Subset (… recall)

Theorem 1. For any set, we have

- $\emptyset \subseteq A$ for any set A
- $A \subseteq A$ for any set A

Proper subset:

$A \subset B$ "A is a proper subset of B"

$$A \subset B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$$

or

$$A \subset B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge \neg \forall x\, (x \in B \rightarrow x \in A)$$

Finite set and cardinality

- **Definition**. For a set S. If there are exactly *n* different elements inside S, with *n* nonnegative integer, then S is a finite set and *n* is the cardinality of S.
- Cardinality of S is expressed as |S|.

Infinite set

- **Definisi.** A set is an infinite set if the set is not a finite set

Cardinality of a Set

Example:

A = {Mercedes, BMW, Porsche}, $|A| = 3$

B = {1, {2, 3}, {4, 5}, 6} $|B| = 4$

C = $\emptyset$ $|C| = 0$

D = $\{ x \in \mathbf{N} \mid x \leq 7000 \}$ $|D| = 7001$

E = $\{ x \in \mathbf{N} \mid x \geq 7000 \}$ E infinite!

Power Set

- **Definition**. For a set S, power set of S is a set consisting of all subsets of S. Power set of S is written as P(S) or 2^S.

Power Set

2^A or $P(A)$ "power set of A" (consists of all subsets of A)

$2^A = \{B \mid B \subseteq A\}$

Example:

$A = \{x, y, z\}$

$2^A = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

$A = \varnothing$

$2^A = \{\varnothing\}$

Note: $|A| = 0$, $|2^A| = 1$

Power Set

Cardinality of power set:

$| 2^A | = 2^{|A|}$

- Imagine every element of A has 2 possibilities "ON/OFF"
- Every possible configuration of A corresponds to one element in the power set 2^A

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For A with 3 elements, there are $2\times2\times2 = 8$ elements in 2^A

Ordered n-tupel

Definisi. An ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection with a_1 as its first element, a_2 as its second element, …, and a_n as its n^{th} element.

Two ordered n-tupels $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if both have exactly the same elements and with the same order, i.e., $a_i = b_i$ for $1 \leq i \leq n$.

For n=2, this n-tupel is called as ordered pair.

Cartesian Multiplication

Definition. Suppose that A and B are sets. Cartesian multiplication of A and B, written as $A \times B$, is all ordered pairs (a,b) where $a \in A$ and $b \in B$. Therefore,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Example

A = {x, y}

B = {a, b, c}

Hence

$A \times B$ = {(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)}

Cartesian Multiplication

Note that:

- $A \times \varnothing = \varnothing$
- $\varnothing \times A = \varnothing$
- For nonempty sets A and B : $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

Cartesian multiplication of two sets or more is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$

Union and Intersection Sets

- **Definition.** Suppose A and B are sets. The union of sets A and B, written as $A \cup B$, is the set containing elements of A, elements of B, or both.

- **Definition.** Suppose A and B are sets. The intersection of sets A and B, written as $A \cap B$, is a set containing elements that are both in A and B.

Operation on Sets

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cup B = \{a, b, c, d\}$

Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cap B = \{b\}$

Disjoint Sets and Set Difference

- **Definition.** Two sets, A and B, is said disjoint if their intersection is an empty set, that is $A \cap B = \emptyset$

- **Definition.** Consider two sets A and B. The set difference between A and B, written as A-B, is the set of elements that are in A but not in B. Hence, $A\text{-}B = \{x \mid x \in A \wedge x \notin B\}$. This is also called complement of B w.r.t. A.

Example: A = {a, b}, B = {b, c, d}, A-B = {a}

Operation on Sets

Definition. Complement of the set A, written as $\underline{A}$, is the set containing all elements in the domain U that are not in A:

$$\underline{A} = U - A$$

Example: U = **N**, B = {250, 251, 252, …}

$$\underline{B} = \{0, 1, 2, \ldots, 248, 249\}$$

Operation on Sets

How to prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

Alternative I:

$x \in A \cup (B \cap C)$

$\Leftrightarrow x \in A \vee x \in (B \cap C)$

$\Leftrightarrow x \in A \vee (x \in B \wedge x \in C)$

$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

(distributive law for mathematical logic)

$\Leftrightarrow x \in (A \cup B) \wedge x \in (A \cup C)$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Operation on Sets

Alternative II: Use members table

1 means "*x* is a member of this set"

0 berarti "*x* is not an elements of this set"

A B C	B∩C	A∪(B∩C)	A∪B	A∪C	(A∪B) ∩(A∪C)
0 0 0	0	0	0	0	0
0 0 1	0	0	0	1	0
0 1 0	0	0	1	0	0
0 1 1	1	1	1	1	1
1 0 0	0	1	1	1	1
1 0 1	0	1	1	1	1
1 1 0	0	1	1	1	1
1 1 1	1	1	1	1	1

Operation on Sets

From the examples given, we conclude that:

Every logic expression can be transformed into an equivalent expression in set theory, vice versa.

