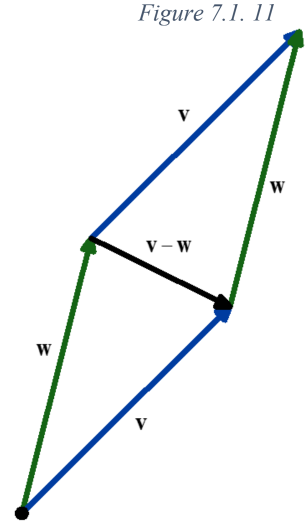


Figure 7.1. 11



Scalar Multiplication

For vectors, **scalar multiplication** is the process of multiplying a vector by a real number. We define scalar multiplication of vectors in the same way we defined it for matrices.

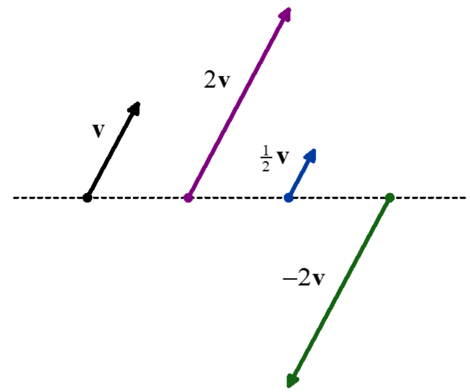
Definition 7.3. Scalar Multiplication: If k is a real number and $\mathbf{v}=\langle v_1, v_2\rangle$, then

$$k\,\mathbf{v} = k\langle v_1, v_2\rangle = \langle k\,v_1, k\,v_2\rangle$$

[6] See **Figure 7.1.8**.

Multiplication of vectors by a scalar k can be understood geometrically as scaling the vector (if $k > 0$) or scaling the vector and reversing its direction (if $k < 0$) as demonstrated to the right.

Figure 7.1. 12



Note that, by definition,

$$\begin{aligned}(-1)\mathbf{v} &= (-1)\langle v_1, v_2\rangle \\ &= \langle (-1)v_1, (-1)v_2\rangle \\ &= \langle -v_1, -v_2\rangle \\ &= -\mathbf{v}\end{aligned}$$

This and other properties of scalar multiplication are summarized below.

Theorem 7.2. Properties of Scalar Multiplication.

- **Associative Property:** For every vector $\mathbf{v}$ and scalars k and r, $(kr)\mathbf{v} = k(r\mathbf{v})$.
- **Identity Property:** For every vector $\mathbf{v}$, $1 \cdot \mathbf{v} = \mathbf{v}$.
- **Additive Inverse Property:** For every vector $\mathbf{v}$, $-\mathbf{v} = (-1)\mathbf{v}$.
- **Distributive Property over Scalar Addition:** For every vector $\mathbf{v}$ and scalars k and r, $(k+r)\mathbf{v} = k\mathbf{v} + r\mathbf{v}$.
- **Distributive Property over Vector Addition:** For all vectors $\mathbf{v}$ and $\mathbf{w}$ and for any scalar k, $k(\mathbf{v}+\mathbf{w}) = k\mathbf{v} + k\mathbf{w}$.
- **Zero Product Property:** If $\mathbf{v}$ is a vector and k is a scalar, then $k\mathbf{v} = \mathbf{0}$ if and only if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

The proof of **Theorem 7.2** ultimately relies on the definition of scalar multiplication and the properties of real numbers. For example, to prove the associative property, we let $\mathbf{v} = \langle v_1, v_2\rangle$. If k and r are scalars, then

$$
\begin{aligned}
(kr)\mathbf{v} &= (kr)\langle v_1, v_2\rangle \\
&= \langle (kr)v_1, (kr)v_2\rangle && \text{definition of scalar multiplication} \\
&= \langle k(rv_1), k(rv_2)\rangle && \text{associative property of real number multiplication} \\
&= k\langle rv_1, rv_2\rangle && \text{definition of scalar multiplication} \\
&= k\left(r\langle v_1, v_2\rangle\right) && \text{definition of scalar multiplication} \\
&= k(r\mathbf{v})
\end{aligned}
$$

The remaining properties are proved similarly and are left as exercises. The next example demonstrates how **Theorem 7.2** allows us to do the same kind of algebraic manipulations with vectors as we do with variables.

Example 7.1.4. Solve $5\mathbf{v} - 2(\mathbf{v} + \langle 1, -2\rangle) = \mathbf{0}$ for $\mathbf{v}$.

Solution.

$$
\begin{aligned}
5\mathbf{v} - 2(\mathbf{v} + \langle 1, -2\rangle) &= \mathbf{0} \\
5\mathbf{v} + (-2)(\mathbf{v} + \langle 1, -2\rangle) &= \mathbf{0} \\
5\mathbf{v} + (-2)\mathbf{v} + (-2)\langle 1, -2\rangle &= \mathbf{0} && \text{distributive property over vector addition} \\
3\mathbf{v} + (-2)\langle 1, -2\rangle &= \mathbf{0} && \text{distributive property over scalar addition} \\
3\mathbf{v} + \langle -2, 4\rangle &= \mathbf{0} && \text{definition of scalar multiplication} \\
3\mathbf{v} + \langle -2, 4\rangle + \langle 2, -4\rangle &= \mathbf{0} + \langle 2, -4\rangle \\
3\mathbf{v} + \langle 0, 0\rangle &= \mathbf{0} + \langle 2, -4\rangle && \text{definition of vector addition} \\
3\mathbf{v} &= \langle 2, -4\rangle && \text{property of additive identity} \\
\left(\frac{1}{3}\right)3\mathbf{v} &= \left(\frac{1}{3}\right)\langle 2, -4\rangle \\
1\cdot\mathbf{v} &= \left\langle \frac{2}{3}, -\frac{4}{3}\right\rangle && \text{associative property, scalar multiplication} \\
\mathbf{v} &= \left\langle \frac{2}{3}, -\frac{4}{3}\right\rangle && \text{property of multiplicative identity}
\end{aligned}
$$

□

Vectors in Standard Position

A vector whose initial point is $(0,0)$ when plotted on the Cartesian coordinate system is said to be in **standard position**.

Definition 7.4. Let $\mathbf{v}$ be a vector in standard position. Let θ be an angle this vector makes with the positive side of the *x*-axis.

- The **magnitude** of $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is the length of the vector $\mathbf{v}$.
- For $\mathbf{v} \neq \mathbf{0}$, θ is a **direction angle** of $\mathbf{v}$.

Note the following:

1. $\|\mathbf{v}\| \geq 0$
2. The direction angle measure is not unique.
3. The direction angle is not defined if $\mathbf{v} = \mathbf{0}$.

If $\mathbf{v} = \langle v_1, v_2 \rangle$ is plotted in standard position, then its terminal point is (v_1, v_2), as shown below.

Figure 7.1. 13

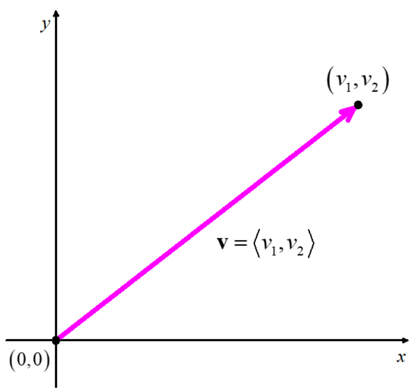
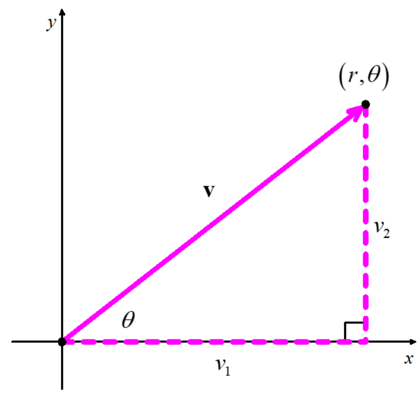


Figure 7.1. 14



$\mathbf{v} = \langle v_1, v_2 \rangle$ in standard position

Let $r = \|\mathbf{v}\|$ and let θ be a direction angle of $\mathbf{v}$. Then (r, θ) are polar coordinates of the point having rectangular coordinates (v_1, v_2). By the Pythagorean Theorem, $r = \sqrt{v_1^2 + v_2^2}$. Using right triangle trigonometry, we also find that

$$v_1 = r\cos(\theta) = \|\mathbf{v}\|\cos(\theta)$$
$$v_2 = r\sin(\theta) = \|\mathbf{v}\|\sin(\theta)$$

From the definition of scalar multiplication and vector equality, we get

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2 \rangle \\ &= \langle \|\mathbf{v}\|\cos(\theta), \|\mathbf{v}\|\sin(\theta) \rangle \\ &= \|\mathbf{v}\| \langle \cos(\theta), \sin(\theta) \rangle \end{aligned}$$

These results are stated in the following theorem, along with properties of magnitude.

Theorem 7.3. Properties of Magnitude and Direction: Suppose $\mathbf{v} = \langle v_1, v_2 \rangle$ is a vector. Let (r, θ) be polar coordinates of the point (v_1, v_2) with $r \geq 0$. Then

- $\|\mathbf{v}\| = r = \sqrt{v_1^{\,2} + v_2^{\,2}}$
- $\mathbf{v} = \|\mathbf{v}\| \langle \cos(\theta), \sin(\theta) \rangle$
- $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- For all scalars k, $\|k\,\mathbf{v}\| = |k| \cdot \|\mathbf{v}\|$.

Although the angle θ in the polar point (r, θ) is not unique, for $\mathbf{v} \neq \langle 0,0 \rangle$, we have $r > 0$ and so all such angles are coterminal and the result $\mathbf{v} = \|\mathbf{v}\| \langle \cos(\theta), \sin(\theta) \rangle$ holds. Additionally, the result holds if $\mathbf{v} = \langle 0,0 \rangle$ since in this case $\|\mathbf{v}\| = 0$.

The proof of the third property in **Theorem 7.3** is a direct consequence of $\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2}}$, which is by definition greater than or equal to zero. Moreover, $\sqrt{v_1^{\,2} + v_2^{\,2}} = 0$ if and only if $v_1^{\,2} + v_2^{\,2} = 0$. Hence, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \langle 0,0 \rangle = \mathbf{0}$, as required.

For the fourth property, if $\mathbf{v} = \langle v_1, v_2 \rangle$ and k is a scalar, then

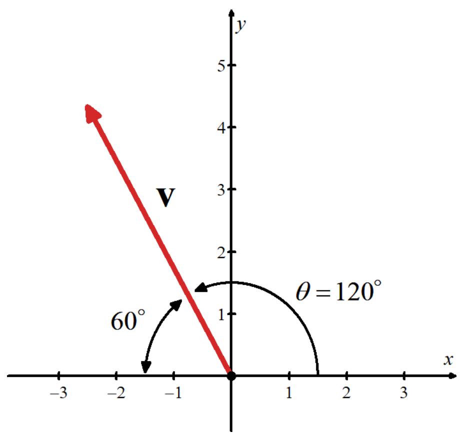
$$\begin{aligned}
\|k\,\mathbf{v}\| &= \|k \langle v_1, v_2 \rangle\| \\
&= \|\langle k\,v_1, k\,v_2 \rangle\| && \text{definition of scalar multiplication} \\
&= \sqrt{(k\,v_1)^2 + (k\,v_2)^2} && \text{definition of magnitude} \\
&= \sqrt{k^2 v_1^{\,2} + k^2 v_2^{\,2}} \\
&= \sqrt{k^2 \left(v_1^{\,2} + v_2^{\,2}\right)} \\
&= \sqrt{k^2}\sqrt{v_1^{\,2} + v_2^{\,2}} && \text{product rule for radicals} \\
&= |k| \sqrt{v_1^{\,2} + v_2^{\,2}} && \text{since } |k| = \sqrt{k^2} \\
&= |k| \cdot \|\mathbf{v}\| && \text{definition of magnitude}
\end{aligned}$$

The equation $\mathbf{v} = \|\mathbf{v}\| \langle \cos(\theta), \sin(\theta) \rangle$ in **Theorem 7.3** says that any given vector is the product of its magnitude and direction, an important concept to keep in mind when studying and using vectors.

Example 7.1.5. Find the component form of the vector $\mathbf{v}$ with $\|\mathbf{v}\| = 5$ so that when $\mathbf{v}$ is plotted in standard position it lies in Quadrant II and makes a 60° angle[7] with the negative *x*-axis.

Solution. We are told that $\|\mathbf{v}\| = 5$ and are given information about its direction, so we can use the formula $\mathbf{v} = \|\mathbf{v}\|\langle\cos(\theta), \sin(\theta)\rangle$ to get the component form of $\mathbf{v}$. To determine θ, since $\mathbf{v}$ lies in Quadrant II and makes a 60° angle with the negative *x*-axis, a polar form of the terminal point of $\mathbf{v}$ is $(5, 120^\circ)$ when $\mathbf{v}$ is plotted in standard position. (See the diagram below.)

Figure 7.1. 15



Thus,

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\langle\cos(120^\circ), \sin(120^\circ)\rangle \\ &= 5\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle \\ &= \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2}\right\rangle\end{aligned}$$

□

Example 7.1.6. For $\mathbf{v} = \langle 3, -3\sqrt{3}\rangle$, find $\|\mathbf{v}\|$ and θ, $0 \le \theta < 2\pi$, so that $\mathbf{v} = \|\mathbf{v}\|\langle\cos(\theta), \sin(\theta)\rangle$.

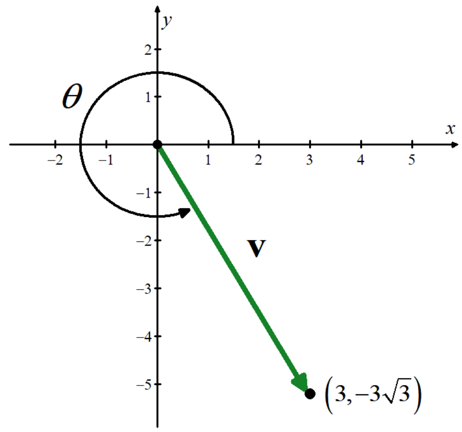
Solution. For $\mathbf{v} = \langle 3, -3\sqrt{3}\rangle$, we get $\|\mathbf{v}\| = \sqrt{(3)^2 + (-3\sqrt{3})^2} = 6$.

[7] Due to the utility of vectors in real-world applications, we will usually use degree measure for the angle when giving the vector's direction.

We can find the θ we are looking for by converting the point with rectangular coordinates $\left(3,-3\sqrt{3}\right)$ to polar form (r,θ), where $r=\|\mathbf{v}\|>0$. From **Section 6.1**, we have

$$\begin{aligned}\tan(\theta)&=\frac{y}{x}\\&=\frac{-3\sqrt{3}}{3}\\&=-\sqrt{3}\end{aligned}$$

Figure 7.1. 16



Since $\left(3,-3\sqrt{3}\right)$ is a point in Quadrant IV, θ is a Quadrant IV angle; we find $\theta=\frac{5\pi}{3}$.

We may check our answer by verifying that $\mathbf{v}=\left\langle 3,-3\sqrt{3}\right\rangle=6\left\langle \cos\left(\frac{5\pi}{3}\right),\sin\left(\frac{5\pi}{3}\right)\right\rangle$.

□

Example 7.1.7. For the vectors $\mathbf{v}=\langle 3, 4\rangle$ and $\mathbf{w}=\langle 1,-2\rangle$, find the following:

1. $\|\mathbf{v}\|-2\|\mathbf{w}\|$
2. $\|\mathbf{v}-2\mathbf{w}\|$

Solution.

1. For $\mathbf{v}=\langle 3, 4\rangle$, we have $\|\mathbf{v}\|=\sqrt{3^2+4^2}=5$. The magnitude of $\mathbf{w}=\langle 1,-2\rangle$ is $\|\mathbf{w}\|=\sqrt{1^2+(-2)^2}=\sqrt{5}$. So, $\|\mathbf{v}\|-2\|\mathbf{w}\|=5-2\sqrt{2}$.
2. In the expression $\|\mathbf{v}-2\mathbf{w}\|$, notice that the arithmetic on the vectors comes first, then the magnitude. Hence, our first step is to find the component form of the vector $\mathbf{v}-2\mathbf{w}$.

$$\begin{aligned}\mathbf{v}-2\mathbf{w}&=\langle 3, 4\rangle-2\langle 1,-2\rangle\\&=\langle 3, 4\rangle+\langle -2, 4\rangle\\&=\langle 1, 8\rangle\end{aligned}$$

Then,

$$\begin{aligned}\|\mathbf{v}-2\mathbf{w}\|&=\|\langle 1, 8\rangle\|\\&=\sqrt{1^2+8^2}\\&=\sqrt{65}\end{aligned}$$

□

7.1 Exercises

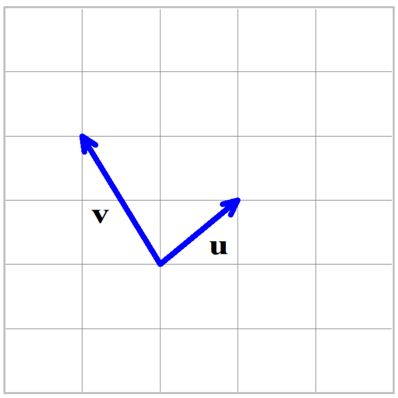
In Exercises 1 – 3, sketch $\mathbf{v}$, $3\mathbf{v}$, and $\frac{1}{2}\mathbf{v}$.

1. $\mathbf{v} = \langle 2, -1 \rangle$
2. $\mathbf{v} = \langle -1, 4 \rangle$
3. $\mathbf{v} = \langle -3, -2 \rangle$

In Exercises 4 – 6, sketch $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{u}$, and $2\mathbf{u}$.

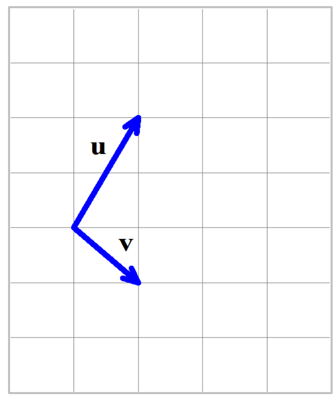
4.

Figure Ex7.1 1



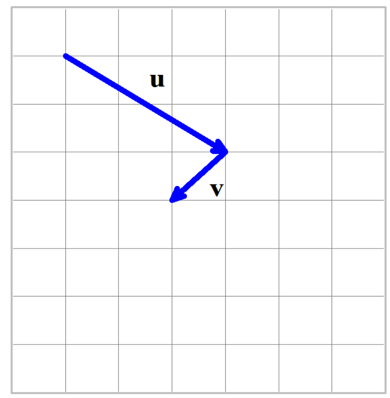
5.

Figure Ex7.1 2



6.

Figure Ex7.1 3



In Exercises 7 – 12, use the given pair of vectors to compute $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $2\mathbf{u}-3\mathbf{v}$.

7. $\mathbf{u} = \langle 2, -3 \rangle$, $\mathbf{v} = \langle 1, 5 \rangle$
8. $\mathbf{u} = \langle -3, 4 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$
9. $\mathbf{u} = \langle -\sqrt{3}, 1 \rangle$, $\mathbf{v} = \langle 2\sqrt{3}, 2 \rangle$
10. $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, $\mathbf{v} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$
11. $\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$, $\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
12. $\mathbf{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, $\mathbf{v} = \langle 1, -\sqrt{3} \rangle$

In Exercises 13 – 16, use the given pair of vectors to find $\mathbf{v}+\mathbf{w}$, $\|\mathbf{v}+\mathbf{w}\|$, and $\|\mathbf{v}\|+\|\mathbf{w}\|$.

13. $\mathbf{v} = \langle 12, -5 \rangle$, $\mathbf{w} = \langle 3, 4 \rangle$
14. $\mathbf{v} = \langle -7, 24 \rangle$, $\mathbf{w} = \langle -5, -12 \rangle$
15. $\mathbf{v} = \langle 2, -1 \rangle$, $\mathbf{w} = \langle -2, 4 \rangle$
16. $\mathbf{v} = \langle 10, 4 \rangle$, $\mathbf{w} = \langle -2, 5 \rangle$

In Exercises 17– 20, verify that the vectors satisfy the Parallelogram Law:

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \frac{1}{2}\left(\|\mathbf{v}+\mathbf{w}\|^2 + \|\mathbf{v}-\mathbf{w}\|^2\right)$$

17. $\mathbf{v} = \langle 12,-5\rangle$, $\mathbf{w} = \langle 3,4\rangle$

18. $\mathbf{v} = \langle -7,24\rangle$, $\mathbf{w} = \langle -5,-12\rangle$

19. $\mathbf{v} = \langle 2,-1\rangle$, $\mathbf{w} = \langle -2,4\rangle$

20. $\mathbf{v} = \langle 10,4\rangle$, $\mathbf{w} = \langle -2,5\rangle$

21. Write the vector with initial point $(5,2)$ and terminal point $(-1,-3)$ in component form $\langle a,b\rangle$.

22. Write the vector with initial point $(-4,2)$ and terminal point $(3,-3)$ in component form $\langle a,b\rangle$.

23. Write the vector with initial point $(7,-1)$ and terminal point $(-1,-7)$ in component form $\langle a,b\rangle$.

In Exercises 24 – 38, find the component form of the vector $\mathbf{v}$ using the information given about its magnitude and direction. Give exact values.

