MAT 1130: MATHEMATICAL IDEAS

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MAT 1130

MATHEMATICAL IDEAS

Revised for Spring 2025

Mirtova and Jones

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About Mathematical Ideas

Written and Compiled by

Helena Mirtova, Ph.D. and Andy D. Jones

Mathematics Department, Prince George's Community College

Mathematical Ideas was developed for the MAT 1130 *Mathematical Ideas* course at Prince George's Community College as an open educational resource (OER). MAT 1130 is a general education course at Prince George's Community College.

This course is a survey of contemporary mathematics useful for students in programs not requiring a mathematics course that emphasizes algebra, calculus, or statistics. It develops mathematical literacy using a variety of everyday problems which can be modeled and solved by quantitative means. Students apply formal logic to analyze reasoning, learn techniques to organize and analyze data using set theory and descriptive statistics, and evaluate chance using probability. Mathematics is also applied to personal finance and societal issues such as voting and apportionment. Technology is incorporated across all topics. This is a terminal course and does not prepare students for algebra, trigonometry, physical or life sciences, engineering, or business courses.

This text addresses the basic ideas and content required by the course outcomes for MAT 1130:

1. Analyze information with tools of set theory.
2. Analyze validity of statements and arguments using formal logic.
3. Apply appropriate formulas and technology to personal finances.
4. Calculate basic and conditional probabilities, odds and expected values.
5. Interpret quantitative data using descriptive statistics.
6. Examine voting and apportionment methods.

This text includes a collection of review exercises for each chapter. It has also been paired with practice exercises for each section in *MyOpenMath*.

Although a major portion of *Mathematical Ideas* is original work by the authors, some parts were derived from other Open Education Resources licensed under CC BY- SA and CC BY-NC. These works have been adapted and revised. Links to the original works are shown throughout the text. A special thanks is offered to these authors:

- *Math in Society* (David Lippman)
- *Math in Society: Mathematics for Liberal Arts Majors* (Jess Brooks, Cara Lee, Sonya Redmond, Cindy Rochester-Gefre)
- *College Mathematics for Everyday Life* (Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, & Kim Sonier)

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CHAPTER OVERVIEW

1: Sets

1.1: Basics of Sets

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects. Individual objects in a set are called **elements,** or members, of the set.

For example, the name of each month is an element of the set of months in a calendar year. The elements of the set are listed inside a pair of braces, $\{\ \}$. The set of months in a calendar year can be written in set notation as follows:

A = {*January, February, March, April, May, June, July, August, September, October, November, December*}.

A set simply specifies the contents so the order in which elements are written makes no difference. Each element is listed only once. Thus, {*March, April, May*} and {*May, April, March*} are considered to be the same set. If two sets contain exactly the same elements, the sets are **equal**.

Notation

Commonly, capital letters are used to name sets to make it easier to refer to that set later.

We show that an element belongs to a set by using the symbol $\in$, which means "is an element of." The symbol $\notin$ means "is not an element of."

A set that contains no elements, $\{\ \}$, is called the **empty set** and is notated $\emptyset$.

Example 1

Let $A = \{1, 2, 3, 4\}$.

To show that 2 is an element of the set A, we'd write $2 \in A$.

To show that 5 is not an element of the set A, we'd write $5 \notin A$.

Sets must be **well-defined**. That is, it must be very clear if an object is an element of the set or not.

For example, the set of letters in the word *ghost* is well-defined because it is clear that the letter *g* is an element of the set and the letter *r* is not an element of the set. However, the set of all diligent students in our class is not well-defined because diligence is matter of perception. Likewise, the set of all days last year with high temperatures below 32 degrees Fahrenheit is well-defined. However, the set of all cold days last year is not well-defined because is whether a day is cold not is subjective.

Sets can be described in three common ways: using a verbal description, listing its elements in roster form, and using set builder notation as seen in these examples.

Example 2

1. Sets can be defined by describing its contents in words. For example,
   a. The set of all even numbers
   b. The set of all books written about travel to Chile
2. Sets can be defined by listing the elements in roster form using set braces. For example,
   a. $\{1, 3, 9, 12\}$
   b. $\{\text{red}, \text{orange}, \text{yellow}, \text{green}, \text{blue}, \text{indigo}, \text{purple}\}$
   c. $\{1, 4, 9, 16, 25, 36, 49, \ldots\}$ This is an example of an *infinite set*. The ellipsis (...) indicates that the set continues in this pattern and there is no end.
3. Sets can be defined using set builder notation. For example,

a. $B = \{x|x \in N \text{ and } 1 < x \leq 8\}$ means Set B is the set of natural numbers greater than 1 but less than or equal to 8, or $B = \{2, 3, 4, 5, 6, 7, 8\}$.
b. $C = \{2x + 1 \text{ where } x = 3, 4, 5\}$ means $C = \{7, 9, 11\}$.

