Function

that the range of a function $f:A \rightarrow B$ is the set of all images of the elements $a \in A$.

For subset $S \subseteq A$, then the set of all images of elements $s \in S$ is called the image of S.

The image of S is denoted by f(S):

$$f(S) = \{f(s) \mid s \in S\}$$

Function

Consider the following functions:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

What is the image of S = {Linda, Max}?

f(S) = {Moscow, Boston}

What is the image of S = {Max, Peter}?

f(S) = {Boston}

Function Property: *one to one/injective*

Definition. A function f:**A**$\rightarrow$**B** is called one-to-one (or injective), if and only if $\forall x, y \in \mathbf{A}$ $(f(x) = f(y) \rightarrow x = y)$

In other words: f is one-to-one if and only if f does not map two distinct elements of A to the same element of B.

Function Property

Example:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

Is f one-to-one?

NO, Max and Peter are mapped to the same element of the image (Boston).

g(Linda) = Moscow

g(Max) = Boston

g(Kathy) = Hong Kong

g(Peter) = New York

Is g one-to-one ?

YES, each element is assigned a unique image element.

Function Property

How to prove that f is one-to-one?

To prove this, first we look at the definition: $\forall x, y \in A$ $(f(x) = f(y) \rightarrow x = y)$

Example:

$f: \mathbf{R} \rightarrow \mathbf{R}$

$f(x) = x^2$

Disproof by counterexample:

$f(3) = f(-3)$, tapi $3 \neq -3$, so *f is not one-to-one*

Function Property

Example:

$f:\mathbf{R}\rightarrow\mathbf{R}$

$f(x) = 3x$

One-to-one: $\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

Will be shown: $f(x) \neq f(y)$ if $x \neq y$

$x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

Hence, if $x \neq y$, then $f(x) \neq f(y)$, as such, f is one-to-one.

Function Property

Definition. A function $f:A \rightarrow B$ with $A,B \subseteq R$ is strictly increasing monotonically, if

$$\forall x,y \in A \ (x < y \rightarrow f(x) < f(y)),$$

and monotonically decreasing (strictly decreasing), if

$$\forall x,y \in A \ (x < y \rightarrow f(x) > f(y)).$$

It is clear that, a strictly increasing or strictly decreasing function is one-to-one.

Properties of functions : *onto (surjective), bijection*

Definition. A function $f:A \to B$ is onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ where $f(a) = b$.

In other words, f is onto if and only if its range is the entire codomain.

Definition. A funcion $f: A \to B$ is said to be one-to-one or bijective if and only if the function is both **one-to-one** and **onto**.
It is clear that, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

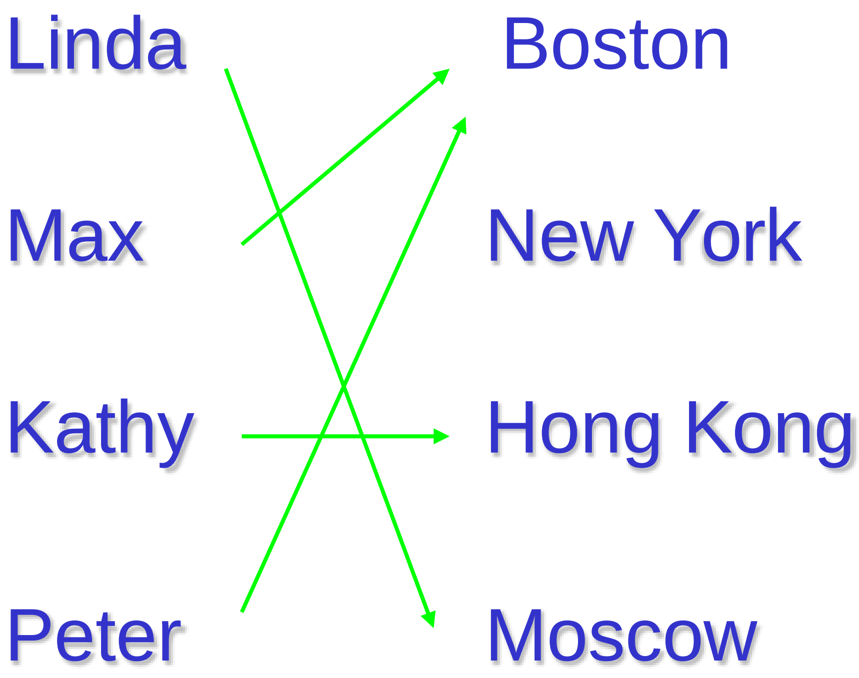
Function Properties

Examples:

In the following examples, we use arrows to represent functions $f:A\rightarrow B$.