24. $\|\mathbf{v}\| = 6$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant I and makes a 60° angle with the positive *x*-axis.

25. $\|\mathbf{v}\| = 3$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant I and makes a 45° angle with the positive *x*-axis.

26. $\|\mathbf{v}\| = \frac{2}{3}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant I and makes a 60° angle with the positive *x*-axis.

27. $\|\mathbf{v}\| = 12$; when drawn in standard position, $\mathbf{v}$ lies along the positive *y*-axis.

28. $\|\mathbf{v}\| = 4$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant II and makes a 30° angle with the negative *x*-axis.

29. $\|\mathbf{v}\| = 2\sqrt{3}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant II and makes a 30° angle with the positive *y*-axis.

30. $\|\mathbf{v}\| = \frac{7}{2}$; when drawn in standard position, $\mathbf{v}$ lies along the negative *x*-axis.

31. $\|\mathbf{v}\| = 5\sqrt{6}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant III and makes a 45° angle with the negative *x*-axis.

32. $\|\mathbf{v}\| = 6.25$; when drawn in standard position, $\mathbf{v}$ lies along the negative y-axis.

33. $\|\mathbf{v}\| = 4\sqrt{3}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant IV and makes a 30° angle with the positive x-axis.

34. $\|\mathbf{v}\| = 5\sqrt{2}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant IV and makes a 45° angle with the negative y-axis.

35. $\|\mathbf{v}\| = 2\sqrt{5}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant I and makes an angle measuring $\arctan\left(\frac{1}{2}\right)$ with the positive x-axis.

36. $\|\mathbf{v}\| = \sqrt{10}$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant II and makes an angle measuring $\arctan(3)$ with the negative x-axis.

37. $\|\mathbf{v}\| = 5$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant III and makes an angle measuring $\arctan\left(\frac{4}{3}\right)$ with the negative x-axis.

38. $\|\mathbf{v}\| = 26$; when drawn in standard position, $\mathbf{v}$ lies in Quadrant IV and makes an angle measuring $\arctan\left(\frac{5}{12}\right)$ with the positive x-axis.

In Exercises 39 – 44, find the component form of the vector $\mathbf{v}$ using the information given about its magnitude and direction. Round each value to two decimal places.

39. $\|\mathbf{v}\| = 392$; when drawn in standard position, $\mathbf{v}$ makes a 117° angle with the positive x-axis.

40. $\|\mathbf{v}\| = 63.92$; when drawn in standard position, $\mathbf{v}$ makes a 78.3° angle with the positive x-axis.

41. $\|\mathbf{v}\| = 5280$; when drawn in standard position, $\mathbf{v}$ makes a 12° angle with the positive x-axis.

42. $\|\mathbf{v}\| = 450$; when drawn in standard position, $\mathbf{v}$ makes a 210.75° angle with the positive x-axis.

43. $\|\mathbf{v}\| = 168.7$; when drawn in standard position, $\mathbf{v}$ makes a 252° angle with the positive x-axis.

44. $\|\mathbf{v}\| = 26$; when drawn in standard position, $\mathbf{v}$ makes a 304.5° angle with the positive x-axis.

In Exercises 45 – 62, find the magnitude and direction of each vector $\mathbf{v}$. Round your answers to two decimal places.

45. $\mathbf{v} = \langle 1, \sqrt{3} \rangle$
46. $\mathbf{v} = \langle 5, 5 \rangle$
47. $\mathbf{v} = \langle -2\sqrt{3}, 2 \rangle$
48. $\mathbf{v} = \langle -\sqrt{2}, \sqrt{2} \rangle$
49. $\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
50. $\mathbf{v} = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
51. $\mathbf{v} = \langle 6, 0 \rangle$
52. $\mathbf{v} = \langle -2.5, 0 \rangle$
53. $\mathbf{v} = \langle 0, \sqrt{7} \rangle$
54. $\mathbf{v} = \langle 3, 4 \rangle$
55. $\mathbf{v} = \langle 12, 5 \rangle$
56. $\mathbf{v} = \langle -4, 3 \rangle$
57. $\mathbf{v} = \langle -7, 24 \rangle$
58. $\mathbf{v} = \langle -2, -1 \rangle$
59. $\mathbf{v} = \langle -2, -6 \rangle$
60. $\mathbf{v} = \langle 123.4, -77.05 \rangle$
61. $\mathbf{v} = \langle 965.15, 831.6 \rangle$
62. $\mathbf{v} = \langle -114.1, 42.3 \rangle$

63. A small boat leaves the dock at Camp DuNuthin and heads across the Nessie River at 17 miles per hour, relative to the water, at a bearing of S68°W. The river is flowing due east at 8 miles per hour. What is the boat's true speed and bearing? Round the speed to the nearest mile per hour and round the bearing to the nearest tenth of a degree.

64. The HMS Sasquatch leaves port with bearing S20°E maintaining a speed of 42 miles per hour, relative to the water. If the ocean current is 5 miles per hour with a bearing of N60°E, find the HMS Sasquatch's true speed and bearing. Round the speed to the nearest mile per hour and round the bearing to the nearest tenth of a degree.

65. The goal of this exercise is to use vectors to describe non-vertical lines in the plane. To that end, consider the line $y = 2x - 4$. Let $\mathbf{v_0} = \langle 0, -4 \rangle$ and let $\mathbf{s} = \langle 1, 2 \rangle$. Let t be any real number. Show that the vector defined by $\mathbf{v} = \mathbf{v_0} + t\,\mathbf{s}$, when drawn in standard position, has its terminal point on the line $y = 2x - 4$. (Hint: Show that $\mathbf{v} = \mathbf{v_0} + t\,\mathbf{s} = \langle t, 2t - 4 \rangle$ for any real number t.)

 Now consider the non-vertical line $y = mx + b$. Repeat the previous analysis with $\mathbf{v_0} = \langle 0, b \rangle$ and let $\mathbf{s} = \langle 1, m \rangle$. Thus, any non-vertical line can be thought of as a collection of terminal points of the vector sum of $\langle 0, b \rangle$ (the position vector of the y-intercept) and a scalar multiple of the slope vector $\mathbf{s} = \langle 1, m \rangle$.

66. Prove the associative and identity properties of vector addition in **Theorem 7.1**.

67. Prove the properties of scalar multiplication in **Theorem 7.2**.

7.2 The Unit Vector and Vector Applications

Learning Objectives

- **Use vectors in component form to solve application problems.**
- **Find the unit vector in a given direction.**
- **Perform operations on vectors given in terms of principal unit vectors.**
- **Use vectors to model forces.**

Using Vectors in Component Form to Solve Applications

We continue our discussion of the component form of a vector from **Section 7.1** and show how it can be used to solve application problems. This next example revisits **Example 7.1.2**, making use of component forms and vector algebra to solve the problem.

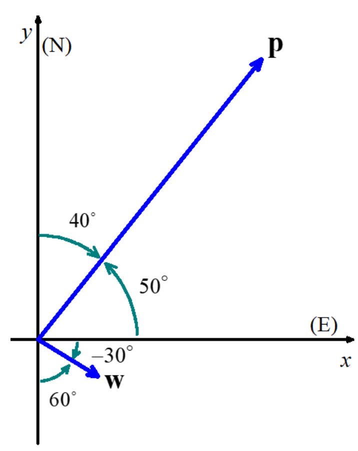
Example 7.2.1. A plane leaves the local airport with an airspeed of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

Solution. As in **Example 7.1.2**, we let $\mathbf{p}$ denote the plane's velocity and $\mathbf{w}$ denote the wind's velocity, and set about determining $\mathbf{p}+\mathbf{w}$. If we regard the airport as being at the origin, the positive *y*-axis acting as due north, and the positive *x*-axis acting as due east, we see that the vectors $\mathbf{p}$ and $\mathbf{w}$ are in standard position and their directions correspond to the angles 50° and -30°, respectively, for component forms:

$$\begin{aligned}\mathbf{p} &= 175\left\langle \cos\left(50^\circ\right), \sin\left(50^\circ\right)\right\rangle \\ &= \left\langle 175\cos\left(50^\circ\right), 175\sin\left(50^\circ\right)\right\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{w} &= 35\left\langle \cos\left(-30^\circ\right), \sin\left(-30^\circ\right)\right\rangle \\ &= \left\langle 35\cos\left(-30^\circ\right), 35\sin\left(-30^\circ\right)\right\rangle\end{aligned}$$

Figure 7.2. 1



Since we have no convenient way to express the exact values of sine and cosine of 50°, we leave both vectors in terms of sines and cosines. Adding corresponding components, the resultant vector is

$$\mathbf{p}+\mathbf{w}=\left\langle 175\cos\left(50^\circ\right)+35\cos\left(-30^\circ\right),\, 175\sin\left(50^\circ\right)+35\sin\left(-30^\circ\right)\right\rangle$$

To find the 'true' speed of the plane, we compute the magnitude of the resultant vector:

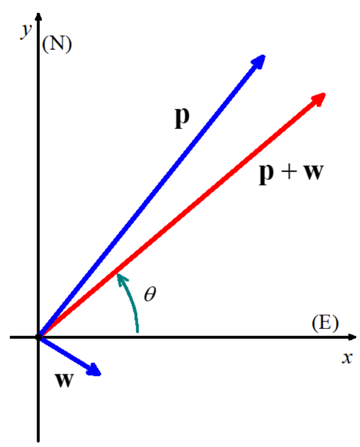
$$\begin{aligned}\left\|\mathbf{p}+\mathbf{w}\right\| &= \sqrt{\left(175\cos\left(50^\circ\right)+35\cos\left(-30^\circ\right)\right)^2+\left(175\sin\left(50^\circ\right)+35\sin\left(-30^\circ\right)\right)^2}\\ &\approx 184.33\end{aligned}$$

Hence, the 'true' speed of the plane is approximately 184 miles per hour. To find the true bearing, we need to find the angle θ that corresponds to the polar form (r,θ), $r>0$, of the point

$$\begin{aligned}(x,y) &= \left(175\cos\left(50^\circ\right)+35\cos\left(-30^\circ\right),\, 175\sin\left(50^\circ\right)+35\sin\left(-30^\circ\right)\right)\\ &\approx (142.799, 116.558)\end{aligned}$$

Since both of these coordinates are positive, we know θ is a Quadrant I angle, as depicted below.

Figure 7.2. 2



Furthermore,

$$\begin{aligned}\tan(\theta) &= \frac{y}{x}=\frac{175\sin\left(50^\circ\right)+35\sin\left(-30^\circ\right)}{175\cos\left(50^\circ\right)+35\cos\left(-30^\circ\right)}\\ &\approx \frac{116.558}{142.799}\end{aligned}$$

Using the arctangent function, we get $\theta \approx 39.223^\circ$. Since, for the purposes of bearing, we need the angle between $\mathbf{p}+\mathbf{w}$ and the positive y-axis, we take the complement of θ and find the 'true' bearing of the plane to be approximately N51°E.

□

The Unit Vector

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as a given vector, but having a magnitude of one. We refer to a vector with a magnitude of one as a **unit vector**.

Definition 7.5. A vector $\mathbf{v}$ is a **unit vector** if $\|\mathbf{v}\| = 1$.

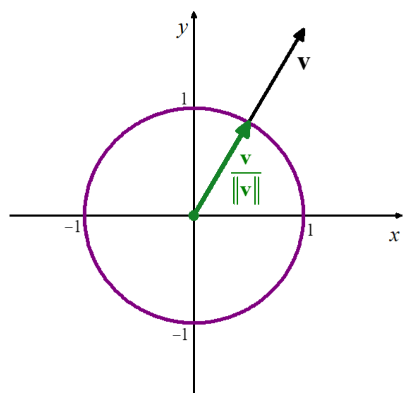
For any nonzero vector $\mathbf{v}$, we can obtain a unit vector in the same direction as $\mathbf{v}$, as follows.

The vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the direction of a vector $\mathbf{v}$. Since dividing a vector by a number is equivalent to multiplying by the reciprocal of that number, $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$.

The process of multiplying a nonzero vector by the reciprocal of its magnitude is called '**normalizing** the vector'. Vectors $\mathbf{v}$ and $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ have the same direction since $\|\mathbf{v}\| > 0$. We leave it as an exercise to show that $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector for any nonzero vector $\mathbf{v}$.

The terminal points of unit vectors, when plotted in standard position, lie on the Unit Circle. (Think about this before moving on.) As a result, we visualize normalizing a nonzero vector in standard position as stretching or compressing the vector so that its terminal point is on the Unit Circle.

Figure 7.2. 3



Example 7.2.2. Find a unit vector in the same direction as $\mathbf{v} = \langle -5, 12 \rangle$.

Solution. We begin by finding the magnitude.

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + (12)^2} = \sqrt{169} = 13$$

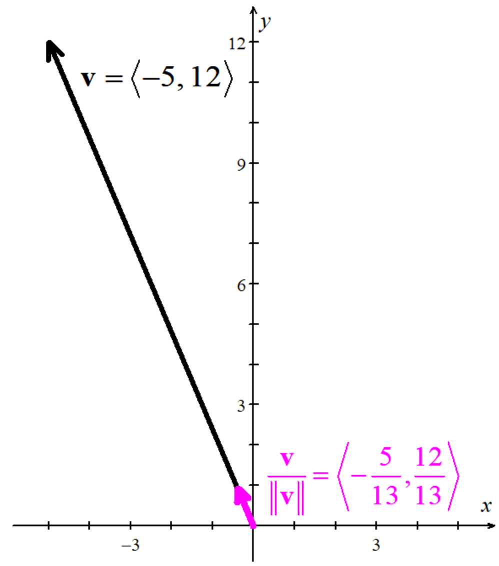
Next, we divide $\mathbf{v}=\langle -5, 12\rangle$ by $\|\mathbf{v}\|=13$.

$$\begin{aligned}\frac{\mathbf{v}}{\|\mathbf{v}\|}&=\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\\&=\frac{1}{13}\langle -5, 12\rangle\\&=\left\langle -\frac{5}{13},\frac{12}{13}\right\rangle\end{aligned}$$

We can check that $\frac{\mathbf{v}}{\|\mathbf{v}\|}=\left\langle -\frac{5}{13},\frac{12}{13}\right\rangle$ is indeed a unit vector by verifying that its magnitude is 1.

Try it!

Figure 7.2. 4



□

Multiplying a unit vector in the direction of a nonzero vector by the magnitude of that vector gives us back the vector: $\frac{\mathbf{v}}{\|\mathbf{v}\|}\cdot\|\mathbf{v}\|=\mathbf{v}$. (Try this with the unit vector we found in **Example 7.2.2**.)

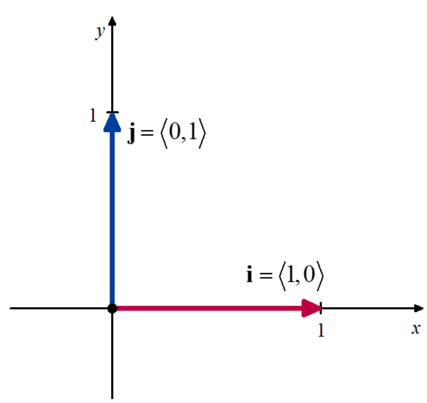
The Principal Unit Vectors

Of all the unit vectors, there are two that deserve special mention.

Definition 7.6. The Principal Unit Vectors in the Plane:

- The vector $\mathbf{i}$ is defined by $\mathbf{i}=\langle 1, 0\rangle$.
- The vector $\mathbf{j}$ is defined by $\mathbf{j}=\langle 0, 1\rangle$.

Figure 7.2. 5



We can think of the vector $\mathbf{i}$ as representing the positive x-direction while the vector $\mathbf{j}$ represents the positive y-direction. We have the following 'decomposition' theorem.

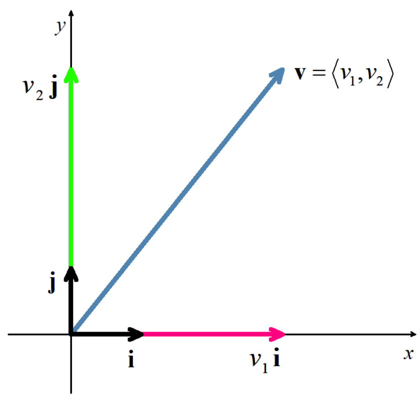
Theorem 7.4. Let $\mathbf{v}$ be a vector with component form $\mathbf{v} = \langle v_1, v_2 \rangle$. Then $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j}$.

The proof of **Theorem 7.4** is straightforward. Since $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$, we use the definitions of scalar multiplication and vector addition to get

$$\begin{aligned} v_1\,\mathbf{i} + v_2\,\mathbf{j} &= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle && \text{definitions of } \mathbf{i} \text{ and } \mathbf{j} \\ &= \langle v_1, 0 \rangle + \langle 0, v_2 \rangle && \text{scalar multiplication} \\ &= \langle v_1, v_2 \rangle && \text{vector addition} \\ &= \mathbf{v} \end{aligned}$$

Geometrically, the situation looks like this:

Figure 7.2. 6



In **Section 7.1**, we found the component form of a vector $\overrightarrow{PQ}$ with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$ to be $\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$. It follows from **Theorem 7.4** that $\overrightarrow{PQ}$ may also be written in terms of $\mathbf{i}$ and $\mathbf{j}$ as $\overrightarrow{PQ} = (x_1 - x_0)\,\mathbf{i} + (y_1 - y_0)\,\mathbf{j}$. An example follows.

Example 7.2.3. Given a vector $\mathbf{v}$ with initial point $P(2, -6)$ and terminal point $Q(-6, 6)$, write the vector in terms of $\mathbf{i}$ and $\mathbf{j}$.

Solution.

$$\begin{aligned} \mathbf{v} &= (-6-2)\mathbf{i} + (6-(-6))\mathbf{j} \\ &= -8\mathbf{i} + 12\mathbf{j} \end{aligned}$$

□

Performing Operations on Vectors in Terms of $\mathbf{i}$ and $\mathbf{j}$

When vectors are written in terms of $\mathbf{i}$ and $\mathbf{j}$, we carry out addition, subtraction, and scalar multiplication by performing operations on principal unit vectors.

Operations on Vectors Written in Terms of $\mathbf{i}$ and $\mathbf{j}$: For vectors $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j}$,

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1)\,\mathbf{i} + (v_2 + w_2)\,\mathbf{j}$
- $\mathbf{v} - \mathbf{w} = (v_1 - w_1)\,\mathbf{i} + (v_2 - w_2)\,\mathbf{j}$
- $k\,\mathbf{v} = (k\,v_1)\mathbf{i} + (k\,v_2)\mathbf{j}$ for any scalar k

These results can be verified using definitions of addition, subtraction, and scalar multiplication from **Section 7.1** along with **Theorem 7.4**, and their verification is left to the student.

Example 7.2.4. Use vectors $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + \mathbf{j}$ to find $3\mathbf{v} + \mathbf{w}$.

Solution.

$$\begin{aligned}
3\,\mathbf{v} + \mathbf{w} &= 3(4\mathbf{i} - 2\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\
&= 3(4\mathbf{i} + (-2)\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\
&= (12\mathbf{i} + (-6)\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\
&= 12\mathbf{i} + (-3)\mathbf{i} + (-6)\mathbf{j} + \mathbf{j} \\
&= (12 + (-3))\mathbf{i} + (-6 + 1)\mathbf{j} \\
&= 9\mathbf{i} - 5\mathbf{j}
\end{aligned}$$

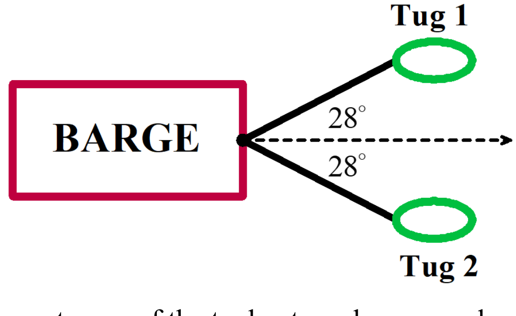
□

Using Vectors to Model Forces

We conclude this section with a couple of examples that demonstrate how vectors are used to model forces. A **force** is defined as a 'push' or a 'pull'. The intensity of the push or pull is the magnitude of the force, and is measured in newtons (N) in the SI system or pounds (lbs.) in the imperial system.