Try it Now 1

Let $D = \{n|n \in N \text{ and } n \text{ is an odd number}\}$. Write set D in roster notation and using a verbal description.

Answer

$D = \{1, 3, 5, 7, 9, 11, \ldots\}$Set D is the set of all odd numbers. This is another example of an infinite set.

Sometimes it is important to know how many elements are in a set. The **cardinality** of a set S, denoted $n(S)$, indicates the number of elements in set S. If $S = \{a, b\}$, the cardinality of set S is 2, and this is written $n(S) = 2$.

Cardinality

The number of elements in a set is the **cardinality** of that set.

The cardinality of the set A is often notated $n(A)$.

Example 3

What is the cardinality of P, the set of English names for the months of the year?

Solution

The cardinality of this set is 12, since there are 12 months in the year. We can write $n(P) = 12$.

Two sets are said to be **equivalent** set if they have both have the same cardinality. That is, set A is equivalent to set B if $n(A) = n(B)$. We write $A \cong B$ to show these two sets are equivalent.

The term *equivalent* should not be confused with *equal*. Remember that two sets are equal when they contain exactly the same elements. The difference between the terms *equal* and *equivalent* is demonstrated in the next example.

Example 4

Consider the four sets: $A = \{p, q, r, s\}$ $\quad B = \{a, b, c\}$ $\quad C = \{x, y, z\}$ $\quad D = \{b, a, c\}$.

Compare the sets using the terms *equal* and *equivalent*.

Solution

- Sets A and B: These sets are not equivalent ($A \ncong B$) because they have different cardinality. They are not equal ($A \neq B$) because they do not contain exactly the same elements.
- Sets B and C: These sets are equivalent ($B \cong C$) because they both have cardinality 3. They are not equal ($B \neq C$) because they do not contain exactly the same elements.
- Sets B and D: These sets are equivalent ($B \cong D$) because they both have cardinality 3. They are equal ($B = D$) because they contain exactly the same elements.
- Sets C and D: These are equivalent ($C \cong D$) because they both have cardinality 3. They are not equal ($C \neq D$) because they do not contain exactly the same elements.

Try it Now 2

Consider the five sets:

$P = \{a, b, c, d\}$ $\quad Q = \{x, y, z, w\}$ $\quad R = \{c, d, a, b\}$ $\quad S = \{z, y, x\}$ $\quad T = \{x|x \in N \text{ and } 1 \leq x \leq 4\}$

a. Which sets are equal?
b. Which sets are equivalent?

Answer

a. $P = R$
b. $P \cong Q$, $P \cong R$, $P \cong T$, $Q \cong R$, and $Q \cong T$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$.

More formally,

1. If every element of set B is also an element of set A, then $B \subseteq A$.
2. If $B \subseteq A$, then from $x \in B$ it follows that $x \in A$.

A **proper subset** is a subset that is not identical to the original set.

If B is a proper subset of A, we write $B \subset A$.

Example 5

Consider these three sets: $A =$ the set of all even numbers $\quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$.

Here, $B \subset A$. Every element of B is also an even number and is also an element of A. However, there are elements in set A that are not in set B.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A.

Example 6

Suppose a set contains the plays "Much Ado About Nothing", "MacBeth", and "A Midsummer's Night Dream". What is a larger set this might be a subset of?

Solution

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all works of British literature.

Try it Now 3

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Answer

There are several answers. For example,

- The set of all odd numbers less than 10.
- The set of all odd numbers.
- The set of all integers.
- The set of all real numbers.

Sometimes it is useful to list all the subsets of a set. In other instances, it is important to know how many subsets there will be without listing them all. To develop a formula for the number of subsets of a set, it will be helpful to consider a few examples and look for a pattern.

- **A set with no elements**: The empty set $\{ \}$ has only one subset -- itself.

- **A set with one element:** Suppose we add one more element, a, to the empty set to form the set $\{a\}$. This new set still has the subset $\{\ \}$ from the previous set. However, there is also a new subset, $\{a\}$. So, a set with one element, has two subsets.
- **A set with two elements:** Suppose we add one more element, b, to the previous set to form the set $\{a, b\}$. This new set has subsets $\{\ \}$, $\{a\}$, $\{b\}$, and $\{a, b\}$. So, a set with two elements, has four subsets.
- **A set with three elements:** Suppose we add one more element, c, to the previous set to form the set $\{a, b, c\}$. This new set has subsets $\{\ \}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. So, a set with three elements, has eight subsets.

