Sequence

Definition

A sequence is an ordered list of elements.

Definition. A sequence is a function from a subset Z , usually {0,1,2, …} or {1,2, …} to a set S. We use the notation a_n to denote the image of n, while a_n is called a term of the sequence.

Example:

a subset of N: 1 2 3 4 5 …

S: 2 4 6 8 10 …

Sequence

We symbolize the sequence with $\{a_n\}$.

Note: Do not confuse it with the {...} symbol in set notation.

A sequence is clearer when given as a formula.

For example, the sequence in the previous slide can be written more specifically as $\{a_n\}$, where $a_n = 2n$.

Write the formula for $a_1, a_2, a_3, \ldots$

1, 3, 5, 7, 9, …	$a_n = 2n - 1$
-1, 1, -1, 1, -1, …	$a_n=(-1)^n$
2, 5, 10, 17, 26, …	$a_n = n^2 + 1$
0.25, 0.5, 0.75, 1, 1.25 …	$a_n = 0.25n$
3, 9, 27, 81, 243, …	$a_n = 3^n$

Geometric Progression

- **Definition.** A geometric progression is a series that is in the form of a, $\mathbf{a}\boldsymbol{r}$, $\mathbf{a}\boldsymbol{r}^2$, …, $\mathbf{a}\boldsymbol{r}^n$,

  where the initial term **a** and the ratio **r** are real numbers.

  Example:
  The series $\{b_n\}$, where $b_n=(-1)^n$
  The series $\{c_n\}$, where $c_n=2 \cdot 5^n$

Arithmetic Progression

- **Definition**. An arithmetic progression is a series of the form a, a+d, a+2d, …., a+nd
- where the initial term is **a** and the common difference factor **d** is a real number.
- Example:
- The series $\{s_n\}$, where $s_n = -1 + 4n$

  The series $\{t_n\}$, where $t_n = 7 - 3n$

Other sequence example

TABLE 1 **Some Useful Sequences.**

*n*th Term	First 10 Terms
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Anyone know the name of the last row…?

String

A finite sequence is also called a string, its notation is $a_1a_2a_3\ldots a_n$.

The length of a string S is the number of terms in the string.

An empty string contains no terms at all. Its length is equal to zero.

Summation of sequence

What does this mean $\sum_{j=m}^{n} a_j$?

The symbol represents addition: $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$.

The variable j is called the summation index, running from the lower limit m to the upper limit n. The index may use other letters..

Summation of sequence

How to write the sum of the first 1000 terms of a series $\{a_n\}$ with $a_n=n^2$ for n = 1, 2, 3, … ?

We write it as $\sum_{j=1}^{1000} j^2$.

What is the value $\sum_{j=1}^{6} j$?

The value is 1 + 2 + 3 + 4 + 5 + 6 = 21.

Calculate $\sum_{j=1}^{100} j$?

The calculation using the previous method is somehow expensive

Summation of sequence

Friedrich Gauss found this formula:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

If there is such a formula, the sum of any length of string can be determined easily, for example:

$$\sum_{j=1}^{100} j = \frac{100\ (100+1)}{2} = \frac{10100}{2} = 5050$$

Double Summation

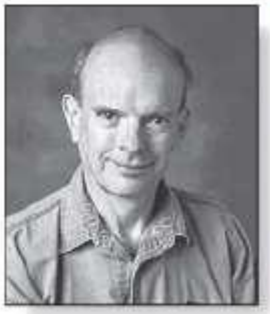
Regarding *nested loops* in programming (e.g. in C, Pascal, Java, …), there are double, triple additions etc.:

Example: $$\sum_{i=1}^{5}\sum_{j=1}^{2} ij$$

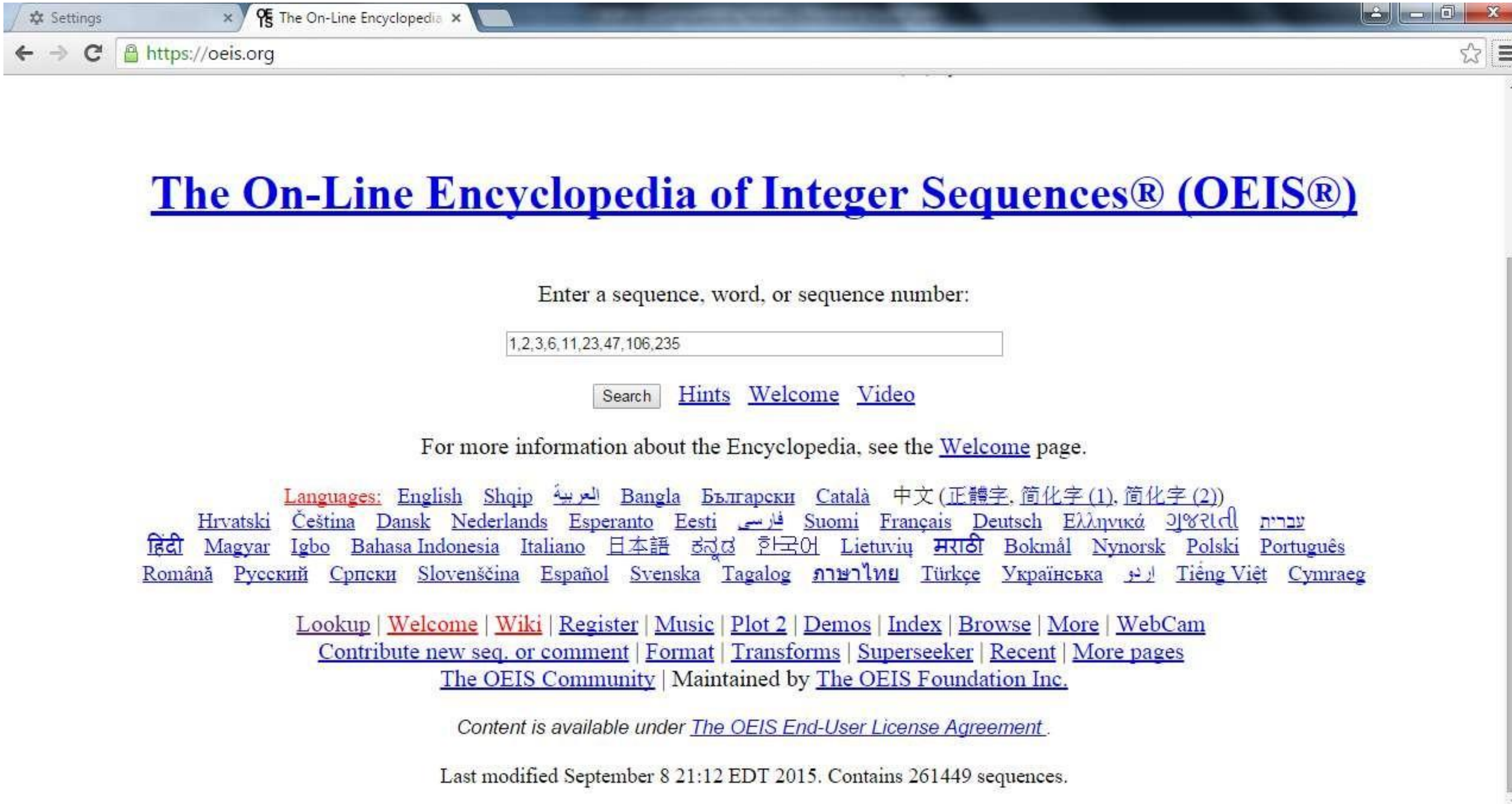
$$= \sum_{i=1}^{5}(i + 2i)$$

$$= \sum_{i=1}^{5} 3i \quad = 3+6+9+12+15 = 45$$

Mathematician of the day



NEIL SLOANE (BORN 1939) Neil Sloane studied mathematics and electrical engineering at the University of Melbourne on a scholarship from the Australian state telephone company. He mastered many telephone-related jobs, such as erecting telephone poles, in his summer work. After graduating, he designed minimal-cost telephone networks in Australia. In 1962 he came to the United States and studied electrical engineering at Cornell University. His Ph.D. thesis was on what are now called neural networks. He took a job at Bell Labs in 1969, working in many areas, including network design, coding theory, and sphere packing. He now works for AT&T Labs, moving there from Bell Labs when AT&T split up in 1996. One of his favorite problems is the **kissing problem** (a name he coined), which asks how many spheres can be arranged in n dimensions so that they all touch a central sphere of the same size. (In two dimensions the answer is 6, because 6 pennies can be placed so that they touch a central penny. In three dimensions, 12 billiard balls can be placed so that they touch a central billiard ball. Two billiard balls that just touch are said to "kiss," giving rise to the terminology "kissing problem" and "kissing number.") Sloane, together with Andrew Odlyzko, showed that in 8 and 24 dimensions, the optimal kissing numbers are, respectively, 240 and 196,560. The kissing number is known in dimensions 1, 2, 3, 4, 8, and 24, but not in any other dimensions. Sloane's books include *Sphere Packings, Lattices and Groups,* 3d ed., with John Conway; *The Theory of Error-Correcting Codes* with Jessie MacWilliams; *The Encyclopedia of Integer Sequences* with Simon Plouffe (which has grown into the famous OEIS website); and *The Rock-Climbing Guide to New Jersey Crags* with Paul Nick. The last book demonstrates his interest in rock climbing; it includes more than 50 climbing sites in New Jersey.



- http://oeis.org/

Exercise

1. Calculate a_0, a_1, a_2, and a_3 from $\{a_n\}$, where a_n equals
   (a) 2^n+1
   (b) $(n+1)^{n+1}$
   (c) $\lfloor n/2 \rfloor$
   (d) $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor$
2. Give formula for the sequence
   (a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102 …
   (b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, …
   (c) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, …
3. Calculate

(a) $$\sum_{j=0}^{8} 3 \cdot 2^j$$

(b) $$\sum_{j=0}^{4} 2 \cdot (-3)^j$$

(c) $$\sum_{i=1}^{2} \sum_{j=1}^{3} (i+j)$$

(d) $$\sum_{i=1}^{2} \sum_{j=1}^{3} i^2 \cdot j^3$$