Example 7.2.5. A barge loaded with merchandise is being towed by two tugboats, as shown below.

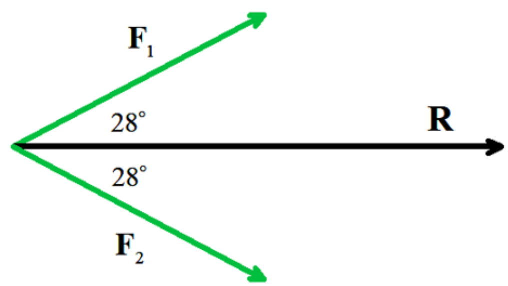
Figure 7.2. 7



Each tow line connecting the barge to one of the tugboats makes an angle of 28° with the path of the barge. Find the tension (magnitude of the 'pulling' force) in the tow lines if the resultant force on the barge has a magnitude of 4,400 pounds.

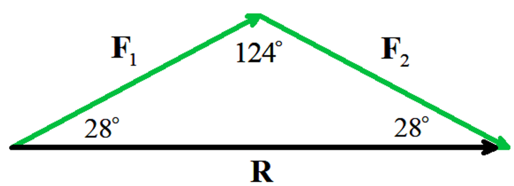
Solution. Using vectors to solve this problem, we let $\mathbf{F}_1$ represent the force exerted by Tug 1 and $\mathbf{F}_2$ represent the force exerted by Tug 2. Then $\mathbf{R}$ is the resultant force vector.

Figure 7.2. 8



Using vector addition, we have $\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{R}$, shown in the following figure.

Figure 7.2. 9



The measure of the angle between vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ is $180^\circ - 28^\circ - 28^\circ = 124^\circ$. We have been given the magnitude of the resultant vector, $\mathbf{R}$, which is 4,400 pounds. Our goal is to find the tensions in the tow lines, or the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$. We can use the Law of Sines here.

$$\frac{\sin\left(28^\circ\right)}{\left\|\mathbf{F}_1\right\|} = \frac{\sin\left(124^\circ\right)}{4400}$$

$$\left\|\mathbf{F}_1\right\|\sin\left(124^\circ\right) = 4400\sin\left(28^\circ\right)$$

$$\left\|\mathbf{F}_1\right\| = \frac{4400\sin\left(28^\circ\right)}{\sin\left(124^\circ\right)}$$

$$\left\|\mathbf{F}_1\right\| \approx 2491.7$$

From properties of isosceles triangles, $\left\|\mathbf{F}_2\right\| \approx 2491.7$, so the tension in each tow line is approximately 2491.7 pounds.

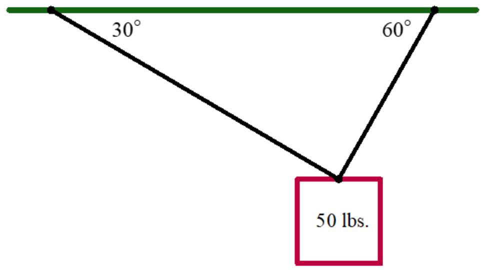
□

The following example should be studied in great detail.

Example 7.2.6. A 50-pound speaker is suspended from the ceiling by two support cables. If one of the cables makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what are the tensions on each of the cables?

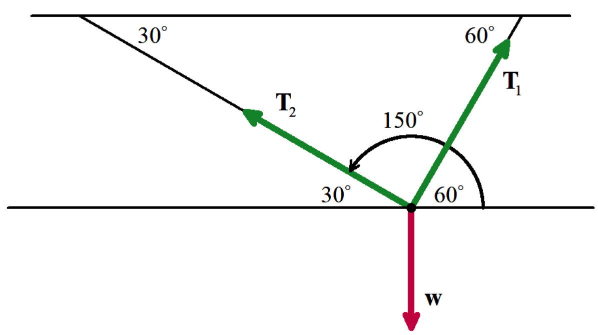
Solution. We first represent the problem schematically.

Figure 7.2. 10



There are three forces working on the speaker; the weight of the speaker, which we will call $\mathbf{w}$, that pulls the speaker directly downward, and the forces on the cables, which we will call $\mathbf{T}_1$ and $\mathbf{T}_2$ (for 'tensions') acting upward at angles 60° and 30°, respectively. Following is a corresponding vector diagram.

Figure 7.2. 11



We are looking for the tensions on the cables, which are the magnitudes $\|\mathbf{T}_1\|$ and $\|\mathbf{T}_2\|$. Since the speaker is stationary,[8] we must have $\mathbf{w}+\mathbf{T}_1+\mathbf{T}_2=\mathbf{0}$. Viewing the common initial point of these vectors as the origin and the horizontal line through this point as the *x*-axis, we find component representations for the three vectors involved.

- We can model the weight of the speaker as a vector pointing directly downward with a magnitude of 50 pounds. That is, $\|\mathbf{w}\|=50$. Since the vector $\mathbf{w}$ is directed strictly downward, $-\mathbf{j}=\langle 0,-1\rangle$ is a unit vector in the direction of $\mathbf{w}$. Hence,

$$\begin{aligned}\mathbf{w}&=50\langle 0,-1\rangle\\&=\langle 0,-50\rangle\end{aligned}$$

- For the force on the first cable, applying **Theorem 7.3**, we get

[8] This is the criteria for 'static equilibrium'.

$$\begin{aligned}\mathbf{T_1} &= \|\mathbf{T_1}\|\langle \cos(60^\circ), \sin(60^\circ)\rangle \\ &= \|\mathbf{T_1}\|\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle \\ &= \left\langle \frac{\|\mathbf{T_1}\|}{2}, \frac{\|\mathbf{T_1}\|\sqrt{3}}{2}\right\rangle\end{aligned}$$

- For the second cable, since the angle 30° is measured from the negative *x*-axis, the angle needed to write $\mathbf{T_2}$ in component form is 150°. Hence,

$$\begin{aligned}\mathbf{T_2} &= \|\mathbf{T_2}\|\langle \cos(150^\circ), \sin(150^\circ)\rangle \\ &= \|\mathbf{T_2}\|\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \\ &= \left\langle -\frac{\|\mathbf{T_2}\|\sqrt{3}}{2}, \frac{\|\mathbf{T_2}\|}{2}\right\rangle\end{aligned}$$

The requirement $\mathbf{w} + \mathbf{T_1} + \mathbf{T_2} = \mathbf{0}$ gives us

$$\langle 0, -50\rangle + \left\langle \frac{\|\mathbf{T_1}\|}{2}, \frac{\|\mathbf{T_1}\|\sqrt{3}}{2}\right\rangle + \left\langle -\frac{\|\mathbf{T_2}\|\sqrt{3}}{2}, \frac{\|\mathbf{T_2}\|}{2}\right\rangle = \langle 0, 0\rangle$$

$$\left\langle \frac{\|\mathbf{T_1}\|}{2} - \frac{\|\mathbf{T_2}\|\sqrt{3}}{2}, \frac{\|\mathbf{T_1}\|\sqrt{3}}{2} + \frac{\|\mathbf{T_2}\|}{2} - 50\right\rangle = \langle 0, 0\rangle$$

Equating the corresponding components of the vectors on each side, we get a system of linear equations in the variables $\|\mathbf{T_1}\|$ and $\|\mathbf{T_2}\|$.

$$\begin{cases} \dfrac{\|\mathbf{T_1}\|}{2} - \dfrac{\|\mathbf{T_2}\|\sqrt{3}}{2} = 0 \\ \dfrac{\|\mathbf{T_1}\|\sqrt{3}}{2} + \dfrac{\|\mathbf{T_2}\|}{2} - 50 = 0 \end{cases}$$

From the first equation, we get $\|\mathbf{T_1}\| = \|\mathbf{T_2}\|\sqrt{3}$. Substituting into the second equation,

$$\begin{aligned}\frac{\left(\|\mathbf{T_2}\|\sqrt{3}\right)\sqrt{3}}{2} + \frac{\|\mathbf{T_2}\|}{2} - 50 &= 0 \\ \frac{3\|\mathbf{T_2}\| + \|\mathbf{T_2}\|}{2} &= 50 \\ \|\mathbf{T_2}\| &= 25\end{aligned}$$

Hence, $\|\mathbf{T_1}\| = \|\mathbf{T_2}\|\sqrt{3} = 25\sqrt{3} \approx 43.3$. The tension on the cable making the 60° angle with the ceiling is approximately 43.3 pounds and the tension on the cable with the 30° angle is 25 pounds.

□

7.2 Exercises

1. Write the vector $\mathbf{v}$ with the initial point $P(-3,1)$ and terminal point $Q(5,2)$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

2. Write the vector $\mathbf{v}$ with the initial point $P(6,0)$ and terminal point $Q(-1,-3)$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

In Exercises 3 – 4, use the vectors $\mathbf{u}=\mathbf{i}+5\mathbf{j}$, $\mathbf{v}=-2\mathbf{i}-3\mathbf{j}$, and $\mathbf{w}=4\mathbf{i}-\mathbf{j}$ to find the following:

3. $\mathbf{u}+(\mathbf{v}-\mathbf{w})$

4. $4\mathbf{v}+2\mathbf{u}$

5. Let $\mathbf{v}=-4\mathbf{i}+3\mathbf{j}$. Find a vector that is half the length and points in the same direction as $\mathbf{v}$.

6. Let $\mathbf{v}=5\mathbf{i}+2\mathbf{j}$. Find a vector that is twice the length and points in the opposite direction of $\mathbf{v}$.

In Exercises 7 – 10, find a unit vector in the same direction as the given vector.

7. $\mathbf{a}=3\mathbf{i}+4\mathbf{j}$

8. $\mathbf{b}=-2\mathbf{i}+5\mathbf{j}$

9. $\mathbf{c}=10\mathbf{i}-\mathbf{j}$

10. $\mathbf{d}=-\frac{1}{2}\mathbf{i}+\frac{2}{3}\mathbf{j}$

In Exercises 11 – 12, use the given pair of vectors to find the following: $\mathbf{v}+\mathbf{w}$, $\|\mathbf{v}+\mathbf{w}\|$, $\|\mathbf{v}\|+\|\mathbf{w}\|$, and $\|\mathbf{w}\|\mathbf{v}$.

11. $\mathbf{v}=3\mathbf{i}+4\mathbf{j}$, $\mathbf{w}=-2\mathbf{j}$

12. $\mathbf{v}=\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$, $\mathbf{w}=\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$

In Exercises 13 – 15, for the given vector $\mathbf{v}$, find the magnitude $\|\mathbf{v}\|$ and an angle θ with $0^\circ \le \theta < 360^\circ$ so that $\mathbf{v}=\|\mathbf{v}\|(\cos(\theta)\mathbf{i}+\sin(\theta)\mathbf{j})$. Round approximations to two decimal places.

13. $\mathbf{v}=-10\mathbf{j}$

14. $\mathbf{v}=\mathbf{i}+\mathbf{j}$

15. $\mathbf{v}=\mathbf{i}-4\mathbf{j}$

16. Let $\mathbf{v}=\langle v_1, v_2\rangle$ be any non-zero vector. Show that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ has length 1.

17. Aida leaves her home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

18. If the captain of the HMS Sasquatch wishes to reach Chupacabra Cove, which is 100 miles away at a bearing of S20°E from port, in three hours, what speed and bearing should she set to take into account an ocean current of 5 miles per hour? (Hint: If $\mathbf{v}$ denotes the velocity of the HMS Sasquatch and $\mathbf{w}$ denotes the velocity of the current, what does $\mathbf{v}+\mathbf{w}$ need to be to reach Chupacabra Cove in three

hours?) Round the speed to the nearest mile per hour and express the bearing rounded to the nearest tenth of a degree.

19. Cliffs of Insanity Point is located 192 miles from the Pedimaxus International Airport at a bearing of N8.2°E. The wind is blowing from the southeast to the northwest at 25 miles per hour. What speed and bearing should the pilot take so that she makes the trip in 2 hours? Round the speed to the nearest hundredth of a mile per hour and your angle to the nearest tenth of a degree.

20. The SS Bigfoot leaves Yeti Bay on a course of N37°W at a speed of 50 miles per hour. After traveling half an hour, the captain determines he is 30 miles from the bay and his bearing back to the bay is S40°E. What is the speed and bearing of the ocean current? Round the speed to the nearest mile per hour and express the bearing rounded to the nearest tenth of a degree.

21. A 600-pound Sasquatch statue is suspended by two cables from a gymnasium ceiling. If each cable makes a 60° angle with the ceiling, find the tension on each cable. Round your answer to the nearest pound.

22. Two cables are to support an object hanging from a ceiling. If the cables are each to make a 42° angle with the ceiling, and each cable is rated to withstand a maximum tension of 100 pounds, what is the heaviest object that can be supported? Round your answer down to the nearest pound.

23. A 300-pound metal star is hanging on two cables that are attached to the ceiling. The left hand cable makes a 72° angle with the ceiling while the right hand cable makes an 18° angle with the ceiling. What is the tension on each of the cables? Round your answers to three decimal places.

24. Two college students have filled a barrel with rocks and tied ropes to it in order to drag it down the street in the middle of the night. The stronger of the two students pulls with a force of 100 pounds at a bearing of N77°E and the other pulls at a bearing of S68°E. What force should the weaker student apply to his rope so that the barrel of rocks heads due east? What resultant force is applied to the barrel? Round your answer to the nearest pound.

25. Emboldened by the success of their late night barrel pull in the previous exercise, our intrepid young scholars have decided to pay homage to the chariot race scene from the movie 'Ben-Hur' by tying three ropes to a couch, loading the couch with all but one of their friends, and pulling it due west down the street. The first rope points N80°W, the second points due west and the third points S80°W. The force applied to the first rope is 100 pounds, the force applied to the second rope is 40 pounds and the force applied (by the non-riding friend) to the third rope is 160 pounds. They need the resultant force to be at least 300 pounds; otherwise, the couch won't move. Does it move? If so, is it heading due west?

7.3 The Dot Product

Learning Objectives

- **Find the dot product of two vectors.**
- **Know and apply properties of the dot product.**
- **Use the dot product to determine the angle between two vectors.**
- **Determine whether two vectors are orthogonal.**
- **Solve application problems using the dot product.**

Thus far in Chapter 7, we have learned how to add and subtract vectors and how to multiply vectors by scalars. In this section, we define a product of vectors.

Definition and Algebraic Properties of the Dot Product

We begin with the following definition.

Definition 7.7. Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors whose component forms are $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$. The **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

Example 7.3.1. Find the dot product of $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle 1, -2 \rangle$.

Solution.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \langle 3, 4 \rangle \cdot \langle 1, -2 \rangle \\ &= (3)(1) + (4)(-2) \\ &= -5 \end{aligned}$$

□

Note that the dot product takes two vectors and produces a scalar. For that reason, the quantity $\mathbf{v} \cdot \mathbf{w}$ is often called the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$.[9] The dot product has the following properties.

[9] The dot product may also be referred to as the **inner product** of two vectors.

Theorem 7.5. Properties of the Dot Product:

- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$.
- **Distributive Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$.
- **Scalar Multiple Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, and scalars k, $(k\,\mathbf{v})\cdot\mathbf{w}=k(\mathbf{v}\cdot\mathbf{w})=\mathbf{v}\cdot(k\,\mathbf{w})$.
- **Magnitude Property:** For all vectors $\mathbf{v}$, $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$.

Like most of the theorems involving vectors, the proof of **Theorem 7.5** amounts to using the definition of the dot product and properties of real number arithmetic. To show the commutative property, for instance, we let $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$. Then

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= \langle v_1,v_2\rangle\cdot\langle w_1,w_2\rangle \\
&= v_1w_1+v_2w_2 && \text{definition of dot product} \\
&= w_1v_1+w_2v_2 && \text{commutativity of real number multiplication} \\
&= \langle w_1,w_2\rangle\cdot\langle v_1,v_2\rangle && \text{definition of dot product} \\
&= \mathbf{w}\cdot\mathbf{v}
\end{aligned}$$

The distributive property is proved similarly and is left as an exercise.

For the scalar multiple property, assume that $\mathbf{v}=\langle v_1,v_2\rangle$, $\mathbf{w}=\langle w_1,w_2\rangle$, and k is a scalar. Then

$$\begin{aligned}
(k\,\mathbf{v})\cdot\mathbf{w} &= \left(k\langle v_1,v_2\rangle\right)\cdot\langle w_1,w_2\rangle \\
&= \langle k\,v_1,k\,v_2\rangle\cdot\langle w_1,w_2\rangle && \text{definition of scalar multiplication} \\
&= (k\,v_1)(w_1)+(k\,v_2)(w_2) && \text{definition of dot product} \\
&= k(v_1w_1)+k(v_2w_2) && \text{associativity of real number multiplication} \\
&= k(v_1w_1+v_2w_2) && \text{distributive law for real numbers} \\
&= k\left(\langle v_1,v_2\rangle\cdot\langle w_1,w_2\rangle\right) && \text{definition of dot product} \\
&= k(\mathbf{v}\cdot\mathbf{w})
\end{aligned}$$

We leave the proof of $k(\mathbf{v}\cdot\mathbf{w})=\mathbf{v}\cdot(k\,\mathbf{w})$ as an exercise.

For the last property, if $\mathbf{v}=\langle v_1,v_2\rangle$ then

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{v} &= \langle v_1,v_2\rangle\cdot\langle v_1,v_2\rangle \\
&= v_1^2+v_2^2 \\
&= \|\mathbf{v}\|^2 && \text{definition of magnitude}
\end{aligned}$$

The following example puts **Theorem 7.5** to good use.

Example 7.3.2. Prove the identity $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$.

Solution. We begin by using **Theorem 7.5** to rewrite $\|\mathbf{v}-\mathbf{w}\|^2$ in terms of the dot product.

$$\begin{aligned}
\|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) && \text{magnitude property}\\
&= (\mathbf{v}+(-\mathbf{w}))\cdot\mathbf{v} + (\mathbf{v}+(-\mathbf{w}))\cdot(-\mathbf{w}) && \text{distributive property}\\
&= \mathbf{v}\cdot(\mathbf{v}+(-\mathbf{w})) + (-\mathbf{w})\cdot(\mathbf{v}+(-\mathbf{w})) && \text{commutative property}\\
&= \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot(-\mathbf{w}) + (-\mathbf{w})\cdot\mathbf{v} + (-\mathbf{w})\cdot(-\mathbf{w}) && \text{distributive property}\\
&= \mathbf{v}\cdot\mathbf{v} + (-1)(\mathbf{v}\cdot\mathbf{w}) + (-1)(\mathbf{v}\cdot\mathbf{w}) + (-1)(-1)(\mathbf{w}\cdot\mathbf{w}) && \text{scalar multiple and commutative properties}\\
&= \mathbf{v}\cdot\mathbf{v} - 2(\mathbf{v}\cdot\mathbf{w}) + \mathbf{w}\cdot\mathbf{w}\\
&= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 && \text{magnitude property}
\end{aligned}$$

Hence, $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$ as required.

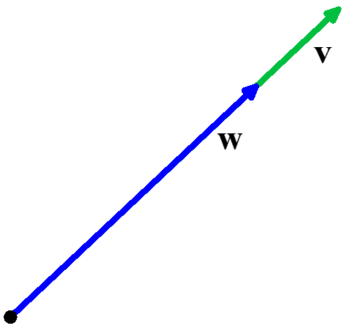
□

Taking a look back at the solution to **Example 7.3.2**, we see that the bulk of the work is needed to show that $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - 2(\mathbf{v}\cdot\mathbf{w}) + \mathbf{w}\cdot\mathbf{w}$. If this looks familiar, it should. Since the dot product possesses many of the same properties as the real numbers, the steps required to expand $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w})$ for vectors $\mathbf{v}$ and $\mathbf{w}$ match those required to expand $(v-w)(v-w)$ for real numbers v and w. Hence, we get similar looking results. The identity verified in **Example 7.3.2** plays a large role in the development of the geometric properties of the dot product, which we now explore.