For each example, sets A and B are shown in full.

Function Properties



Is *f* injective ? NO.

Is *f* surjective? NO.

Is *f* bijective? NO.

Function Properties

Is f injective?

NO.

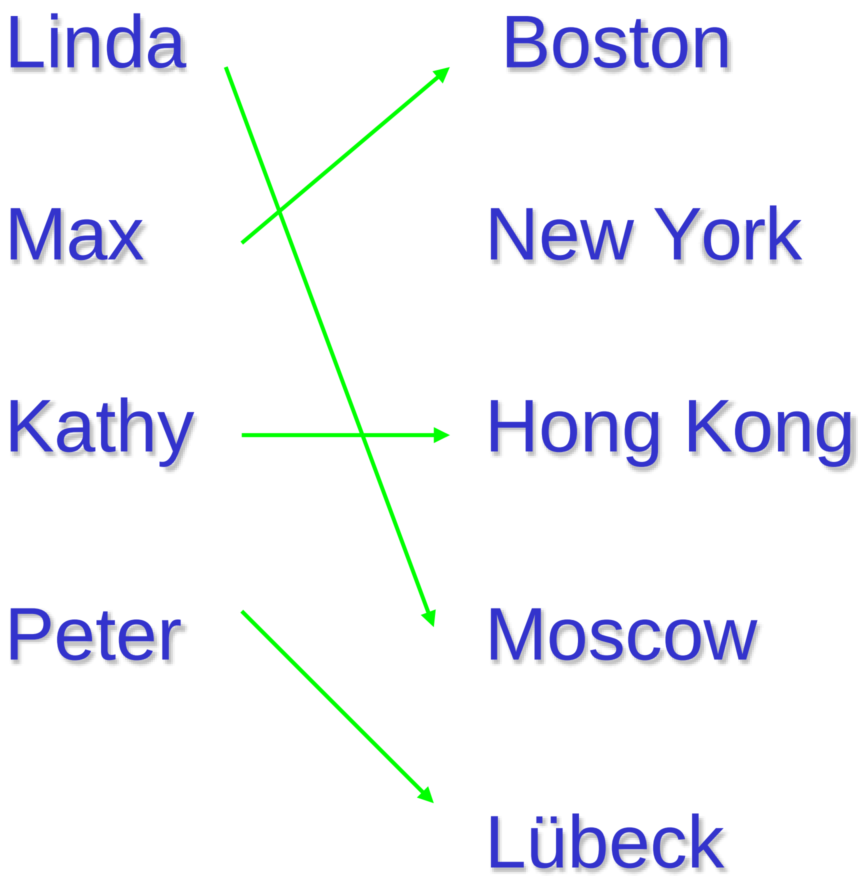
Is f surjective?

YES.

Is f bijective?

NO.

Function Properties



Is f injective?

YES.

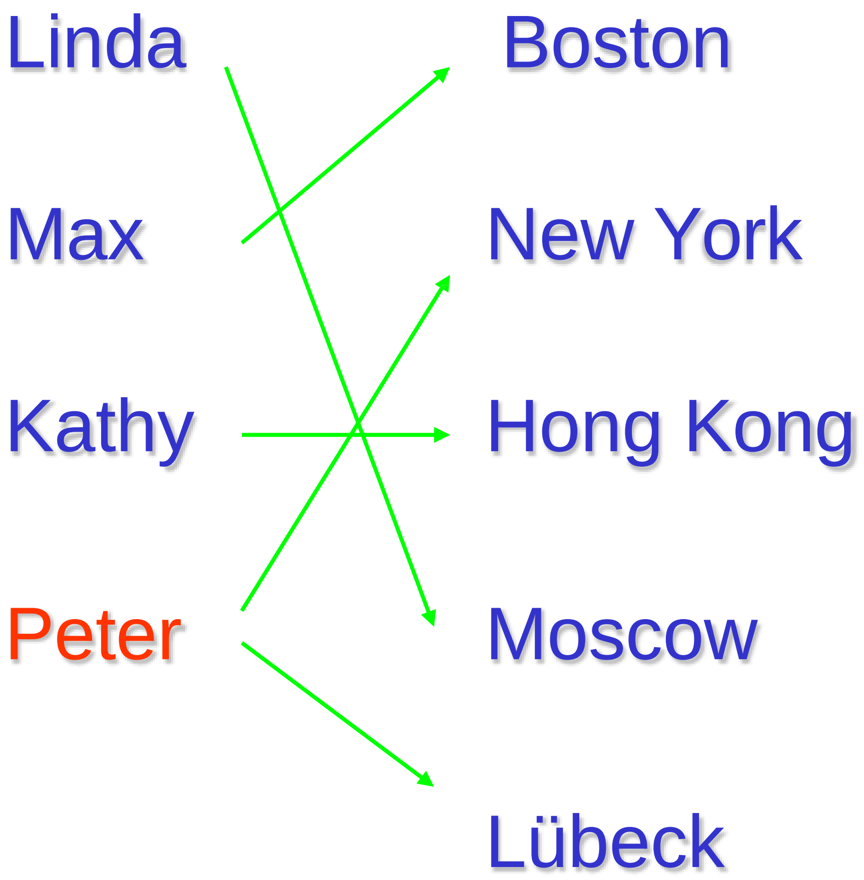
Is f surjective?