Problem

- Given two finite sets: $A=\{1,2\}$ and $B=\{2,3,4\}$.
  Calculate:
  a) $|A|$ and $|B|$
  b) $P(A)$ and $P(B)$
  c) $|P(A)|$ and $|P(B)|$
  d) $A \times B$
  e) $|A \times B|$
  f) $A \cap B$, $A-B$, and $A \cup B$

Function

Function

Definition. A function *f* from set A to set B assigns exactly one element of **B to each element of A**.

We write

$$f(a) = b$$

where for every element in A, there exists a unique element in B.

If *f* is a function from A to B, we write

$$f: A \rightarrow B$$

(Note, notation “$\rightarrow$“ is not related to the implication logical operator : if…. then…)

Function

Given f:A→B, we say A as the *domain* of f and B as the *codomain* of f.

If f(a) = b, we say b as the *image* of a and a as the *pre-image* of b.

Range of f:A→B is a set containing all images of the elements of A.

We say f:A→B maps A to B.

Function

Note the function f: P→C where

P = {Linda, Max, Kathy, Peter}

C = {Boston, New York, Hong Kong, Moscow}

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = New York

Here, the range of f is C.

Function

Suppose *f* is given as:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

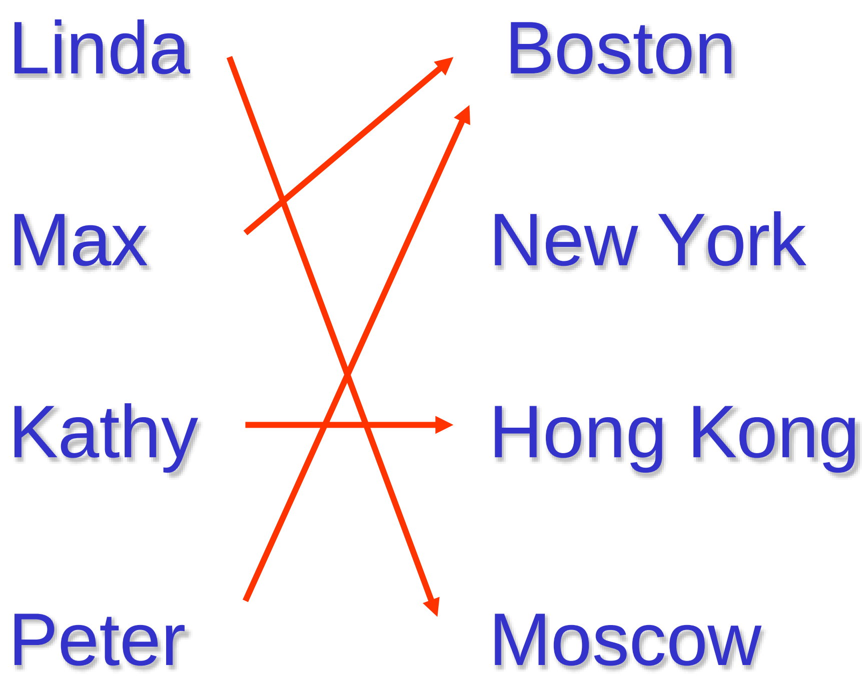
Is *f* still a funciton? yes

What is the *range*? {Moscow, Boston, Hong Kong}

Function

Other representation of f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Function

If the domain of a function is very large, it is better to define *f* by a formula, example:

$f:\mathbf{R}\rightarrow\mathbf{R}$

$f(x) = 2x$

which results in

$f(1) = 2$

$f(3) = 6$

$f(-3) = -6$

…

Function

Suppose f_1 and f_2 are functions from A to **R**.

Then the summation and multiplication of f_1 and f_2 are also functions from A to **R** defined as:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x)\, f_2(x)$$

Example:

$$f_1(x) = 3x, \quad f_2(x) = x + 5$$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$$

$$(f_1 f_2)(x) = f_1(x)\, f_2(x) = 3x\,(x + 5) = 3x^2 + 15x$$

Function

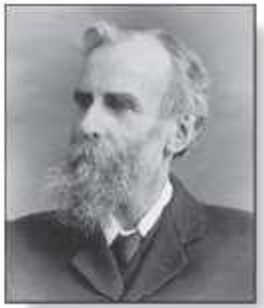
We know that *range* of a function $f:A \rightarrow B$ is the set of all images from the elements $a \in A$.

If we only consider the subset $S \subseteq A$, the set of all images from the elements $s \in S$ is the image of S.

Image of S is denoted as f(S):

$$f(S) = \{f(s) \mid s \in S\}$$

Mathematician of the day



JOHN VENN (1834–1923) John Venn was born into a London suburban family noted for its philanthropy. He attended London schools and got his mathematics degree from Caius College, Cambridge, in 1857. He was elected a fellow of this college and held his fellowship there until his death. He took holy orders in 1859 and, after a brief stint of religious work, returned to Cambridge, where he developed programs in the moral sciences. Besides his mathematical work, Venn had an interest in history and wrote extensively about his college and family.

Venn's book *Symbolic Logic* clarifies ideas originally presented by Boole. In this book, Venn presents a systematic development of a method that uses geometric figures, known now as *Venn diagrams*. Today these diagrams are primarily used to analyze logical arguments and to illustrate relationships between sets. In addition to his work on symbolic logic, Venn made contributions to probability theory described in his widely used textbook on that subject.