Geometric Properties of the Dot Product

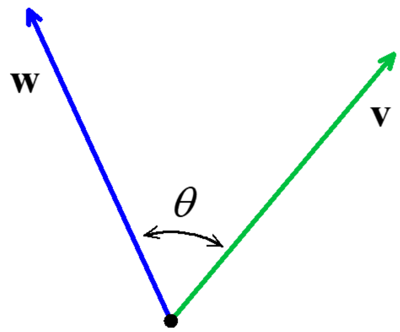
Suppose $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors. If we draw $\mathbf{v}$ and $\mathbf{w}$ with the same initial point, there are two angles determined by the rays containing the vectors $\mathbf{v}$ and $\mathbf{w}$. We define the **angle between $\mathbf{v}$ and $\mathbf{w}$** to be the angle θ, with $0 \le \theta \le \pi$, determined by those rays, as illustrated below.

Figure 7.3. 1



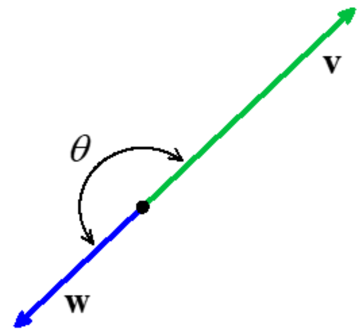
$\theta = 0$

Figure 7.3. 2



$0 < \theta < \pi$

Figure 7.3. 3



$\theta = \pi$

The following theorem gives some insight into the geometric role that the dot product plays.

Theorem 7.6. Geometric Interpretation of the Dot Product: If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, then $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

We prove **Theorem 7.6** in cases.

Case 1: $\theta = 0$

In this case, we need to show that $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(0) = \|\mathbf{v}\|\|\mathbf{w}\|$. If $\theta = 0$, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction, so $\mathbf{w} = k\,\mathbf{v}$ for some positive constant k. Then

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= \mathbf{v}\cdot(k\,\mathbf{v}) \\
&= k(\mathbf{v}\cdot\mathbf{v}) && \text{scalar multiple property} \\
&= k\|\mathbf{v}\|^2 && \text{magnitude property} \\
&= k\|\mathbf{v}\|\|\mathbf{v}\| \\
&= \|\mathbf{v}\|(k\|\mathbf{v}\|) \\
&= \|\mathbf{v}\|(|k|\|\mathbf{v}\|) && \text{since } k > 0 \\
&= \|\mathbf{v}\|\|k\,\mathbf{v}\| && \text{Theorem 7.3} \\
&= \|\mathbf{v}\|\|\mathbf{w}\|
\end{aligned}$$

This proves the formula holds for $\theta = 0$.

Case 2: $\theta = \pi$

In this case, we need to show that $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\pi) = -\|\mathbf{v}\|\|\mathbf{w}\|$. If $\theta = \pi$, then $\mathbf{v}$ and $\mathbf{w}$ have opposite directions. It follows that $\mathbf{w} = k\,\mathbf{v}$ for some negative constant k, and we have

$$\begin{aligned} \mathbf{v}\cdot\mathbf{w} &= \mathbf{v}\cdot(k\,\mathbf{v}) \\ &= k(\mathbf{v}\cdot\mathbf{v}) && \text{scalar multiple property} \\ &= k\|\mathbf{v}\|^2 && \text{magnitude property} \\ &= k\|\mathbf{v}\|\|\mathbf{v}\| \\ &= \|\mathbf{v}\|(k\|\mathbf{v}\|) \\ &= \|\mathbf{v}\|(-|k|\|\mathbf{v}\|) && \text{since } k<0 \\ &= -\|\mathbf{v}\|\|k\,\mathbf{v}\| && \text{Theorem 7.3} \\ &= -\|\mathbf{v}\|\|\mathbf{w}\| \end{aligned}$$

Thus, the formula holds for $\theta = \pi$.

Case 3: $0 < \theta < \pi$

Here, the vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}-\mathbf{w}$ determine a triangle with side lengths $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and $\|\mathbf{v}-\mathbf{w}\|$, respectively, as shown in the following diagrams.

Figure 7.3. 4

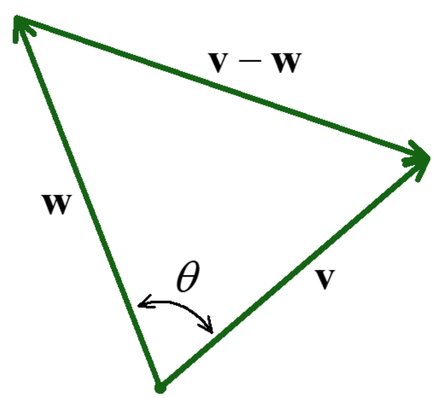
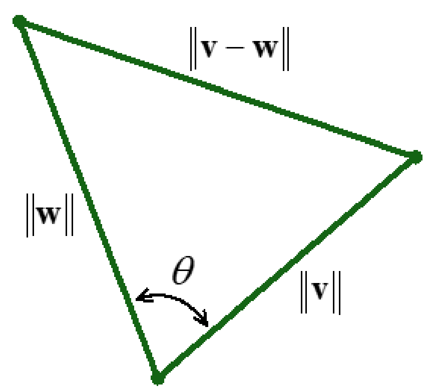


Figure 7.3. 5



The Law of Cosines yields $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$. From **Example 7.3.2**, we know $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$. Equating these two expressions for $\|\mathbf{v}-\mathbf{w}\|^2$ gives

$$\begin{aligned} \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta) &= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \\ -2\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta) &= -2(\mathbf{v}\cdot\mathbf{w}) \\ \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta) &= \mathbf{v}\cdot\mathbf{w} \end{aligned}$$

Thus, $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, as required.

Determining the Angle Between Two Vectors

An immediate consequence of **Theorem 7.6** is the following.

Theorem 7.7. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

The formula in **Theorem 7.7** is obtained by solving the equation given in **Theorem 7.6** for θ. Since $\mathbf{v}$ and $\mathbf{w}$ are nonzero, so are $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$. Hence, we may divide both sides of $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$ by $\|\mathbf{v}\|\|\mathbf{w}\|$ to get

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

Since $0 \le \theta \le \pi$ by definition, the values of θ exactly match the range of the arccosine function. Hence,

$$\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

An example follows.

Example 7.3.3. Find the angle between the following pairs of vectors.

1. $\mathbf{v} = \left\langle 3, -3\sqrt{3}\right\rangle$ and $\mathbf{w} = \left\langle -\sqrt{3}, 1\right\rangle$
2. $\mathbf{v} = \langle 2, 2\rangle$ and $\mathbf{w} = \langle 5, -5\rangle$
3. $\mathbf{v} = \langle 3, -4\rangle$ and $\mathbf{w} = \langle 2, 1\rangle$

Solution.

1. For $\mathbf{v} = \left\langle 3, -3\sqrt{3}\right\rangle$ and $\mathbf{w} = \left\langle -\sqrt{3}, 1\right\rangle$,

$$\mathbf{v}\cdot\mathbf{w} = \left\langle 3, -3\sqrt{3}\right\rangle \cdot \left\langle -\sqrt{3}, 1\right\rangle = -3\sqrt{3} - 3\sqrt{3} = -6\sqrt{3}$$

$$\|\mathbf{v}\| = \sqrt{3^2 + \left(-3\sqrt{3}\right)^2} = \sqrt{36} = 6$$

$$\|\mathbf{w}\| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$$

Then

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)\\ &= \arccos\left(\frac{-6\sqrt{3}}{(6)(2)}\right)\\ &= \arccos\left(-\frac{\sqrt{3}}{2}\right)\\ &= \frac{5\pi}{6}\end{aligned}$$

2. We have $\mathbf{v} = \langle 2,2 \rangle$ and $\mathbf{w} = \langle 5,-5 \rangle$, so that

$$\mathbf{v} \cdot \mathbf{w} = \langle 2,2 \rangle \cdot \langle 5,-5 \rangle = 10 - 10 = 0$$

It follows that

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \\ &= \arccos\left(\frac{0}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \\ &= \arccos(0) && \mathbf{v} \neq \mathbf{0} \text{ and } \mathbf{w} \neq \mathbf{0}\text{, so } \|\mathbf{v}\| \neq 0 \text{ and } \|\mathbf{w}\| \neq 0 \\ &= \frac{\pi}{2}\end{aligned}$$

3. We find, for $\mathbf{v} = \langle 3,-4 \rangle$ and $\mathbf{w} = \langle 2,1 \rangle$,

$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= \langle 3,-4 \rangle \cdot \langle 2,1 \rangle = 6 - 4 = 2 \\ \|\mathbf{v}\| &= \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5 \\ \|\mathbf{w}\| &= \sqrt{2^2 + 1^2} = \sqrt{5}\end{aligned}$$

So,

$$\begin{aligned}\arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) &= \arccos\left(\frac{2}{5\sqrt{5}}\right) \\ &= \arccos\left(\frac{2\sqrt{5}}{25}\right)\end{aligned}$$

Since $\frac{2\sqrt{5}}{25}$ is not the cosine of one of the standard angles, we leave the answer as

$\theta = \arccos\left(\frac{2\sqrt{5}}{25}\right)$.

□

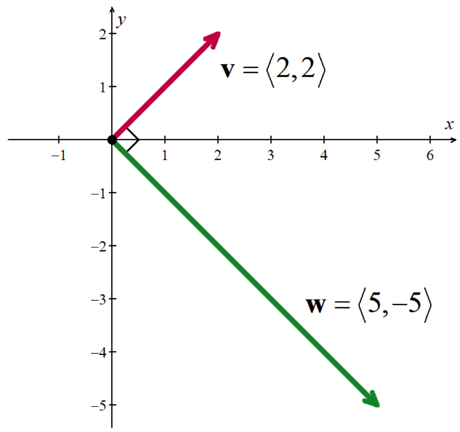
Orthogonal Vectors

We begin with a definition.

Definition 7.8. Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are called **orthogonal** and denoted as $\mathbf{v} \perp \mathbf{w}$ if the angle between them is $\frac{\pi}{2}$ radians or 90°.

In **Example 7.3.3**, we found that the angle between the vectors $\mathbf{v} = \langle 2,2 \rangle$ and $\mathbf{w} = \langle 5,-5 \rangle$ is $\frac{\pi}{2}$, verifying that these two vectors are orthogonal. Geometrically, when orthogonal vectors are sketched with the same initial point, the lines containing the vectors are perpendicular. The vectors $\mathbf{v} = \langle 2,2 \rangle$ and $\mathbf{w} = \langle 5,-5 \rangle$ are sketched below.

Figure 7.3. 6



The relationship between orthogonal vectors and their dot product follows.

Theorem 7.8. The Dot Product Detects Orthogonality: Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. Then $\mathbf{v} \perp \mathbf{w}$ if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

A proof of **Theorem 7.8** follows.

- Assume $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors with $\mathbf{v} \perp \mathbf{w}$. By definition, the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{2}$. Then, from **Theorem 7.6**, $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\left(\frac{\pi}{2}\right) = 0$.
- Conversely, if $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors and $\mathbf{v} \cdot \mathbf{w} = 0$, by **Theorem 7.7**,

$$\begin{aligned} \theta &= \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \\ &= \arccos\left(\frac{0}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \\ &= \arccos(0) \\ &= \frac{\pi}{2} \end{aligned}$$

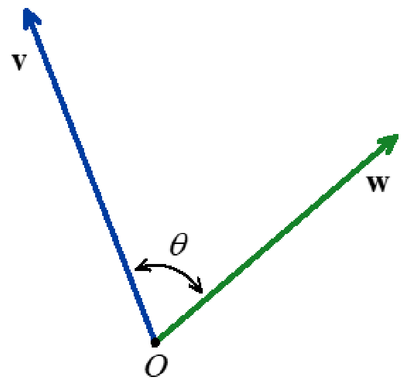
  This verifies that $\mathbf{v} \perp \mathbf{w}$.

While **Theorem 7.8** certainly gives us some insight into what the dot product means geometrically, there is more to the story of the dot product.

Orthogonal Projection

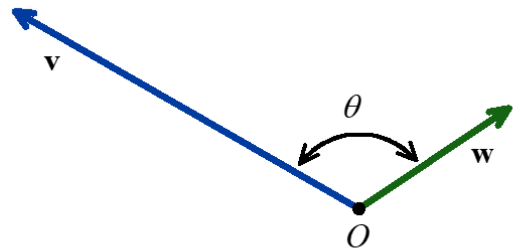
Consider two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ drawn with a common initial point O.

Figure 7.3. 7



Case 1: $0 < \theta < \frac{\pi}{2}$

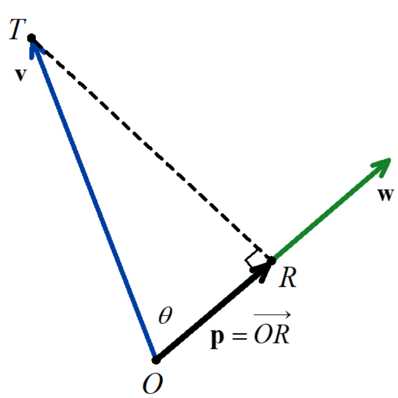
Figure 7.3. 8



Case 2: $\frac{\pi}{2} < \theta < \pi$

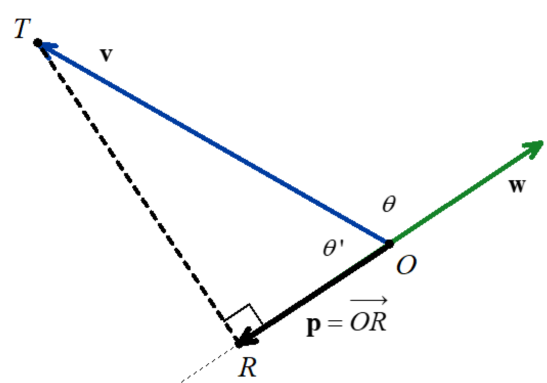
The angle between $\mathbf{v}$ and $\mathbf{w}$, denoted as θ, is shown for the case where $0 < \theta < \frac{\pi}{2}$ and the case where $\frac{\pi}{2} < \theta < \pi$. In each of these cases, to visualize the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$, we drop a perpendicular from the terminal point of $\mathbf{v}$, labeled as T, to the line containing the vector $\mathbf{w}$. The point of intersection of the perpendicular line segment with the line containing $\mathbf{w}$ is labeled as R. The vector $\mathbf{p} = \overrightarrow{OR}$ is called the **orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$.**

Figure 7.3. 9



$0 < \theta < \frac{\pi}{2}$

Figure 7.3. 10

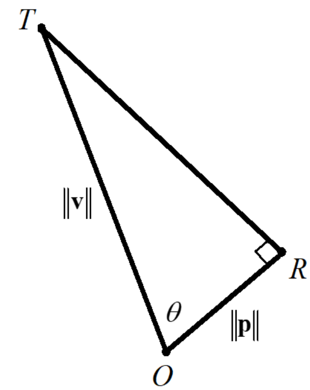


$\frac{\pi}{2} < \theta < \pi$

Like any vector, $\mathbf{p}$ is determined by its magnitude $\|\mathbf{p}\|$ and its direction.

Case 1: $0 < \theta < \frac{\pi}{2}$

Figure 7.3. 11



To determine the magnitude, note that $\cos(\theta) = \frac{\|\mathbf{p}\|}{\|\mathbf{v}\|}$, from which

$$\|\mathbf{p}\| = \|\mathbf{v}\|\cos(\theta)$$

$$\|\mathbf{p}\| = \|\mathbf{v}\|\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \quad \text{from } \mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)\text{, Theorem 7.6}$$

$$\|\mathbf{p}\| = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$$

We determine the direction of $\mathbf{p}$ by finding the unit vector in the direction of $\mathbf{w}$, which is $\frac{\mathbf{w}}{\|\mathbf{w}\|}$.[10] It follows that

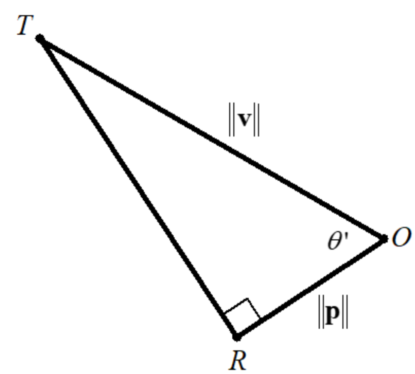
$$\mathbf{p} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}\right)\left(\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) \quad \text{magnitude of } \mathbf{p} \text{ times unit vector in direction of } \mathbf{p}$$

$$= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|\|\mathbf{w}\|}\right)\mathbf{w}$$

$$= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

Case 2: $\frac{\pi}{2} < \theta < \pi$

Figure 7.3. 12



Here, we have $\cos(\theta') = \frac{\|\mathbf{p}\|}{\|\mathbf{v}\|}$, so that

$$\|\mathbf{p}\| = \|\mathbf{v}\|\cos(\theta')$$

$$= \|\mathbf{v}\|\cos(\pi - \theta) \quad \text{since } \theta + \theta' = \pi\text{, Figure 7.3.10}$$

$$= \|\mathbf{v}\|(-\cos(\theta)) \quad \text{difference identity for cosine}$$

$$= -\|\mathbf{v}\|\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \quad \text{from } \mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$$

$$= -\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$$

[10] See **Figure 7.3.9**.

The unit vector in the direction of $\mathbf{p}$ is the unit vector in the opposite direction of $\mathbf{w}$, which is $-\frac{\mathbf{w}}{\|\mathbf{w}\|}$.[11] We find

$$\begin{aligned}\mathbf{p} &= \left(-\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}\right)\left(-\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) && \text{magnitude times direction}\\ &= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}\end{aligned}$$

Note that we have the same formula, $\mathbf{p}=\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$, for both cases. Furthermore, this formula also holds for $\theta=0$, $\theta=\frac{\pi}{2}$, or $\theta=\pi$.

- For $\theta=0$, graphically $\mathbf{p}=\mathbf{v}$ and we observe that

$$\begin{aligned}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} &= \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos(0)}{\|\mathbf{w}\|^2}\mathbf{w}\\ &= \|\mathbf{v}\|\left(\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) && \cos(0)=1\\ &= \mathbf{v} && \mathbf{v}\text{ has same direction as }\mathbf{w}\end{aligned}$$

- For $\theta=\frac{\pi}{2}$, graphically $\mathbf{p}=\mathbf{0}$ and our formula $\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$ also gives the zero vector since $\mathbf{v}\perp\mathbf{w}$ implies that $\mathbf{v}\cdot\mathbf{w}=0$.
- For $\theta=\pi$, graphically $\mathbf{p}=\mathbf{v}$. We find

$$\begin{aligned}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} &= \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos(\pi)}{\|\mathbf{w}\|^2}\mathbf{w}\\ &= \|\mathbf{v}\|\left(-\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) && \cos(\pi)=-1\\ &= \mathbf{v} && \mathbf{v}\text{ is opposite direction of }\mathbf{w}\end{aligned}$$

Finally, we have the following theorem.

Theorem 7.9. If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, then the **orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$** is

$$\text{proj}_{\mathbf{w}}(\mathbf{v})=\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

[11] See **Figure 7.3.10**.

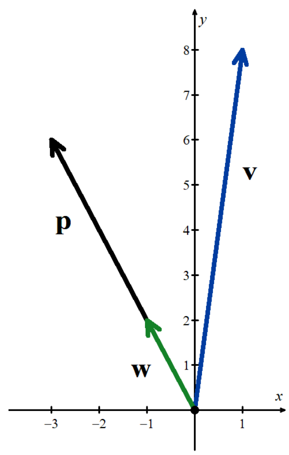
Example 7.3.4. Let $\mathbf{v} = \langle 1,8 \rangle$ and $\mathbf{w} = \langle -1,2 \rangle$. Find $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v})$ and plot $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{p}$ in standard position.

Solution.

$$\begin{aligned} \text{proj}_{\mathbf{w}}(\mathbf{v}) &= \left(\frac{\langle 1,8 \rangle \cdot \langle -1,2 \rangle}{\|\langle -1,2 \rangle\|^2} \right) \langle -1,2 \rangle \qquad \text{since } \text{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \right) \mathbf{w} \\ &= \left(\frac{(1)(-1)+(8)(2)}{\left(\sqrt{(-1)^2+(2)^2}\right)^2} \right) \langle -1,2 \rangle \\ &= \left(\frac{-1+16}{\left(\sqrt{5}\right)^2} \right) \langle -1,2 \rangle \\ &= 3\langle -1,2 \rangle \end{aligned}$$

Hence, $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v}) = \langle -3,6 \rangle$. We next plot the three vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{p}$ in standard position.