NO.

Is f bijective?

NO.

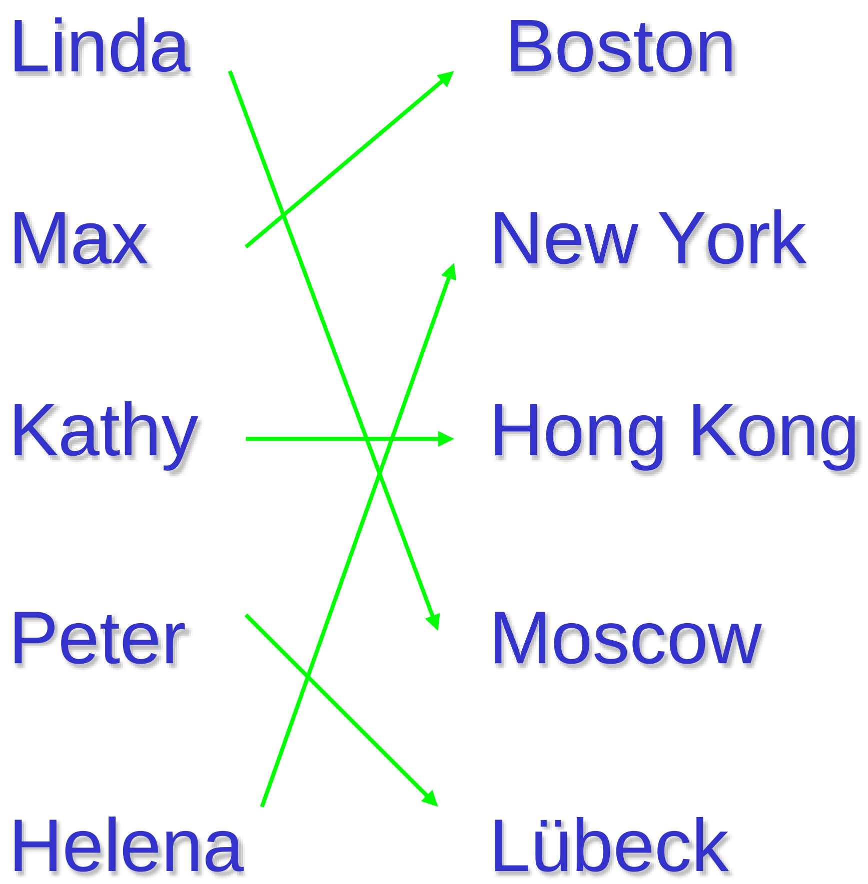
Function Properties



Is f injective? NO!

f is not even a function!

Function Properties



Is f injective?

YES.

Is f surjective?

YES.

Is f bijective?

