Proposition and Propositional Function

Propositional function (open sentence):

A statement containing one or more variables

Example: $x - 3 > 5$.

Suppose we call this propositional function $P(x)$, where P is the predicate and x is the variable.

What is the truth value of $P(2)$? False

What is the truth value of $P(8)$? False

What is the truth value of $P(9)$? True

Propositional function

Consider the propositional function Q(x, y, z) defined:

x + y = z.

Here, Q is the predicate and x, y, and z are the variables.

What is the truth value of Q(2, 3, 5) ? True

What is the truth value of Q(0, 1, 2) ? False

What is the truth value of Q(9, -9, 0) ? True

Universal Quantification

Consider the propositional function P(x).

Definition 1. The universal quantifier of P(x) is the proposition "P(x) is true for all values of x in the universe of discourse",

or

"For all x in the universe of discourse, P(x) holds"

With the universal quantifier □:

$\forall x$P(x) "for all x P(x)" or "for every x P(x)"

(Note: $\forall x$P(x) can be either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Example:

S(x): x is (an) ITB student.
G(x): x is a smart person.

What is the meaning of $\forall x\ (S(x) \rightarrow G(x))$?

“If x is an ITB student, then x is a smart person” or “All ITB students are smart.”

Universal Existential

Definition 2. The existential quantification of P(x) is the proposition
"There is an element x in the universe of discourse such that P(x) is true"

By existential quantification **:**$\exists$:

$\exists$x P(x) "There is an x such that P(x)."
"There is at least one x such that P(x)."

(Note: $\exists$x P(x) can be true or false, so it is a proposition, but not a propositional function.)

Universal Existential

Example:

P(x): x is an Indonesian.

G(x): x is a smart person.

What does it mean $\exists x\ (P(x) \wedge G(x))$?

“There is x such that x is an Indonesian and x is a smart person.”

or

“There is at least one Indonesian who is smart.”

Quantification

Another example:

Suppose the universe of discourse is the real numbers.

What is the meaning of $\forall$**x**$\exists$**y (x + y = 320)** ?

"For every x there is a y such that **x + y = 320**."

Is this statement true? **Yes**

Is this true (applicable) for whole numbers? **No**

Counterexample

The counterexample of $\forall$ x P(x) is an object c such that P(c) is false.

Such statements $\forall$x (P(x) $\rightarrow$ Q(x)) can be disproved simply by providing their counterexample.

The statement: “All birds can fly.” is refuted by the refutation: Penguins.

Negation

$\neg(\forall x\ P(x))$ logically equivalent with $\exists x\ (\neg P(x))$.

$\neg(\exists x\ P(x))$ logically equivalent with $\forall x\ (\neg P(x))$.

Mathematician of the day

BERTRAND RUSSELL (1872–1970) Bertrand Russell was born into a prominent English family active in the progressive movement and having a strong commitment to liberty. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science. He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica,* written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for literature in 1950.

Set

Although they may seem different at first glance, mathematical logic and set theory actually have a very close relationship.

What is a Set ?

- **Definition 1.** A set is an unordered collection of objects.

- **Definition 2.** Objects in a set are also called elements or members of the set. A set is said to contain elements.

Himpunan

- $a \in A$ — "a is an element of A" "a is a member of A"
- $a \notin A$ — "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ — "A contains …"

– The order in which the elements are mentioned does not matter.

– How often the same element is mentioned does not matter.

Kesamaan Himpunan

Definition 3. Sets A and B are said to be equal if and only if they have exactly the same elements.

Example:

- $A = \{9, 2, 7, -3\}$, $B = \{7, 9, -3, 2\}$: $A = B$

- $A = \{dog, cat, horse\}$,

  $B = \{cat, horse, squirrel, dog\}$: $A \neq B$

- $A = \{dog, cat, horse\}$,

  $B = \{cat, horse, dog, dog\}$: $A = B$

Examples of Set

Standard set:

Natural numbers **N** = {0, 1, 2, 3, …}

Integers **Z** = {…, -2, -1, 0, 1, 2, …}

Positive Integers **Z+** = {1, 2, 3, 4, …}

Real numbers **R** = {47.3, -12, π, …}

Rational numbers **Q** = {1.5, 2.6, -3.8, 15, …}
(the “exact” definition will be discussed later)

Examples of Set

- $A = \emptyset$ "empty set/zero set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$
- $A = \{\{x, y\}\}$
  Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid P(x)\}$
  "the set of all x such that P(x)"
- $A = \{x \mid x \in \mathbf{N} \land x > 7\} = \{8, 9, 10, \ldots\}$
  "set formation notation"

Examples of Set

Now we define rational set Q:

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z}^{+}\}$

atau

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z} \wedge b \neq 0\}$

Bagaimana dengan bilangan riil R?

$\mathbf{R} = \{r \mid r \text{ is riil}\}$

Yet no other way to define it better.

Subset

- **Definition 4**. A set A is called a subset of B if and only if every element of A is also an element of B. We write $A \subseteq B$ to denote that A is a subset of the set B.

Subset

$A \subseteq B$ "A is a subset of B"

$A \subseteq B$ if and only if every element of A is also an element of B.

Formally written as

$A \subseteq B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$

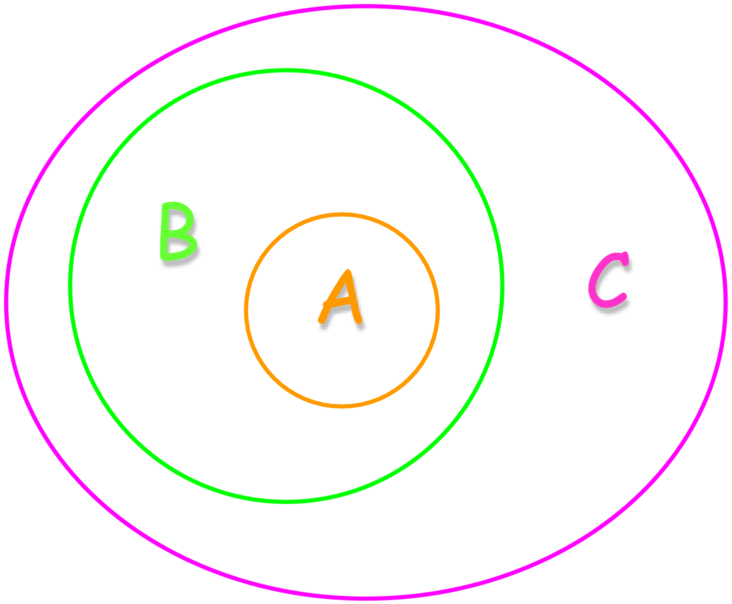
Example:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$,	$A \subseteq B$?	True
$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$,	$A \subseteq B$?	True
$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$,	$A \subseteq B$?	False

Subset

Some useful rules:

- $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (see Venn Diagram)



Subset

Some useful rules:

- $\emptyset \subseteq A$ for any set A
- $A \subseteq A$ for any set A

Proper subset:

$A \subset B$ "A is a proper subset of B"

$$A \subset B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$$

or

$$A \subset B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge \neg \forall x\, (x \in B \rightarrow x \in A)$$