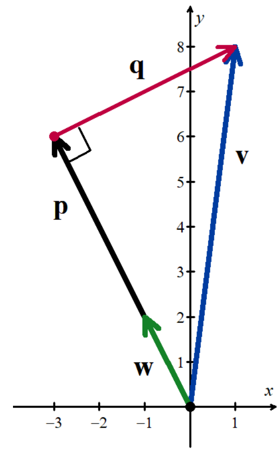
Figure 7.3. 13



□

Suppose we want to verify that our solution, $\mathbf{p} = \langle -3,6 \rangle$, in **Example 7.3.4** is indeed the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$. Since $\mathbf{p} = 3\mathbf{w}$, $\mathbf{p}$ is a scalar multiple of $\mathbf{w}$ and therefore has the correct direction. It remains to check the orthogonality condition. Consider the vector $\mathbf{q}$ whose initial point is the terminal point of $\mathbf{p}$, and whose terminal point is the terminal point of $\mathbf{v}$.

Figure 7.3. 14



From the definition of vector arithmetic, $\mathbf{p}+\mathbf{q}=\mathbf{v}$, so that $\mathbf{q}=\mathbf{v}-\mathbf{p}$. In the case of **Example 7.3.4**, with $\mathbf{v}=\langle 1,8\rangle$ and $\mathbf{p}=\langle -3,6\rangle$, we find $\mathbf{q}=\langle 1,8\rangle-\langle -3,6\rangle=\langle 4,2\rangle$. Then

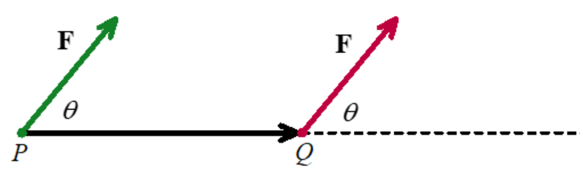
$$\begin{aligned}\mathbf{q}\cdot\mathbf{w}&=\langle 4,2\rangle\cdot\langle -1,2\rangle\\&=(-4)+4\\&=0\end{aligned}$$

This shows that $\mathbf{q}\perp\mathbf{w}$, as required.

Work

We close this section with an application of the dot product. In physics, the work done by a constant scalar force F applied along the direction of motion, when the object moves a distance of d, is $\text{Work}=F\,d$. If the vector force $\mathbf{F}$ applied is not in the direction of the motion, we must consider the magnitude of its component along the direction of motion. Consider the scenario below where the constant vector force $\mathbf{F}$ is applied to an object that moves from the point P to the point Q.

Figure 7.3. 15



We calculate work as follows.

$$\begin{aligned}\text{Work} &= \left\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\right\|\left\|\overrightarrow{PQ}\right\| \\ &= \left\|\left(\frac{\mathbf{F}\cdot\overrightarrow{PQ}}{\left\|\overrightarrow{PQ}\right\|^2}\right)\overrightarrow{PQ}\right\|\left\|\overrightarrow{PQ}\right\| \\ &= \frac{\left|\mathbf{F}\cdot\overrightarrow{PQ}\right|}{\left\|\overrightarrow{PQ}\right\|^2}\left\|\overrightarrow{PQ}\right\|\left\|\overrightarrow{PQ}\right\| \\ &= \left|\mathbf{F}\cdot\overrightarrow{PQ}\right|\end{aligned}$$

The absolute value sign gives the size of the work. However, if the force $\mathbf{F}$ is resisting the movement then its work is negative, compared to the motion. This occurs when $\frac{\pi}{2} < \theta < \pi$, and in this case $\mathbf{F}\cdot\overrightarrow{PQ}$ is negative. Thus, if we allow both positive and negative work, then simply work W is $\mathrm{W} = \mathbf{F}\cdot\overrightarrow{PQ}$.

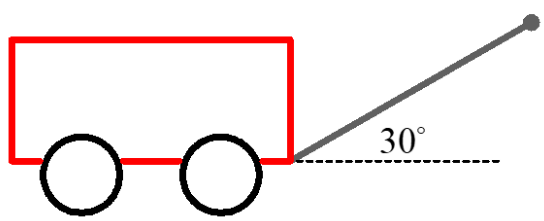
Work as a Dot Product: Suppose a constant force $\mathbf{F}$ is applied to an object moving along a line from point P to point Q. The work W done by $\mathbf{F}$ is given by

$$\mathrm{W} = \mathbf{F}\cdot\overrightarrow{PQ} = \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|\cos(\theta)$$

where θ is the angle between $\mathbf{F}$ and $\overrightarrow{PQ}$.

Example 7.3.5. Taylor pulls her red wagon a distance of 50 feet by exerting a force of 10 pounds along the handle of the wagon, which makes a 30° angle with the horizontal. Find the work done by the force exerted by Taylor.

Figure 7.3. 16



Solution. There are two ways to solve this problem.

- One way is to find the vectors $\mathbf{F}$ and $\overrightarrow{PQ}$ so that we can compute $\mathrm{W} = \mathbf{F}\cdot\overrightarrow{PQ}$. To do this, we assume the origin is at the point where the handle of the wagon meets the wagon and the positive x-axis lies along the dashed line in the figure above. Since the force applied is a constant 10 pounds, we have $\|\mathbf{F}\| = 10$. The force is being applied at a constant angle of $\theta = 30^\circ$ with respect to the positive x-axis, so **Theorem 7.3** gives us

$$\begin{aligned}\mathbf{F} &= 10\left\langle \cos\left(30^\circ\right), \sin\left(30^\circ\right)\right\rangle \\ &= 10\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \\ &= \left\langle 5\sqrt{3}, 5\right\rangle\end{aligned}$$

Since the wagon is being pulled along 50 feet in the positive direction, the displacement vector is

$$\begin{aligned}\overrightarrow{PQ} &= 50\mathbf{i} \\ &= 50\langle 1,0\rangle \\ &= \langle 50,0\rangle\end{aligned}$$

We get

$$\begin{aligned}\mathrm{W} &= \mathbf{F}\cdot\overrightarrow{PQ} \\ &= \left\langle 5\sqrt{3},5\right\rangle\cdot\langle 50,0\rangle \\ &= 250\sqrt{3}\end{aligned}$$

Since force is measured in pounds and distance is measured in feet, $\mathrm{W} = 250\sqrt{3}$ foot-pounds.

- Alternately, we can use the formulation $\mathrm{W} = \|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos(\theta)$ to get

$$\begin{aligned}\mathrm{W} &= (10\text{ pounds})(50\text{ feet})\cos\left(30^\circ\right) \\ &= (500)\left(\frac{\sqrt{3}}{2}\right)\text{ foot-pounds} \\ &= 250\sqrt{3}\text{ foot-pounds}\end{aligned}$$

□

7.3 Exercises

1. Given $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$, calculate $\mathbf{u} \cdot \mathbf{v}$.

2. Given $\mathbf{u} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 5\mathbf{j}$, calculate $\mathbf{u} \cdot \mathbf{v}$.

3. Given $\mathbf{u} = \langle -2, 4 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$, calculate $\mathbf{u} \cdot \mathbf{v}$.

4. Given $\mathbf{u} = \langle -1, 6 \rangle$ and $\mathbf{v} = \langle 6, -1 \rangle$, calculate $\mathbf{u} \cdot \mathbf{v}$.

In Exercises 5 – 24, use the given pair of vectors, $\mathbf{v}$ and $\mathbf{w}$, to answer the following:

(a) Calculate $\mathbf{v} \cdot \mathbf{w}$, $\operatorname{proj}_{\mathbf{w}}(\mathbf{v})$, and the angle θ (in degrees) between $\mathbf{v}$ and $\mathbf{w}$.

(b) Find the vector $\mathbf{q} = \mathbf{v} - \operatorname{proj}_{\mathbf{w}}(\mathbf{v})$. Show that $\mathbf{q}$ and $\mathbf{w}$ are orthogonal by verifying that $\mathbf{q} \cdot \mathbf{w} = 0$.

5. $\mathbf{v} = \langle -2, -7 \rangle$ and $\mathbf{w} = \langle 5, -9 \rangle$

6. $\mathbf{v} = \langle -6, -5 \rangle$ and $\mathbf{w} = \langle 10, -12 \rangle$

7. $\mathbf{v} = \langle 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 1, -\sqrt{3} \rangle$

8. $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle -6, -8 \rangle$

9. $\mathbf{v} = \langle -2, 1 \rangle$ and $\mathbf{w} = \langle 3, 6 \rangle$

10. $\mathbf{v} = \langle -3\sqrt{3}, 3 \rangle$ and $\mathbf{w} = \langle -\sqrt{3}, -1 \rangle$

11. $\mathbf{v} = \langle 1, 17 \rangle$ and $\mathbf{w} = \langle -1, 0 \rangle$

12. $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle 5, 12 \rangle$

13. $\mathbf{v} = \langle -4, -2 \rangle$ and $\mathbf{w} = \langle 1, -5 \rangle$

14. $\mathbf{v} = \langle -5, 6 \rangle$ and $\mathbf{w} = \langle 4, -7 \rangle$

15. $\mathbf{v} = \langle -8, 3 \rangle$ and $\mathbf{w} = \langle 2, 6 \rangle$

16. $\mathbf{v} = \langle 34, -91 \rangle$ and $\mathbf{w} = \langle 0, 1 \rangle$

17. $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 4\mathbf{j}$

18. $\mathbf{v} = -24\mathbf{i} + 7\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i}$

19. $\mathbf{v} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$

20. $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$

21. $\mathbf{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

22. $\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\mathbf{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$

23. $\mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

24. $\mathbf{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\mathbf{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

25. A force of 1500 pounds is required to tow a trailer. Find the work done towing the trailer 300 feet along a flat stretch of road. Assume the force is applied in the direction of the motion.

26. Find the work done in lifting a 10-pound book 3 feet straight up into the air. Assume the force of gravity is acting downward.

27. Suppose Taylor fills her wagon with rocks and must exert a force of 13 pounds to pull her wagon across the yard. If she maintains a 15° angle between the handle of the wagon and the horizontal, compute how much work Taylor does pulling her wagon 25 feet. Round your answer to two decimal places.

28. Two college students have filled a barrel with rocks and attached two ropes so that they can drag it down the street. The stronger of the two students pulls with a force of 100 pounds on the rope that makes a 13° angle with the direction of motion. Find the work done by this student if the barrel is dragged 42 feet. Round your answer to two decimal places. (This scenario was first introduced in the **Section 7.2 Exercises**.)

29. Find the work done in pushing a 200-pound barrel 10 feet along a 12.5° incline. (Hint: Find the angle between the incline and the gravitational force.) Round your answer to two decimal places.

30. Prove the distributive property of the dot product in **Theorem 7.5**.

31. Finish the proof of the scalar multiple property of the dot product in **Theorem 7.5**.

32. Use the identity in **Example 7.3.2**, $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$, to prove the Parallelogram Law, $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \frac{1}{2}\left(\|\mathbf{v}+\mathbf{w}\|^2 + \|\mathbf{v}-\mathbf{w}\|^2\right)$.

33. We know that $|x+y| \le |x| + |y|$ for all real numbers x and y by the Triangle Inequality (from a prior algebra course). We can now establish a Triangle Inequality for vectors. In this exercise, we prove that $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

   a) Show that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$.

   b) Show that $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$, known as the Cauchy-Schwarz Inequality. (Hint: To show this, start with the fact that $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\||\cos(\theta)|$ and use the fact that $|\cos(\theta)| \le 1$ for all θ.)

   c) Justify the following:

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2\end{aligned}$$

d) Use part c to show that $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ for all pairs of vectors $\mathbf{u}$ and $\mathbf{v}$.

e) As an added bonus, we can now show that the Triangle Inequality $|z+w| \le |z| + |w|$ holds for all complex numbers z and w as well. Start by identifying the complex number $z = a + b\,i$ with the vector $\mathbf{u} = \langle a, b \rangle$ and identifying the complex number $w = c + d\,i$ with the vector $\mathbf{v} = \langle c, d \rangle$.

CHAPTER 8
PARAMETRIC EQUATIONS

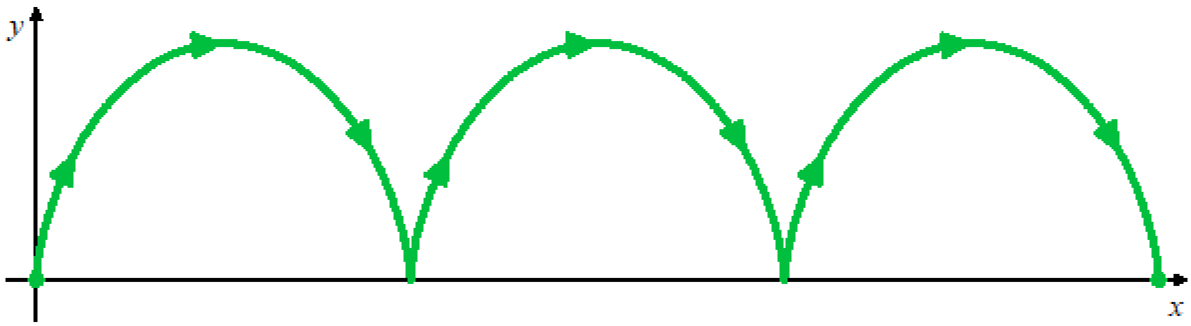


Figure 8.0. 1

Chapter Outline

Introduction

You have seen curves described by equations in Cartesian coordinates and polar coordinates. However, some curves cannot be easily described by a function in either one of these coordinate systems. Here we will use a more general method. In this method, we will describe the coordinates of each point using separate equations called parametric equations. These equations will express each coordinate as a function of a third variable called a parameter.

In Section 8.1, we will graph curves in the plane that are defined by parametric equations. We then get some practice eliminating the parameter in a pair of parametric equations, leading to rewriting the parametric equations for a curve as a single Cartesian equation. In Section 8.2, we will learn to go the opposite direction – rewriting a Cartesian equation as a pair of parametric equations. Finally, we will spend some time altering the parametric equations that define a plane curve to change the orientation and/or starting point.

While this may sound a little overwhelming, know that this is an introduction to these topics. We will discuss a few ideas now, then revisit with a more thorough approach in calculus and physical science courses.

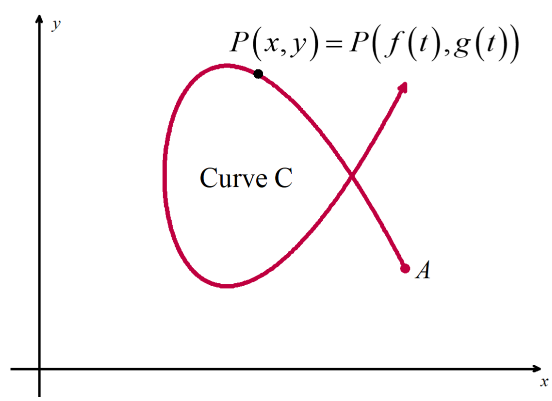
8.1 Sketching Curves Described by Parametric Equations

Learning Objectives

- **Graph plane curves described by parametric equations.**
- **Analyze the behavior of graphs of parametric equations.**
- **Eliminate the parameter in a pair of parametric equations.**

As we have seen, most recently in **Section 6.2**, there are many interesting curves that do not represent traditionally defined functions. In this section, we will represent such curves using two functions. To motivate the idea, imagine a bug crawling across a tabletop starting at point A and tracing out a curve C in the plane, as shown below.

Figure 8.1. 1



The curve C does not represent a function. However, since the bug can be in only one place, $P(x,y)$, at any given time t, we can define the *x*-coordinates and *y*-coordinates of P as functions of t, say $x=f(t)$ and $y=g(t)$.

Plane Curves and Parametric Equations

Definition 8.1. Let f and g be two functions with the same domain. The collection of points $(x,y)=(f(t),g(t))$ in the plane is a **plane curve** and the equations $x=f(t)$ and $y=g(t)$ are called **parametric equations** with the **parameter** t.

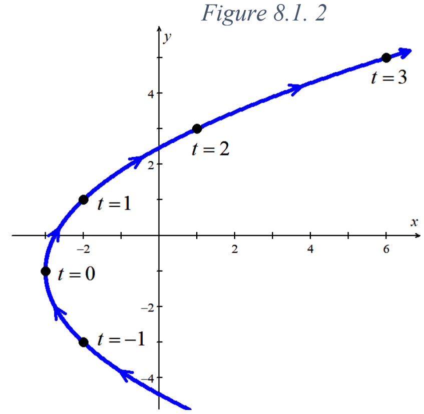
Conversely, if we describe a plane curve C using parametric equations, we say that we have **parametrized** the curve. We refer to the equations as a **parametrization** of the curve C. As we will see later, a parametrization is not unique. A curve can be described by different sets of parametric equations.

Example 8.1.1. Sketch the parametric curve: $\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$

Solution. We follow the same basic procedure as before, when asked to graph anything new. After choosing convenient values for t, we determine the x- and y-values, plot the corresponding points in the plane, and connect the points with a curve. For example, if we choose $t = 0$, we find $x = 0^2 - 3 = -3$ and $y = 2(0) - 1 = -1$. These values, along with others, are included in the following table.

t	$x(t) = t^2 - 3$	$y(t) = 2t - 1$	$(x(t), y(t))$
-1	-2	-3	$(-2,-3)$
0	-3	-1	$(-3,-1)$
1	-2	1	$(-2,1)$
2	1	3	$(1,3)$
3	6	5	$(6,5)$

Figure 8.1. 2



Notice that as t increases, a particle whose position is given by these parametric equations moves along the curve in the direction of the arrows.

□

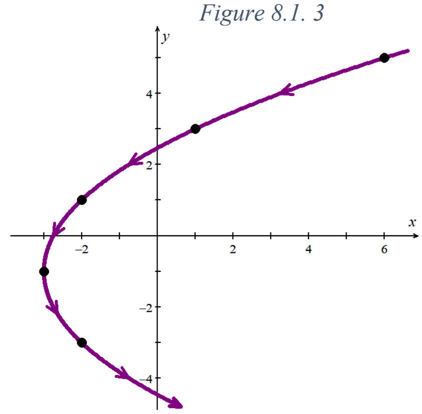
As you can see above, parametric equations not only represent a curve, but also indicate a direction along the curve for increasing values of t. It is important to note, however, that the curve itself is a set of points and as such is devoid of any orientation. The following example is a different parametrization for the curve in **Example 8.1.1**.

Example 8.1.2. Sketch the parametric curve: $\begin{cases} x = t^2 - 3 \\ y = -2t - 1 \end{cases}$

Solution. As in the previous example, we choose values for t to find points on the graph of this curve.

t	$x(t)=t^2-3$	$y(t)=-2t-1$	$(x(t),y(t))$
-3	6	5	$(6,5)$
-2	1	3	$(1,3)$
-1	-2	1	$(-2,1)$
0	-3	-1	$(-3,-1)$
1	-2	-3	$(-2,-3)$

Figure 8.1. 3



□

As you can see, the parametric equations in this example describe the same curve, but the direction corresponding to the increasing values of t is opposite to that in **Example 8.1.1**. The following example is yet another parametrization of the same curve.

Example 8.1.3. Sketch the parametric curve: $\begin{cases} x=\frac{1}{4}t^2-3 \\ y=-t-1 \end{cases}$

Solution. We select values for t and determine the corresponding *x*- and *y*-coordinates of points on the graph.

t	$x(t)=\frac{1}{4}t^2-3$	$y(t)=-t-1$	$(x(t),y(t))$
-6	6	5	$(6,5)$
-4	1	3	$(1,3)$
-2	-2	1	$(-2,1)$
0	-3	-1	$(-3,-1)$
2	-2	-3	$(-2,-3)$

Note that these parametric equations describe the same curve as in **Example 8.1.2** (refer to **Figure 8.1.3**). The difference is in the values for t that were used to determine the points on the graph. In this example, we can say that we move from one point to the next in 'twice the time'.

□

Note that the curve in the previous three examples looks like a parabola. We will use the technique of substitution later in this section to eliminate the parameter t and get an equation involving just x and y.

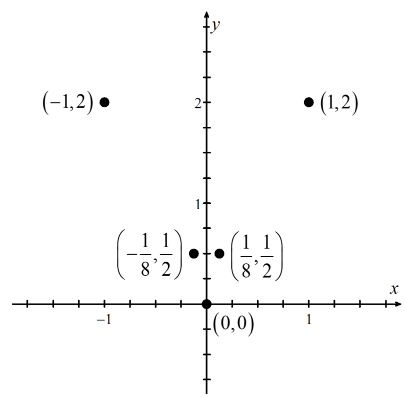
The resulting Cartesian equation will be $(y+1)^2 = 4(x+3)$, which describes a horizontal parabola with vertex $(-3,-1)$.