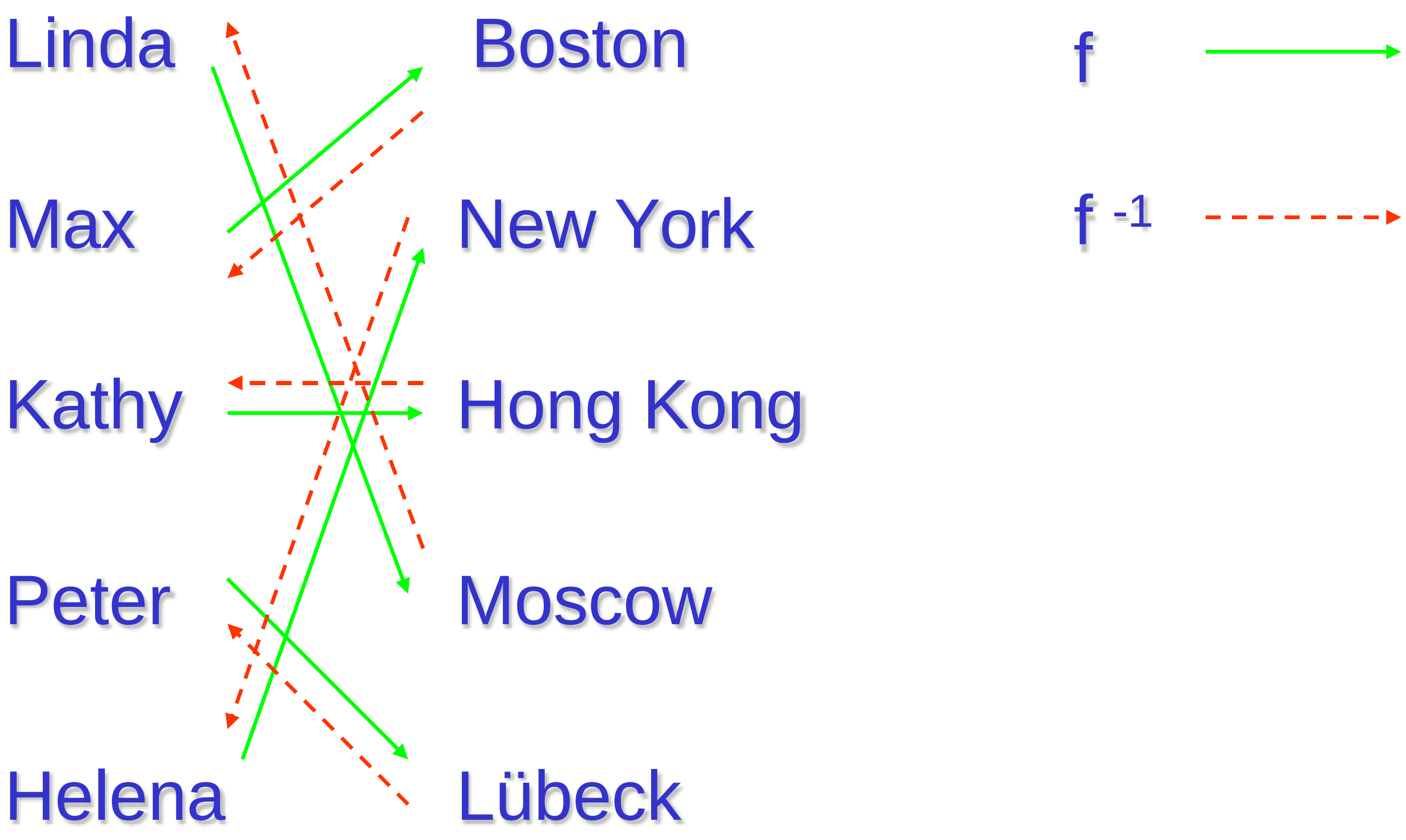
YES.

Inverse

One of the important properties of a bijective function is that it has an inverse.

Definition. Inverse function of a bijection f:A→B is a function f^{-1}:B→A with f^{-1}(b) = a for f(a) = b.

Inverse


Linda
Max
Kathy
Peter
Helena
Boston
New York
Hong Kong
Moscow
Lübeck
f
f -1

Inverse

Example:

$f(\text{Linda}) = \text{Moscow}$

$f(\text{Max}) = \text{Boston}$

$f(\text{Kathy}) = \text{Hong Kong}$

$f(\text{Peter}) = \text{Lübeck}$

$f(\text{Helena}) = \text{New York}$

Hence, f is bijective.

Inverse function f^{-1} is given as:

$f^{-1}(\text{Moscow}) = \text{Linda}$

$f^{-1}(\text{Boston}) = \text{Max}$

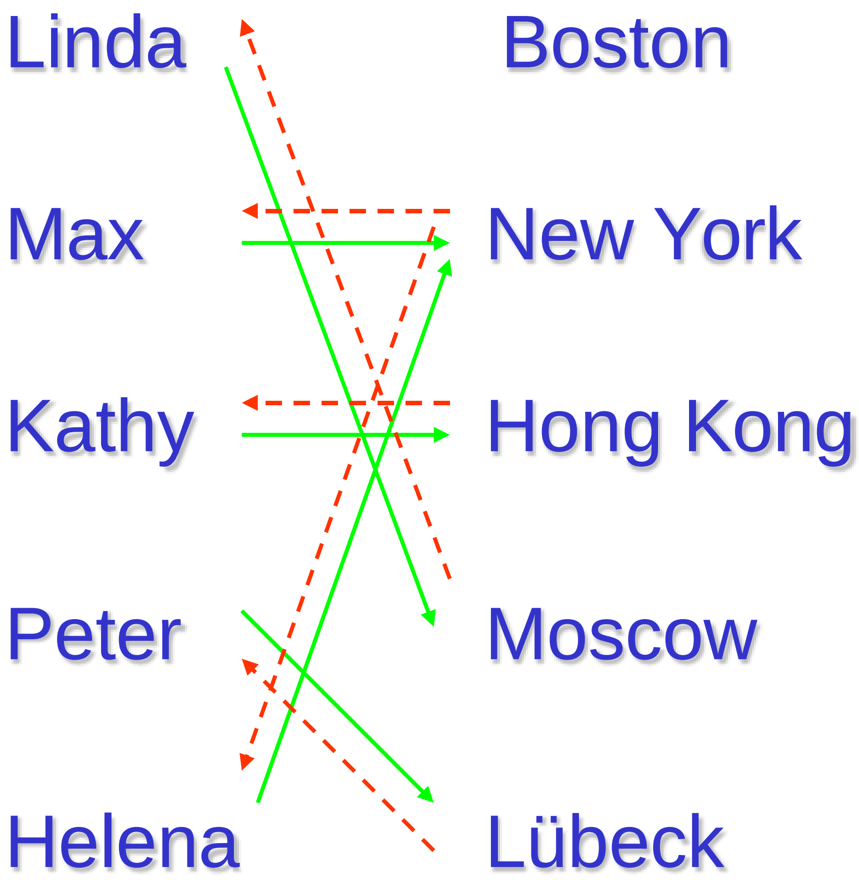
$f^{-1}(\text{Hong Kong}) = \text{Kathy}$

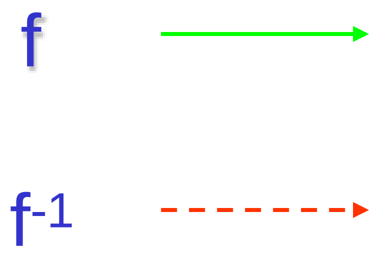
$f^{-1}(\text{Lübeck}) = \text{Peter}$

$f^{-1}(\text{New York}) = \text{Helena}$

Inversion is only possible for bijections (=invertible functions).

Inverse





$f^{-1}: C \rightarrow P$ is not a function, because not everything is defined for the C elements and assigns two images to the New York pre-image.

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Composition

The composition of two functions $g:A \to B$ and $f:B \to C$, denoted as $f \circ g$, is defined as

$$(f \circ g)(a) = f(g(a))$$

This means that
first, function g is applied to an element $a \in A$,
which maps it to an element of B,
then, function f is applied to this element of B, which maps it to an element of C.
Thus, the composite function maps from A to C.

Composition

Example:

$f(x) = 7x – 4$, $g(x) = 3x$,

$f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$

$(f\circ g)(5) = f(g(5)) = f(15) = 105 – 4 = 101$

$(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

Function composition and its invers:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

The composition of a function with its inverse is the identity function. $i(x) = x$.

Graph

Definition. Graph of a function f:A$\rightarrow$B is a set of ordered pairs $\{(a, b) \mid a \in A$ dan $f(a) = b\}$.

The graph is a subset of AXB that can be used to visualize f in a two-dimensional coordinate system.

Floor and Ceiling

floor and **ceiling** map real numbers to integers($\mathbf{R} \rightarrow \mathbf{Z}$).

floor assigns real numbers $r \in \mathbf{R}$ to the smallest integer $z \in \mathbf{Z}$ with $z \leq r$. Notation is $\lfloor r \rfloor$.

Example: $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$

ceiling assigns real numbers $r \in \mathbf{R}$ to the largest integer $z \in \mathbf{Z}$ dengan $z \geq r$. Notation is $\lceil r \rceil$.

Contoh: $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Problem

1. Calculate values for these funcitons:

   (a) $\lceil 3/4 \rceil$ (b) $\lfloor 7/8 \rfloor$ (c) $\lceil -3/4 \rceil$

   (d) $\lfloor -7/8 \rfloor$ (e) $\lceil 3 \rceil$ (f) $\lfloor -1 \rfloor$

   (g) $\lfloor 0.5 + \lceil 3/2 \rceil \rfloor$ (h) $\lfloor 0.5 \cdot \lfloor 5/2 \rfloor \rfloor$

2. The following function f maps a finite set A={a,b,c,d} to A itself. Determine whether the following f is one-to-one (injective):

   f(a) = b, f(b) = a, f(c)=c, f(d) = d

   a) f(a) = b, f(b) = b, f(c)=d, f(d) = c

   b) f(a) = d, f(b)=b, f(c)=c, f(d) = d

3. Determine from question no. 2, which one is onto (surjective) ?
4. From question no. 2, which one is bijective?