Example 8.1.4. Sketch the parametric curve: $\begin{cases} x = t^3 \\ y = 2t^2 \end{cases}$ for $-1 \le t \le 1$

Solution. We plot a few points to get a sense of the position and orientation of the curve.

t	$x(t)=t^3$	$y(t)=2t^2$	$(x(t),y(t))$
-1	-1	2	$(-1,2)$
$-\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{2}$	$\left(-\frac{1}{8},\frac{1}{2}\right)$
0	0	0	$(0,0)$
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\left(\frac{1}{8},\frac{1}{2}\right)$
1	1	2	$(1,2)$

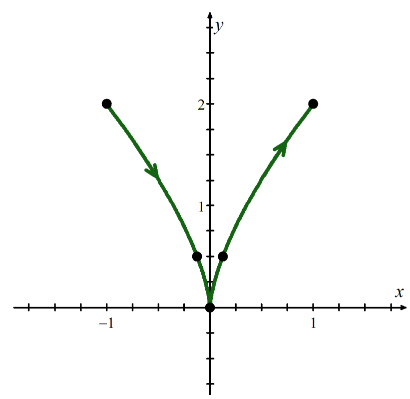
Figure 8.1. 4



To trace out the path described by the parametric equations:

- We start at $(-1,2)$ where $t=-1$, then move to the right (since x is increasing) and down (since y is decreasing) through $\left(-\frac{1}{8},\frac{1}{2}\right)$ to $(0,0)$.
- We continue to move to the right (since x is still increasing) but now move upwards (since y is now increasing) until we reach $(1,2)$, where $t=1$.

Figure 8.1. 5



$\begin{cases} x = t^3 \\ y = 2t^2 \end{cases}$ for $-1 \le t \le 1$

□

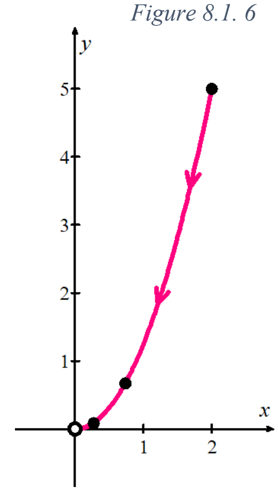
Example 8.1.5. Sketch the parametric curve: $\begin{cases} x = 2e^{-t} \\ y = 5e^{-2t} \end{cases}$ for $t \ge 0$.

Solution. We substitute enough values for t to get a sense of the position, shape, and orientation of the curve. For values of t other than 0, the corresponding values for x and y are approximations, obtained from a calculator.

t	$x(t)=2e^{-t}$	$y(t)=5e^{-2t}$	$(x(t),y(t))$
0	2	5	$(2,5)$
1	0.73576	0.67668	$(0.73576,0.67668)$
2	0.27067	0.09158	$(0.27067,0.09158)$

- Note that as t gets large, the values of both $x(t)$ and $y(t)$ approach 0, but the point $(0,0)$ is not part of the graph. The number 0 is not in the range of either $x(t)$ or $y(t)$.
- Since both $x=2e^{-t}$ and $y=5e^{-2t}$ are decreasing for $t\geq 0$, the graph will start at $(2,5)$, where $t=0$, and move consistently to the left (since x is decreasing) and down (since y is decreasing) to approach the origin.

Figure 8.1. 6



$$\begin{cases} x=2e^{-t} \\ y=5e^{-2t} \end{cases}$$

□

Example 8.1.6. Sketch the parametric curve: $\begin{cases} x=a\cos(t) \\ y=a\sin(t) \end{cases}$ for $a>0$.

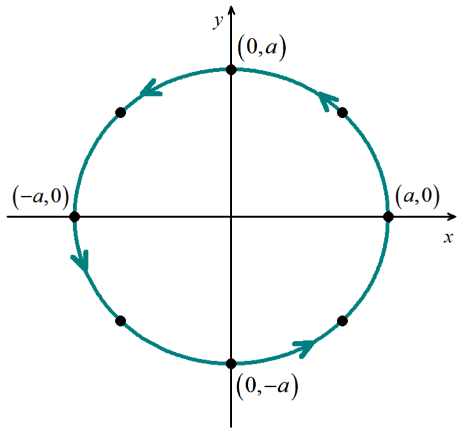
Solution. Since there is no restriction on values for t, we choose to start with $t=-\frac{\pi}{2}$. (Other starting values for t would also work here.) We identify enough points on the graph to envision its behavior. Note that $x(t)=a\cos(t)$ and $y(t)=a\sin(t)$.

t	$(x(t),y(t))$
$-\frac{\pi}{2}$	$(0,-a)$
$-\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}a,-\frac{\sqrt{2}}{2}a\right)$
0	$(a,0)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}a,\frac{\sqrt{2}}{2}a\right)$
$\frac{\pi}{2}$	$(0,a)$

t	$(x(t),y(t))$
$\frac{3\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}a,\frac{\sqrt{2}}{2}a\right)$
π	$(-a,0)$
$\frac{5\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}a,-\frac{\sqrt{2}}{2}a\right)$
$\frac{3\pi}{2}$	$(0,-a)$

Observing that the points (x, y) have begun repeating, we can stop here and plot the points in the table, sketching the curve and adding arrows to indicate direction. The resulting graph is a circle. (Justification for the shape of this graph will be given in **Example 8.1.11**.)

Figure 8.1. 7



$$\begin{cases} x = a\cos(t) \\ y = a\sin(t) \end{cases} \text{ for } a > 0$$

□

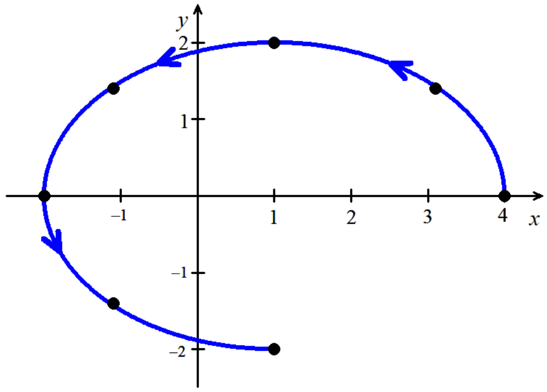
Example 8.1.7. Sketch the parametric curve: $\begin{cases} x = 1 + 3\cos(t) \\ y = 2\sin(t) \end{cases}$ for $0 \le t \le \frac{3\pi}{2}$.

Solution. Plugging in the values $t = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, and $\frac{3\pi}{2}$ gives the following (x, y) coordinates.

t	$(x(t), y(t)) = (1 + 3\cos(t), 2\sin(t))$
0	$(4, 0)$
$\frac{\pi}{4}$	$\left(1 + \frac{3\sqrt{2}}{2}, \sqrt{2}\right) \approx (3.1, 1.4)$
$\frac{\pi}{2}$	$(1, 2)$
$\frac{3\pi}{4}$	$\left(1 - \frac{3\sqrt{2}}{2}, \sqrt{2}\right) \approx (-1.1, 1.4)$
π	$(-2, 0)$
$\frac{5\pi}{4}$	$\left(1 - \frac{3\sqrt{2}}{2}, -\sqrt{2}\right) \approx (-1.1, -1.4)$
$\frac{3\pi}{2}$	$(1, -2)$

Following is the resulting graph.

Figure 8.1. 8



$$\begin{cases} x = 1 + 3\cos(t) \\ y = 2\sin(t) \end{cases} \text{ for } 0 \le t \le \frac{3\pi}{2}$$

□

The graph looks suspiciously like a portion of an ellipse; we will find shortly that this is indeed the case.

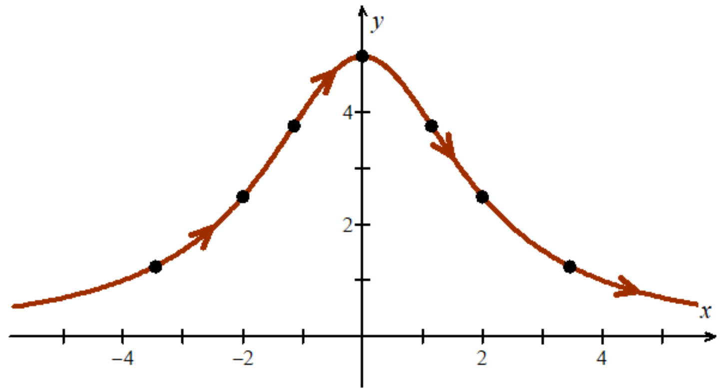
Example 8.1.8. Sketch the parametric curve: $\begin{cases} x = 2\tan(t) \\ y = 5\cos^2(t) \end{cases}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Solution. We select enough values for t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ to get a good grasp on the shape and orientation of the curve.

t	$x(t) = 2\tan(t)$	$y(t) = 5\cos^2(t)$	$(x(t), y(t))$
$-\frac{\pi}{3}$	$-2\sqrt{3}$	$\frac{5}{4}$	$\left(-2\sqrt{3}, \frac{5}{4}\right)$
$-\frac{\pi}{4}$	-2	$\frac{5}{2}$	$\left(-2, \frac{5}{2}\right)$
$-\frac{\pi}{6}$	$-\frac{2}{\sqrt{3}}$	$\frac{15}{4}$	$\left(-\frac{2}{\sqrt{3}}, \frac{15}{4}\right)$
0	0	5	$(0,5)$
$\frac{\pi}{6}$	$\frac{2}{\sqrt{3}}$	$\frac{15}{4}$	$\left(\frac{2}{\sqrt{3}}, \frac{15}{4}\right)$
$\frac{\pi}{4}$	2	$\frac{5}{2}$	$\left(2, \frac{5}{2}\right)$
$\frac{\pi}{3}$	$2\sqrt{3}$	$\frac{5}{4}$	$\left(2\sqrt{3}, \frac{5}{4}\right)$

Note that as t approaches $-\frac{\pi}{2}$, x approaches $-\infty$, and as t approaches $\frac{\pi}{2}$, x approaches ∞. The curve follows.

Figure 8.1. 9



□

This curve is referred to as the 'Witch of Agnesi', named after the Italian mathematician Maria Gaetana Agnesi. The title is a mistranslation of 'averisera' from Italian. It is an interesting curve to research. Other interesting parametric curves include the 'Folium of Descartes' the 'Bicorn', and the 'Astroid'.

Eliminating the Parameter in a Pair of Parametric Equations

Several curves in the previous examples resemble graphs we have seen before. We revisit some of these examples and eliminate the parameter to determine a Cartesian equation. **Eliminating the parameter** t refers to finding a single equation for the curve relating x and y in which t does not appear.

Example 8.1.9. Eliminate the parameter t to determine a Cartesian equation for the curve from **Example 8.1.1**:

$$\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$$

Solution. We use the technique of substitution to eliminate the parameter in the system of equations. The first step is to solve $y = 2t - 1$ for t.

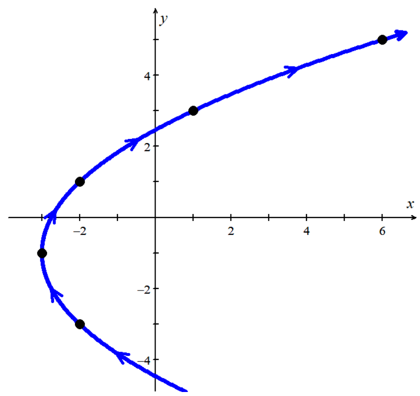
$$\begin{aligned} y &= 2t - 1 \\ y + 1 &= 2t \\ t &= \frac{y+1}{2} \end{aligned}$$

Substituting this result into the equation $x = t^2 - 3$ yields

$$\begin{aligned} x &= \left(\frac{y+1}{2}\right)^2 - 3 \\ x + 3 &= \frac{(y+1)^2}{4} \\ (y+1)^2 &= 4(x+3) \end{aligned}$$

We note that the graph of the equation $(y+1)^2 = 4(x+3)$ is a horizontal parabola with vertex $(-3,-1)$ that opens to the right. While eliminating the parameter does not affect the shape of the curve, the orientation of the curve is lost in the transition to a Cartesian equation. Recall the following graph from **Example 8.1.1**:

Figure 8.1. 10



$$\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$$

□

For practice, try eliminating the parameter in **Example 8.1.2** and **Example 8.1.3**. The resulting Cartesian equation in each of these should be $(y+1)^2 = 4(x+3)$, since the same curve is being sketched in each of the first three examples.

Example 8.1.10. Eliminate the parameter t to determine a Cartesian equation for the curve from **Example 8.1.5**:

$$\begin{cases} x = 2e^{-t} \\ y = 5e^{-2t} \end{cases} \text{ for } t \geq 0$$

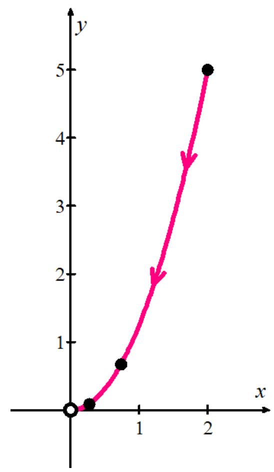
Solution. To eliminate the parameter, one way to proceed is to solve $x = 2e^{-t}$ for t.

$$x = 2e^{-t}$$
$$\frac{x}{2} = e^{-t}$$
$$\ln\left(\frac{x}{2}\right) = \ln e^{-t}$$
$$\ln\left(\frac{x}{2}\right) = -t$$
$$t = -\ln\left(\frac{x}{2}\right)$$

Substituting $t = -\ln\left(\frac{x}{2}\right)$ for t in $y = 5e^{-2t}$ gives

$$\begin{aligned} y &= 5e^{-2\left(-\ln\left(\frac{x}{2}\right)\right)} \\ &= 5e^{2\ln\left(\frac{x}{2}\right)} \\ &= 5e^{\ln\left(\frac{x}{2}\right)^2} \\ &= 5\left(\frac{x}{2}\right)^2 \\ &= \frac{5x^2}{4} \end{aligned}$$

Figure 8.1. 11



$\begin{cases} x = 2e^{-t} \\ y = 5e^{-2t} \end{cases}$ for $t \geq 0$

We get $y = \frac{5}{4}x^2$ as the Cartesian equation for the curve. This is a parabola with vertex $(0,0)$ that opens upward. The parametrized curve is only a portion of this parabola. From the solution to **Example 8.1.5**, we know the parametrized function is defined for $x > 0$ and $y > 0$. Additionally, the restriction $t \geq 0$ requires x to be less than or equal to 2 since $x = 2e^{-t} \leq 2e^0 = 2$, and y to be less than or equal to 5 since, similarly, $y = 5e^{-2t} \leq 5e^0 = 5$. The parametrized curve is the portion of the parabola $y = \frac{5}{4}x^2$ that starts at the point $(2,5)$ and heads toward, but never reaches, $(0,0)$.

□

When eliminating a parameter from a set of parametric equations, it is not always advisable to completely solve for the parameter and then use direct substitution. For example, eliminating the parameter in **Example 8.1.10** could have been achieved in fewer steps as follows:

$$x = 2e^{-t}$$

$$\frac{x}{2} = e^{-t} \qquad \text{Solve for } e^{-t}.$$

$$\left(\frac{x}{2}\right)^2 = \left(e^{-t}\right)^2 \qquad \text{Square both sides of the equation.}$$

$$\frac{x^2}{4} = e^{-2t} \qquad \text{Rewrite in preparation for substituting into } y = 5e^{-2t}.$$

Substituting this result in $y = 5e^{-2t}$, we get $y = 5 \cdot \frac{x^2}{4} = \frac{5x^2}{4}$. The following example uses another technique that works well as an alternate to solving one of the parametric equations for t. This method is particularly useful for eliminating parameters in trigonometric equations.

Example 8.1.11. Eliminate the parameter t to determine a Cartesian equation for the curve from **Example 8.1.6**:

$$\begin{cases} x = a\cos(t) \\ y = a\sin(t) \end{cases} \text{ for } a > 0$$

Solution. To eliminate the parameter, note that the trigonometric functions involved, namely $\cos(t)$ and $\sin(t)$, are related by the Pythagorean identity $\cos^2(t)+\sin^2(t)=1$. After rewriting $x = a\cos(t)$ to get $\cos(t)=\frac{x}{a}$ and $y = a\sin(t)$ to get $\sin(t)=\frac{y}{a}$, we substitute in $\cos^2(t)+\sin^2(t)=1$ with the result $\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2=1$. After simplifying, we have the Cartesian equation $x^2+y^2=a^2$, the equation of a circle with center $(0,0)$ and radius a. This confirms our statement from **Example 8.1.6** that the equation is that of a circle.

□

Example 8.1.12. Eliminate the parameter t to determine a Cartesian equation for the curve from **Example 8.1.7**:

$$\begin{cases} x = 1+3\cos(t) \\ y = 2\sin(t) \end{cases} \text{ for } 0 \le t \le \frac{3\pi}{2}$$

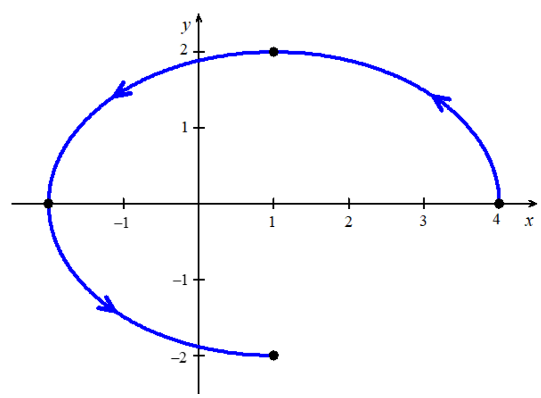
Solution. To eliminate the parameter here, as in the previous example, we make use of the Pythagorean identity.

We solve $x = 1+3\cos(t)$ for $\cos(t)$ to get $\cos(t)=\frac{x-1}{3}$ and $y = 2\sin(t)$ for $\sin(t)$ to get $\sin(t)=\frac{y}{2}$. Substituting these expressions into $\cos^2(t)+\sin^2(t)=1$ gives

$$\left(\frac{x-1}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$$

$$\frac{(x-1)^2}{9}+\frac{y^2}{4}=1$$

Figure 8.1. 12



$$\begin{cases} x = 1+3\cos(t) \\ y = 2\sin(t) \end{cases} \text{ for } 0 \le t \le \frac{3\pi}{2}$$

The equation $\frac{(x-1)^2}{9}+\frac{y^2}{4}=1$ is that of an ellipse centered at $(1,0)$ with vertices at $(-2,0)$ and $(4,0)$. The parametric equations trace out three-quarters of this ellipse in a counter-clockwise direction.

Note that we would need to restrict the domain and range of the ellipse $\frac{(x-1)^2}{9}+\frac{y^2}{4}=1$ to result in the part of the ellipse that is graphed above. One way to do that would be to include only the *x*- and *y*-coordinates such that $-2 \le y \le 2$ when $-2 \le x \le 1$ and $0 \le y \le 2$ when $1 \le x \le 4$. Another solution would be to solve the equation of the ellipse for y; that is $y=\pm\sqrt{4-\frac{4}{9}(x-1)^2}$. Now, the top and bottom portions of the curve can be described, separately, as functions. The top portion is $y=\sqrt{4-\frac{4}{9}(x-1)^2}$ for $-2 \le x \le 4$ and the bottom portion is $y=-\sqrt{4-\frac{4}{9}(x-1)^2}$ for $-2 \le x \le 1$.

□

8.1 Exercises

In Exercises 1 – 6, graph the parametric curve by completing the suggested table of values. Include the orientation on the graph.

1. $\begin{cases} x = t \\ y = t^2 - 1 \end{cases}$

t	$x(t)$	$y(t)$
-3		
-2		
-1		
0		
1		
2		
3		

2. $\begin{cases} x = t - 1 \\ y = t^2 \end{cases}$

t	$x(t)$	$y(t)$
-3		
-2		
-1		
0		
1		
2		

3. $\begin{cases} x = 2 + t \\ y = 3 - 2t \end{cases}$

t	$x(t)$	$y(t)$
-2		
-1		
0		
1		
2		
3		

4. $\begin{cases} x = -2 - 2t \\ y = 3 + t \end{cases}$

t	$x(t)$	$y(t)$
-3		
-2		
-1		
0		
1		

5. $\begin{cases} x = t^3 \\ y = t + 2 \end{cases}$

t	$x(t)$	$y(t)$
-2		
-1		
0		
1		
2		

6. $\begin{cases} x = t^2 \\ y = t + 3 \end{cases}$

t	$x(t)$	$y(t)$
-2		
-1		
0		
1		
2		

In Exercises 7 – 28, sketch the curve defined by the parametric equations by hand and indicate the orientation of the curve.

7. $\begin{cases} x = 4t-3 \\ y = 6t-2 \end{cases}$ for $0 \le t \le 1$

8. $\begin{cases} x = 4t-1 \\ y = 3-4t \end{cases}$ for $0 \le t \le 1$

9. $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ for $-1 \le t \le 2$

10. $\begin{cases} x = t^2 \\ y = 3t \end{cases}$ for $0 \le t \le 5$

11. $\begin{cases} x = t-1 \\ y = 3+2t-t^2 \end{cases}$ for $0 \le t \le 3$

12. $\begin{cases} x = t^2+2t+1 \\ y = t+1 \end{cases}$ for $t \ge 1$

13. $\begin{cases} x = \frac{1}{9}\left(18-t^2\right) \\ y = \frac{1}{3}t \end{cases}$ for $t \ge -3$

14. $\begin{cases} x = t \\ y = \sqrt{25-t^2} \end{cases}$ for $0 \le t \le 5$

15. $\begin{cases} x = -t \\ y = \sqrt{t} \end{cases}$ for $t \ge 0$

16. $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

17. $\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases}$ for $0 \le t \le \pi$

18. $\begin{cases} x = -2\cos(t) \\ y = 6\sin(t) \end{cases}$ for $0 \le t \le \pi$

19. $\begin{cases} x = -1+3\cos(t) \\ y = 4\sin(t) \end{cases}$ for $0 \le t \le 2\pi$

20. $\begin{cases} x = 3\cos(t) \\ y = 2\sin(t)+1 \end{cases}$ for $\frac{\pi}{2} \le t \le 2\pi$

21. $\begin{cases} x = 2\cos(t) \\ y = \sec(t) \end{cases}$ for $0 \le t \le \frac{\pi}{2}$

22. $\begin{cases} x = 2\tan(t) \\ y = \cot(t) \end{cases}$ for $0 < t < \frac{\pi}{2}$

23. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

24. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$

25. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

26. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$

27. $\begin{cases} x = \cos(t) \\ y = t \end{cases}$ for $0 \le t \le \pi$

28. $\begin{cases} x = \sin(t) \\ y = t \end{cases}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

In Exercises 29 – 37, sketch the parametric curve by hand and indicate the orientation of the curve.

29. $\begin{cases} x = t \\ y = t^3 \end{cases}$

30. $\begin{cases} x = t^3 \\ y = t \end{cases}$

31. $\begin{cases} x = 5 - |t| \\ y = t + 2 \end{cases}$

32. $\begin{cases} x = -t + 2 \\ y = 5 - |t| \end{cases}$

33. $\begin{cases} x = 4\sin(t) \\ y = 2\cos(t) \end{cases}$

34. $\begin{cases} x = 2\sin(t) \\ y = 4\cos(t) \end{cases}$

35. $\begin{cases} x = 3\cos^2(t) \\ y = -3\sin(t) \end{cases}$

36. $\begin{cases} x = 3\cos^2(t) \\ y = -3\sin^2(t) \end{cases}$

37. $\begin{cases} x = 2\cos(t) \\ y = -\sin(t) \end{cases}$

In Exercises 38 – 55, eliminate the parameter t to determine a Cartesian equation for the curve. In Exercises 50 – 53, your answer should not include trigonometric functions.

38. $\begin{cases} x = 5 - t \\ y = 8 - 2t \end{cases}$

39. $\begin{cases} x = 6 - 3t \\ y = 10 - t \end{cases}$

40. $\begin{cases} x = 2t + 1 \\ y = 3\sqrt{t} \end{cases}$

41. $\begin{cases} x = 3t - 1 \\ y = 2t^2 \end{cases}$

42. $\begin{cases} x = 2e^t \\ y = 1 - 5t \end{cases}$

43. $\begin{cases} x = e^{-2t} \\ y = 2e^{-t} \end{cases}$

44. $\begin{cases} x = 4\log(t) \\ y = 3 + 2t \end{cases}$

45. $\begin{cases} x = \log(2t) \\ y = \sqrt{t - 1} \end{cases}$

46. $\begin{cases} x = t^3 - 1 \\ y = 2t \end{cases}$

47. $\begin{cases} x = t - t^4 \\ y = t + 2 \end{cases}$

48. $\begin{cases} x = e^{2t} \\ y = e^{6t} \end{cases}$

49. $\begin{cases} x = t^5 \\ y = t^{10} \end{cases}$

50. $\begin{cases} x = 4\cos(t) \\ y = 4\sin(t) \end{cases}$

51. $\begin{cases} x = 3\sin(t) \\ y = 6\cos(t) \end{cases}$

52. $\begin{cases} x = 2\cos^2(t) \\ y = -\sin(t) \end{cases}$

53. $\begin{cases} x = \cos(t) + 4 \\ y = 2\sin^2(t) \end{cases}$

54. $\begin{cases} x = t - 1 \\ y = t^2 \end{cases}$

55. $\begin{cases} x = -t \\ y = t^3 + 1 \end{cases}$

In Exercises 56 – 59, sketch the parametric curve for the indicated parameter values with the help of a graphing utility and indicate the orientation of the curve.

56. $\begin{cases} x = t^3 - 3t \\ y = t^2 - 4 \end{cases}$ for $-2 \le t \le 2$

57. $\begin{cases} x = 4\cos^3(t) \\ y = 4\sin^3(t) \end{cases}$ for $0 \le t \le 2\pi$

58. $\begin{cases} x = e^{t} + e^{-t} \\ y = e^{t} - e^{-t} \end{cases}$ for $-2 \le t \le 2$

59. $\begin{cases} x = \cos(3t) \\ y = \sin(4t) \end{cases}$ for $0 \le t \le 2\pi$

In Exercises 60 – 63, use a graphing utility to view the graph of each of the four sets of parametric equations. Although they look unusual and beautiful, they are so common they have names, as indicated in each exercise.

60. An epicycloid: $\begin{cases} x = 14\cos(t) - \cos(14t) \\ y = 14\sin(t) + \sin(14t) \end{cases}$ for $0 \le t \le 2\pi$

61. A hypocycloid: $\begin{cases} x = 6\sin(t) + 2\sin(6t) \\ y = 6\cos(t) - 2\cos(6t) \end{cases}$ for $0 \le t \le 2\pi$

62. A hypotrochoid: $\begin{cases} x = 2\sin(t) + 5\cos(6t) \\ y = 5\cos(t) - 2\sin(6t) \end{cases}$ for $0 \le t \le 2\pi$

63. A rose: $\begin{cases} x = 5\sin(2t)\sin(t) \\ y = 5\sin(2t)\cos(t) \end{cases}$ for $0 \le t \le 2\pi$

8.2 Parametric Descriptions for Oriented Curves

Learning Objectives

- **Parametrize curves given in Cartesian coordinates.**
- **Reverse orientation and shift the starting point of a curve described by parametric equations.**
- **Apply parametric equations to applications involving projectile motion.**

We next turn to the problem of finding parametric representations for curves.

Parametrizing Curves

Parametrizations of Common Curves

1. To parametrize $y = f(x)$, one option is to use the parametric equations $x = t$ and $y = f(t)$.
2. To parametrize $x = g(y)$, one option is to set $x = g(t)$ and $y = t$.
3. To parametrize a 'directed' line segment with initial point (x_0, y_0) and terminal point (x_1, y_1), one option is to let $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t$ for $0 \le t \le 1$.
4. To parametrize $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a > 0$ and $b > 0$, one option is to let $x = h + a\cos(t)$ and $y = k + b\sin(t)$ for $0 \le t < 2\pi$. (This will impart a counterclockwise orientation.)

The reader is encouraged to verify the above formulas by eliminating the parameter and, when indicated, checking the orientation. Further explanation for each of these parametrizations follows.

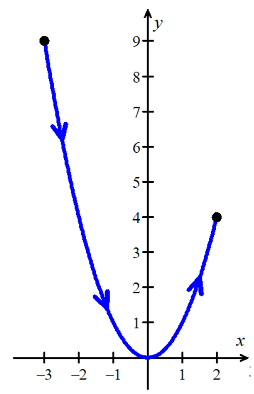
1. For $y = f(x)$, setting $x = t$ results in $y = f(x) = f(t)$.
2. Given $x = g(y)$, setting $y = t$ results in $x = g(y) = g(t)$.
3. For the 'directed' line segment, $x_1 - x_0$ is the displacement in the *x*-direction and $y_1 - y_0$ is the displacement in the *y*-direction. We can think of the right side of the formulas $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t$, for $0 \le t \le 1$, as 'starting point + (displacement)$\cdot t$', where $t = 0$ is the initial point (x_0, y_0) and $t = 1$ is the terminal point (x_1, y_1). (See the **7.1 Exercises**, **#65**.)

4. The parametrization for $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ is based on the Pythagorean identity $\cos^2(t)+\sin^2(t)=1$. We set $\cos(t)=\frac{x-h}{a}$ and $\sin(t)=\frac{y-k}{b}$, then solve for x and y to get, respectively, $x=h+a\cos(t)$ and $y=k+b\sin(t)$. By periodicity of the cosine and sine functions, we only need one period of each, and have chosen $0\le t<2\pi$. The choice of using the cosine function for the *x*-values and the sine function for the *y*-values results in the counterclockwise rotation. Switching this choice would result in a clockwise rotation and a different starting point.

We put these formulas to good use in the following examples.

Example 8.2.1. Find a parametrization for the curve $y=x^2$ from $x=-3$ to $x=2$.

Figure 8.2. 1



Solution. The curve, along with its orientation, is graphed to the right. To parametrize the curve, we let $x=t$ so that $y=x^2=t^2$. For $x=t$, the bounds on t match precisely the bounds on x so that we get

$$\begin{cases} x=t \\ y=t^2 \end{cases} \text{ for } -3\le t\le 2$$

□

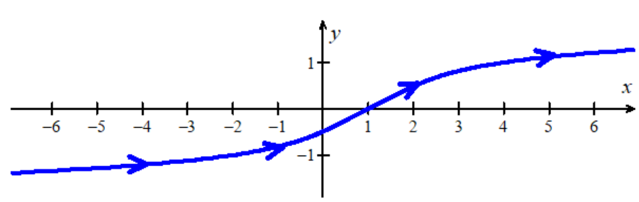
Example 8.2.2. Find a parametrization for the curve $x=y^5+2y+1$.

Solution. A graph of the curve that includes its orientation is shown below. We parametrize the curve by setting $y=t$ and get $x=y^5+2y+1=t^5+2t+1$. Since $y=t$, and there are no bounds placed on y, it follows that there are no bounds placed on t.

Our final answer is

$$\begin{cases} x=t^5+2y+1 \\ y=t \end{cases} \text{ for } -\infty<t<\infty$$

Figure 8.2. 2

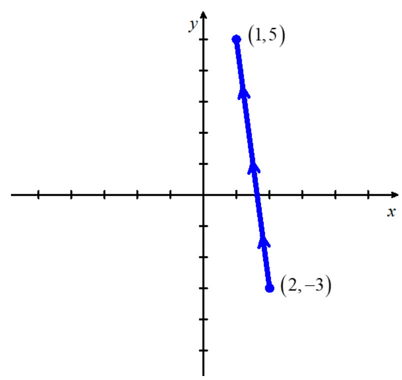


□

Example 8.2.3. Find a parametrization for the line segment that starts at $(2,-3)$ and ends at $(1,5)$.

Solution. A graph of the oriented line segment appears to the right. To parametrize the directed line segment, we begin by noting that the displacement in the x-direction is $x_1 - x_0 = 1 - 2 = -1$ and the displacement in the y-direction is $y_1 - y_0 = 5 - (-3) = 8$. Using the initial point $(x_0, y_0) = (2,-3)$, we find the parametrization is $x = x_0 + (x_1 - x_0)\,t = 2 - t$ and $y = y_0 + (y_1 - y_0)\,t = -3 + 8t$ for $0 \le t \le 1$. Our final answer is

Figure 8.2. 3



$$\begin{cases} x = 2 - t \\ y = -3 + 8t \end{cases} \text{ for } 0 \le t \le 1$$

□

Example 8.2.4. Find a parametrization for the circle $x^2 + 2x + y^2 - 4y = 4$.

Solution. In order to use the formulas $x = h + a\cos(t)$ and $y = k + b\sin(t)$ to parametrize the circle $x^2 + 2x + y^2 - 4y = 4$, we first need to put it into the correct form, $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

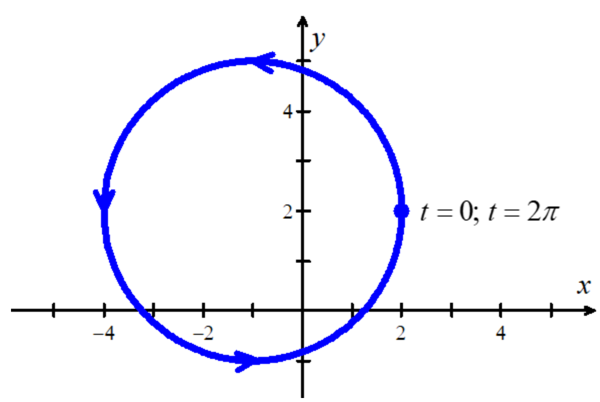
$$\begin{aligned} x^2 + 2x + y^2 - 4y &= 4 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) &= 4 + 1 + 4 \\ (x+1)^2 + (y-2)^2 &= 9 \\ \frac{(x+1)^2}{9} + \frac{(y-2)^2}{9} &= 1 \end{aligned}$$

Then $\dfrac{(x-(-1))^2}{3^2} + \dfrac{(y-2)^2}{3^2} = 1$, from which $h = -1$, $k = 2$, and $a = b = 3$.

A parametrization for the circle, graphed to the right, is

$$\begin{cases} x = -1 + 3\cos(t) \\ y = 2 + 3\sin(t) \end{cases} \text{ for } 0 \le t < 2\pi$$

Figure 8.2. 4



□

Example 8.2.5. Find a parametrization for the left half of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution. In the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$, we can either use the formulas or think back to the Pythagorean identity $\cos^2(t)+\sin^2(t)=1$, along with $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$, to write x and y as functions of t.

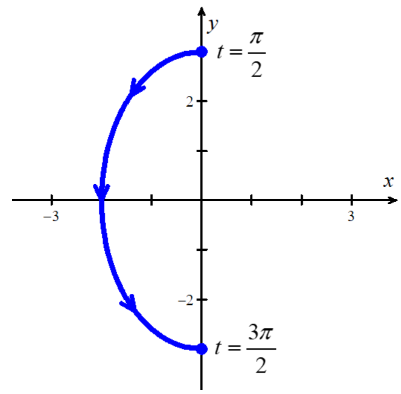
$$\frac{x}{2}=\cos(t) \qquad \frac{y}{3}=\sin(t)$$
$$x=2\cos(t) \qquad y=3\sin(t)$$

The normal range on the parameter in this case is $0\le t<2\pi$, but since we are interested in only the left half of the ellipse, we restrict t to the values that correspond to Quadrant II and Quadrant III angles, namely $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$.

A parametrization for this curve, graphed to the right, is

$$\begin{cases} x=2\cos(t) \\ y=3\sin(t) \end{cases} \text{ for } \frac{\pi}{2}\le t\le\frac{3\pi}{2}$$

Figure 8.2. 5



□

Note that the formulas provided prior to these examples offer only one of literally infinitely many ways to parametrize the common curves listed there.

Adjusting Parametric Equations

At times, the formulas that define a parametric curve need to be altered to suit the situation. Two easy ways to alter the parametrizations are given below.

Adjusting Parametric Equations

- **Reversing Orientation**: Replacing every occurrence of t with $-t$ in a parametric description for a curve (including any inequalities that describe the bounds on t) reverses the orientation of the curve.
- **Shift of Parameter**: Replacing every occurrence of t with $t-c$ in a parametric description for a curve (including any inequalities that describe the bounds on t) shifts the start of the parameter t ahead by c units.

These techniques are demonstrated in the next example.

Example 8.2.6. Find a parametrization for the following curves.

1. The curve that starts at $(2,4)$, follows the parabola $y=x^2$, and ends at $(-1,1)$, with the parameter shifted so that the path starts at $t=0$.
2. The two part path that starts at $(0,0)$, travels along a straight line to $(3,4)$, and then travels along another straight line to $(5,0)$.
3. The Unit Circle, oriented clockwise, with $t=0$ corresponding to $(0,-1)$

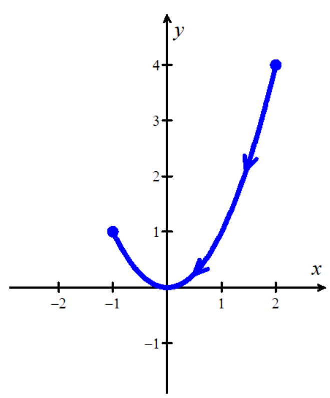
Solution.

1. The desired curve is shown below. We can parametrize $y=x^2$ from $x=-1$ to $x=2$ as $\begin{cases} x=t \\ y=t^2 \end{cases}$ for $-1 \le t \le 2$. This parametrization, however, starts at $(-1,1)$ and ends at $(2,4)$. We need to reverse the orientation. To do so, we replace every occurrence of t with $-t$ to get $x=-t$ and $y=(-t)^2$ for $-1 \le -t \le 2$. After simplifying, we have $\begin{cases} x=-t \\ y=t^2 \end{cases}$ for $-2 \le t \le 1$.

   We would next like to begin at $t=0$ instead of $t=-2$. The problem here is that the parametrization we have starts 2 units too soon, so we need to introduce a time delay of 2. This may be accomplished by replacing every occurrence of t with $t-2$, resulting in $x=-(t-2)$ and $y=(t-2)^2$ for $-2 \le t-2 \le 1$. Simplifying yields

   $$\begin{cases} x=2-t \\ y=t^2-4t+4 \end{cases} \text{ for } 0 \le t \le 3$$

   Figure 8.2. 6

2. When parametrizing line segments, we think: starting point + (displacement)$\cdot t$. For the first part of the path, that starts at $(0,0)$ and travels along a line to $(3,4)$, we get

   $$\begin{cases} x=3t \\ y=4t \end{cases} \text{ for } 0 \le t \le 1$$

   For the second part, that starts at $(3,4)$ and travels to $(5,0)$, we get

   $$\begin{cases} x=3+2t \\ y=4-4t \end{cases} \text{ for } 0 \le t \le 1$$

Since the first parametrization leaves off at $t=1$, we shift the parameter in the second part so that it starts at $t=1$. Our current description of the second part starts at $t=0$, so we introduce a time delay of 1 unit to the second set of parametric equations by replacing t with $t-1$. The second set of parametric equations becomes $x=3+2(t-1)$ and $y=4-4(t-1)$ for $0\le t-1\le 1$. Simplifying yields

$$\begin{cases} x=1+2t \\ y=8-4t \end{cases} \text{ for } 1\le t\le 2$$

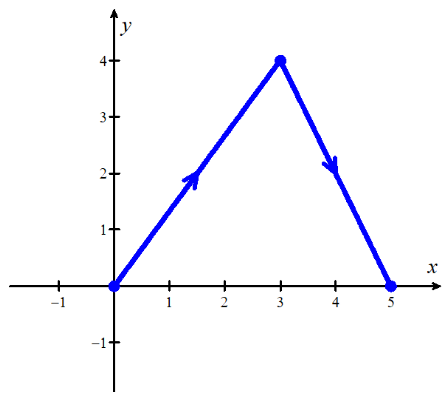
In summary, the path, shown in the figure to the right, may be parametrized as follows:

$$\begin{cases} x=f(t) \\ y=g(t) \end{cases} \text{ for } 0\le t\le 2 \text{, where}$$

$$f(t)=\begin{cases} 3t \text{ for } 0\le t\le 1 \\ 1+2t \text{ for } 1\le t\le 2 \end{cases} \text{ and}$$

$$g(t)=\begin{cases} 4t \text{ for } 0\le t\le 1 \\ 8-4t \text{ for } 1\le t\le 2 \end{cases}$$

Figure 8.2. 7



3. To parametrize the Unit Circle with a clockwise orientation and 'starting point' of $(0,-1)$ corresponding to $t=0$, we first note that a counterclockwise orientation is given by

$$\begin{cases} x=\cos(t) \\ y=\sin(t) \end{cases} \text{ for } 0\le t<2\pi$$

We reverse the direction by replacing t with $-t$. This results in $x=\cos(-t)$ and $y=\sin(-t)$ for $0\le -t<2\pi$, which (after applying the even/odd identities) simplifies to

$$\begin{cases} x=\cos(t) \\ y=-\sin(t) \end{cases} \text{ for } -2\pi<t\le 0$$

This parametrization gives a clockwise orientation, but $t=0$ corresponds to the point $(1,0)$; the point $(0,-1)$ is reached when $t=-\frac{3\pi}{2}$. Our strategy is to first get the parametrization to start at the point $(0,-1)$ and then shift the parameter accordingly so the start coincides with $t=0$.

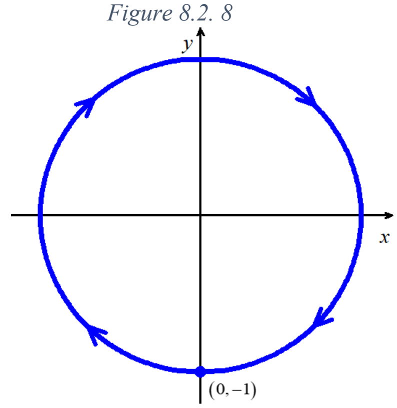
- We know that any interval of length 2π will parametrize the entire circle, so we keep the equations $x=\cos(t)$ and $y=-\sin(t)$, but start the parameter t at $-\frac{3\pi}{2}$, and find the upper bound by adding 2π so that

$$\begin{cases} x = \cos(t) \\ y = -\sin(t) \end{cases} \text{ for } -\frac{3\pi}{2} \le t < \frac{\pi}{2}$$

The reader can verify that the Unit Circle is traced out clockwise starting at the point $(0,-1)$.

- To shift the parameter so that the start coincides with $t = 0$, we introduce a time delay of $\frac{3\pi}{2}$ units by replacing each occurrence of t with $t - \frac{3\pi}{2}$. We get $x = \cos\left(t - \frac{3\pi}{2}\right)$ and $y = -\sin\left(t - \frac{3\pi}{2}\right)$ for $-\frac{3\pi}{2} \le t - \frac{3\pi}{2} < \frac{\pi}{2}$.

Figure 8.2. 8

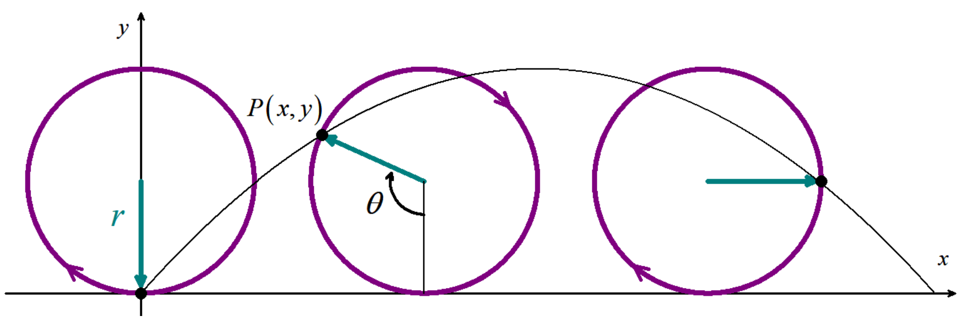


Applying sum/difference formulas, this simplifies to $\begin{cases} x = -\sin(t) \\ y = -\cos(t) \end{cases}$ for $0 \le t < 2\pi$. The parametrized curve is shown above.

□

We put the answer to **Example 8.2.6**, part 3, to good use to derive the equation of a cycloid. Suppose a circle of radius r rolls along the positive x-axis at a constant speed v as pictured below. Let θ be the angle in radians that measures the amount of clockwise rotation experienced by the radius highlighted in the following figure.

Figure 8.2. 9



Our goal is to find parametric equations for the coordinates of the point $P(x, y)$ in terms of θ. From our work in **Example 8.2.6**, part 3, we know that clockwise motion along the Unit Circle starting at the point $(0,-1)$ can be modeled by the equations

$$\begin{cases} x = -\sin(\theta) \\ y = -\cos(\theta) \end{cases} \text{ for } 0 \le \theta < 2\pi$$

(We have renamed the parameter as θ to match the context of this problem.) To model this motion on a circle of radius r, all we need to do[1] is multiply both x and y by the factor r, which yields

$$\begin{cases} x = -r\sin(\theta) \\ y = -r\cos(\theta) \end{cases}$$

We now need to adjust for the fact that the circle is not stationary but is rolling along the positive *x*-axis, and its center is not $(0,0)$. Since the speed v is constant, we know that at time t the center of the circle has traveled a distance vt down the positive *x*-axis. Furthermore, since the radius of the circle is r and the circle is not moving vertically, we know that the center of the circle is always r units above the *x*-axis. Putting these two facts together, at time t the center of the circle is at the point (vt, r).

From **Section 2.3**, we know the angular speed is $\omega = \frac{\theta}{t}$ and the linear speed is $v = r\omega$. Putting these together, we have $v = \frac{r\theta}{t}$, or $vt = r\theta$. Hence, the center of the circle, in terms of the parameter θ, is $(r\theta, r)$.

As a result, we need to modify the equations $x = -r\sin(\theta)$ and $y = -r\cos(\theta)$ by shifting the *x*-coordinates to the right $r\theta$ units (by adding $r\theta$ to the expression for x) and the *y*-coordinates up r units (by adding r to the expression for y). We get $x = -r\sin(\theta) + r\theta$ and $y = -r\cos(\theta) + r$, which can be written as

$$\begin{cases} x = r(\theta - \sin(\theta)) \\ y = r(1 - \cos(\theta)) \end{cases}$$

Since the motion starts at $\theta = 0$ and proceeds indefinitely, we set $\theta \ge 0$.

Example 8.2.7. Find the parametric equations of a cycloid which results from a circle of radius 3 rolling down the positive *x*-axis.

---

[1] If we replace x with $\frac{x}{r}$ and y with $\frac{y}{r}$ in the equation for the Unit Circle, we obtain $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$, which reduces to $x^2 + y^2 = r^2$. Note that we are 'stretching the graph by a factor of r in both the *x*- and *y*-directions. Hence, we multiply both the *x*- and *y*-coordinates of points on the graph by r.

Solution. We completed the major part of our work above. With $r = 3$, we have the equations

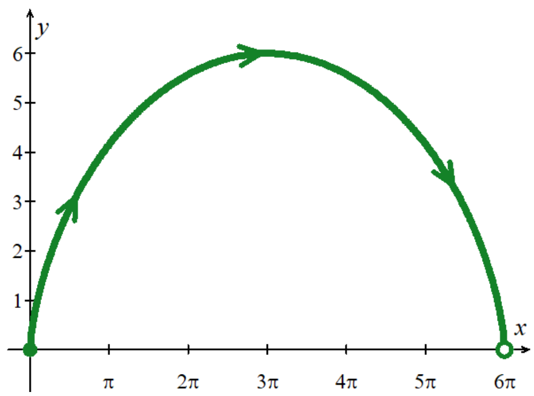
$$\begin{cases} x = 3(t - \sin(t)) \\ y = 3(1 - \cos(t)) \end{cases} \text{ for } t \geq 0$$

(Here we have returned to the convention of using t as the parameter.)

□

In the previous example, we note that one full revolution of the circle occurs over the interval $0 \leq t < 2\pi$, as shown in the following figure. As t ranges between 0 and 2π, we see that x ranges between 0 and 6π. The values of y range between 0 and 6.

Figure 8.2. 10

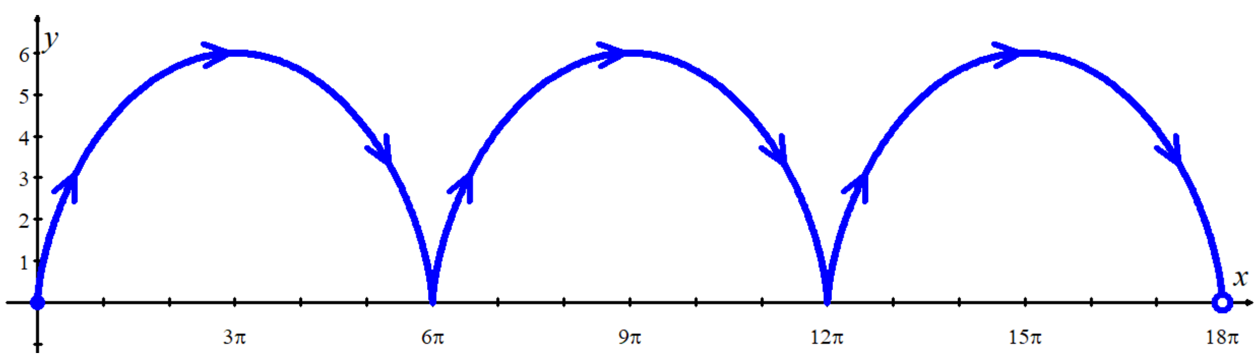


$$\begin{cases} x = 3(t - \sin(t)) \\ y = 3(1 - \cos(t)) \end{cases} \text{ for}$$

$0 \leq t < 2\pi$

Below, we extend t to range from 0 to 6π, which forces x to range from 0 to 18π, yielding three arches of the cycloid.

Figure 8.2. 11



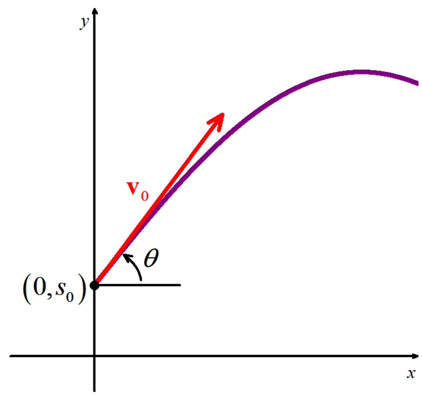
$$\begin{cases} x = 3(t - \sin(t)) \\ y = 3(1 - \cos(t)) \end{cases} \text{ for } 0 \leq t < 6\pi$$

□

Projectile Motion

Projectile motion occurs when an object is launched into the air. Suppose an object is launched at time zero from the initial point $(0, s_0)$ with a velocity of magnitude v_0 and the direction angle θ, as shown to the right.

Figure 8.2. 12



Not taking into account the air resistance, the coordinates of the object at time t are given below.

Equations for Projectile Motion

The position of an object launched at time $t = 0$ from the initial position $(0, s_0)$, with initial velocity of magnitude v_0 and direction angle θ, is

$$\begin{cases} x = \left[v_0 \cos(\theta)\right] t \\ y = s_0 + \left[v_0 \sin(\theta)\right] t - \frac{1}{2} g\, t^2 \end{cases}$$

at time t, where g is the constant acceleration due to gravity.

Example 8.2.8. From a height 4 feet above the ground, a baseball is hit with an initial speed of 110 miles per hour at an angle of 30°. How far will the baseball travel (horizontally), and what is its maximum height?

Solution. To use the equations for projectile motion, with $s_0 = 4$ feet, $v_0 = 110$ miles/hour, and $\theta = 30^\circ$, we start by converting v_0 to feet per second.

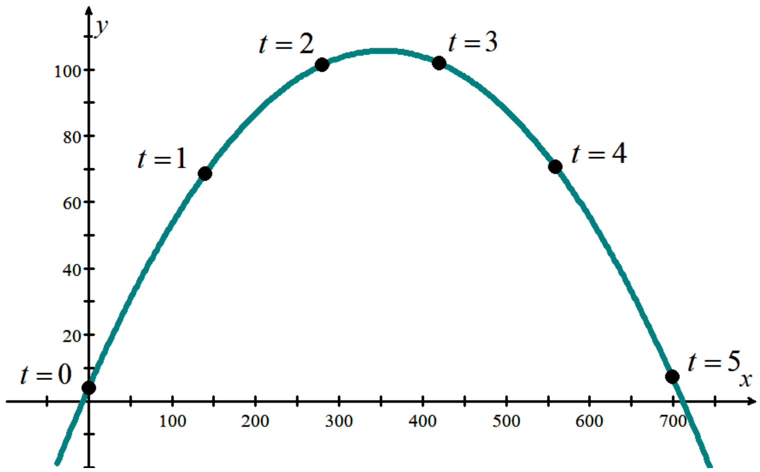
$$\begin{aligned} v_0 &= \frac{110 \text{ miles}}{\text{hour}} \cdot \frac{5280 \text{ feet}}{\text{mile}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \\ &= \frac{484}{3} \text{ feet/second} \end{aligned}$$

Noting that acceleration due to gravity is $g = 32$ ft/sec^2, and using the equations for projectile motion, we find $x = \left[\frac{484}{3}\cos(30^\circ)\right] t$ and $y = 4 + \left[\frac{484}{3}\sin(30^\circ)\right] t - \frac{1}{2}(32)\, t^2$. After evaluating the sine and cosine values and simplifying, the equations are

$$x = \frac{242}{\sqrt{3}}\, t \text{ and } y = -16t^2 + \frac{242}{3}\, t + 4$$

The equations are graphed below (with the aid of a calculator to determine *x*- and *y*-values).

Figure 8.2. 13



$$\begin{cases} x = \dfrac{242}{\sqrt{3}}t \\ y = -16t^2 + \dfrac{242}{3}t + 4 \end{cases}$$

To determine how far the baseball will travel horizontally, we look for the time when the height of the ball is zero. The time is found by setting y equal to zero: $y = -16t^2 + \frac{242}{3}t + 4 = 0$. After applying the quadratic formula,

$$t = \frac{-\frac{242}{3} \pm \sqrt{\left(\frac{242}{3}\right)^2 - 4(-16)(4)}}{2(-16)}$$

Then $t \approx -0.4911$ or $t \approx 5.0908$. The time we are looking for is when the ball lands, so we choose $t \approx 5.0908$ seconds. We plug this value into $x = \frac{242}{\sqrt{3}}t$ to get $x \approx 711.28$, and conclude that the ball travels a horizontal distance of approximately 711.28 feet.

Next, noting that the maximum height occurs at the largest *y*-value, we find the *y*-coordinate of the vertex for the quadratic function $y = -16t^2 + \frac{242}{3}t + 4$. At the vertex,

$$t = -\frac{b}{2a} = -\frac{\frac{242}{3}}{2(-16)} = \frac{121}{48}$$

Then $y = -16t^2 + \frac{242}{3}t + 4 = -16\left(\frac{121}{48}\right)^2 + \left(\frac{242}{3}\right)\left(\frac{121}{48}\right) + 4$, from which $y \approx 105.67$. So, the maximum height of the ball is approximately 105.67 feet.

□

8.2 Exercises

In Exercises 1 – 4, parametrize (write parametric equations for) each Cartesian equation by setting the independent variable equal to t.

1. $y = 3x^2 + 3$
2. $y = 2\sin(x) + 1$
3. $x = 3\log y + y$
4. $x = \sqrt{y} + 2y$

In Exercises 5 – 8, parametrize (write parametric equations for) each Cartesian equation by using $x = a\cos(t)$ and $y = b\sin(t)$, for appropriate a and b values. Identify the curve.

5. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
6. $\frac{x^2}{16} + \frac{y^2}{36} = 1$
7. $x^2 + y^2 = 16$
8. $x^2 + y^2 = 10$

In Exercises 9 – 23, find a parametric description for the given oriented curve.

9. The directed line segment from $(3,-5)$ to $(-2,2)$

10. The directed line segment from $(-2,-1)$ to $(3,-4)$

11. The curve $y = 4 - x^2$ from $(-2,0)$ to $(2,0)$

12. The curve $y = 4 - x^2$ from $(2,0)$ to $(-2,0)$

    (Shift the parameter so $t = 0$ corresponds to $(2,0)$.)

13. The curve $x = y^2 - 9$ from $(-5,-2)$ to $(0,3)$

14. The curve $x = y^2 - 9$ from $(0,3)$ to $(-5,-2)$

    (Shift the parameter so $t = 0$ corresponds to $(0,3)$.)

15. The circle $x^2 + y^2 = 25$, oriented counterclockwise

16. The circle $(x-1)^2 + y^2 = 4$, oriented counterclockwise

17. The circle $x^2 + y^2 - 6y = 0$, oriented counterclockwise

18. The circle $x^2 + y^2 - 6y = 0$, oriented clockwise

    (Shift the parameter so t begins at 0.)

19. The circle $(x-3)^2 + (y+1)^2 = 117$, oriented counterclockwise

20. The ellipse $(x-1)^2 + 9y^2 = 9$, oriented counterclockwise

21. The ellipse $9x^2+4y^2+24y=0$, oriented counterclockwise

22. The ellipse $9x^2+4y^2+24y=0$, oriented clockwise

(Shift the parameter so $t=0$ corresponds to $(0,0)$.)

23. The triangle with vertices $(0,0)$, $(3,0)$ and $(0,4)$, oriented counterclockwise

(Shift the parameter so $t=0$ corresponds to $(0,0)$.)

24. Use parametric equations and a graphing utility to graph the inverse of $f(x)=x^3+3x-4$.

25. Every polar curve $r=f(\theta)$ can be translated to a system of parametric equations with parameter θ by $\{x=r\cos(\theta)=f(\theta)\cdot\cos(\theta), y=r\sin(\theta)=f(\theta)\cdot\sin(\theta)\}$. Convert $r=6\cos(2\theta)$ to a system of parametric equations. Check your answer by graphing $r=6\cos(2\theta)$ by hand and then graphing the parametric equations you found using a graphing utility.

26. A dart is thrown upward from the ground level with an initial velocity of 65 feet/second at an angle of elevation of 52°. Consider the position of the dart at any time t. Neglect air resistance.

a) Find parametric equations $x=f(t)$ and $y=g(t)$ that model the position of the dart. For gravity, use 32 feet per second squared.
b) How far away from its initial launch point does the dart hit the ground?
c) When will the dart hit the ground?
d) Find the maximum height of the dart.
e) At what time will the dart reach maximum height?

27. Suzanne's friend Jason competes in Highland Games Competitions across the country. In one event, the 'hammer throw', he throws a 56 pound weight for distance. If the weight is released 6 feet above the ground at an angle of 42° with respect to the horizontal, with an initial speed of 33 feet per second, find parametric equations for the flight of the hammer. For gravity, use 32 feet per second squared. When will the hammer hit the ground? Check your answer with a graphing utility.

28. Recall the equations for projectile motion:

$$\begin{cases} x = \left[v_0 \cos(\theta)\right] t \\ y = s_0 + \left[v_0 \sin(\theta)\right] t - \frac{1}{2} g t^2 \end{cases}$$

Eliminate the parameter in the equations for projectile motion to show that the path of the projectile follows the curve $y = -\frac{g \sec^2(\theta)}{2v_0^2} x^2 + \tan(\theta) x + s_0$.

Recall that for a quadratic function $f(x) = ax^2 + bx + c$, the vertex can be determined using the formula $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. Use this formula to show the maximum height of the projectile is $y = \frac{v_0^2 \sin^2(\theta)}{2g} + s_0$ when $x = \frac{v_0^2 \sin(2\theta)}{2g}$.

29. In another event, the 'sheaf toss', Jason throws a 20 pound weight for height. If the weight is released 5 feet above the ground at an angle of 85° with respect to the horizontal and the sheaf reaches a maximum height of 31.5 feet, use your results from the previous exercise to determine how fast the sheaf was launched into the air. Once again, gravity is $g = 32 \text{ ft/s}^2$.

In Exercises 30 – 33, we explore the **hyperbolic sine** function, denoted $\sinh(t)$, and the **hyperbolic cosine** function, denoted $\cosh(t)$, defined below:

$$\sinh(t) = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh(t) = \frac{e^t + e^{-t}}{2}$$

30. Using a graphing utility as needed, verify that the domain of $\cosh(t)$ is $(-\infty, \infty)$ and the range of $\cosh(t)$ is $[1, \infty)$.

31. Using a graphing utility as needed, verify that the domain and range of $\sinh(t)$ are both $(-\infty, \infty)$.

32. Show that $\{x(t) = \cosh(t), y(t) = \sinh(t)\}$ parameterize the right half of the 'unit' hyperbola $x^2 - y^2 = 1$.

Hence, the use of the adjective 'hyperbolic'. They are referred to as 'sine' and 'cosine' because $\cosh^2(t) - \sinh^2(t) = 1$ while $\sin^2(\theta) + \cos^2(\theta) = 1$. The hyperbolic sine and cosine mirror the traditional trigonometric identities but are built around $\cosh^2(t) - \sinh^2(t) = 1$.

33. Four other hyperbolic functions are waiting to be defined. Following the definitions of secant, cosecant, tangent, and cotangent, define the hyperbolic secant, $\operatorname{sech}(t)$, the hyperbolic cosecant, $\operatorname{csch}(t)$, the hyperbolic tangent, $\tanh(t)$, and the hyperbolic cotangent, $\coth(t)$. Define these functions in terms of $\sinh(t)$ and $\cosh(t)$, then convert them to formulas involving e^{t} and e^{-t}. Consult a suitable reference and spend some time reliving the thrills of trigonometry with these hyperbolic functions.

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