The table summarizes the number of subsets that can be formed based on the cardinality of the set.

How Number of Elements Determines Number of Subsets

Number of elements in set	Number of subsets
0	1
1	2
2	4
3	8

Notice that each time an extra element is included in a set, its number of subsets double. Since the empty set has one subset and each additional element doubles the number of subsets, a set with n elements has 2^n subsets.

Number of Subsets

If set A has n elements, then set A has 2^n subsets.

Example 7

Suppose the set V is the set of all vowels in the English alphabet: $V = \{a, e, i, o, u\}$.

How many subsets can be formed using the elements of set V?

Solution

Set V contains 5 elements. The number of subsets that can be formed can be found using $2^n = 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.

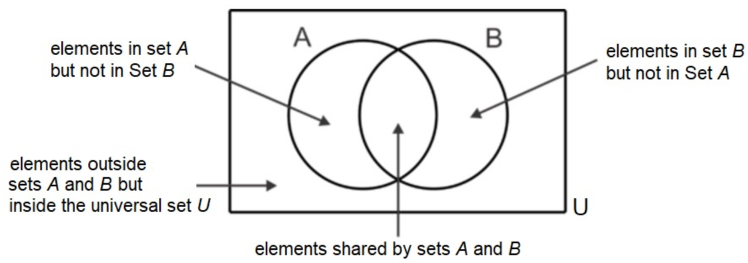
1.2: Operations with Sets

Sets can interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

To visualize the interactions and operations with sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

Venn Diagram

A **Venn diagram** represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas of circles indicate elements common to both sets. Non-overlapping areas of circles indicate that the sets have no elements in common.



Universal Set

The **universal set** is a set that contains all elements of interest and is usually denoted, U. The universal set is defined by the context of the problem.

Example 1

a. If we were searching for books, the universal set might be all the books in the library.
b. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
c. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers.

There are three common operations that can be performed on sets: *complement*, *union*, and *intersection*. Each of these operations is defined below and is illustrated using a shaded area of a Venn diagram.

The universal set is necessary to find the complement of set.

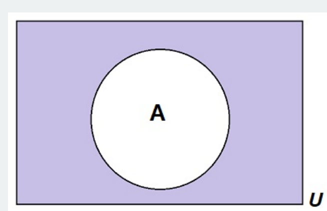
Complement

The **complement** of a set contains everything that is *not* in the set but still within the universal set.

The complement of set A is notated A'.

More formally, we write $x \in A'$ if $x \in U$ and $x \notin A$.

The shaded region of the Venn diagram to the right shows A'.



Note: Occasionally you may see other notation such as A^c and $\overline{A}$ used to represent the complement of A.

Example 2

Suppose the universal set is $U =$ all whole numbers from 1 to 9.

If $A = \{ 1, 2, 4 \}$, then $A' = \{ 3, 5, 6, 7, 8, 9 \}$.

The union and intersection are operations that work on two sets.

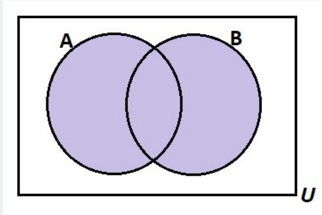
Union and Intersection

The **union** of two sets contains all the elements contained in either (or both) sets.

The union of sets A and B is notated $A \cup B$.

More formally, we write $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both).

The shaded region of the Venn diagram to the right shows $A \cup B$.

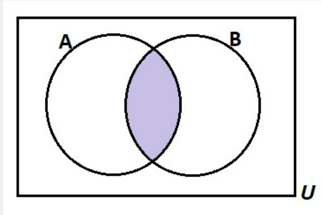


The **intersection** of two sets contains only the elements that are in both sets.

The intersection of sets A and B is notated $A \cap B$.

More formally, we write $x \in A \cap B$ if $x \in A$ and $x \in B$.

The shaded region of the Venn diagram to the right shows $A \cap B$.



Example 3

Suppose the universal set is the letters in the word *elastic:* $U = \{ \text{e, l, a, s, t, i, c} \}$.

Consider these sets: $A = \{ \text{s, c, a, l, e} \}$ $B = \{ \text{c, a, t} \}$

a. Find $A \cup B$.
b. Find $A \cap B$.
c. Find A'.

Solution

a. The union contains all the elements in either set:

$A \cup B = \{ \text{s, c, a, l, e, t} \}$.

Notice we only list *c* and *a* once.

b. The intersection contains all the elements in both sets:

$A \cap B = \{ \text{c, a} \}$.

c. Here look for all the elements that are not in set A but still in U:

$A' = \{ \text{t, i} \}$.

Even though letters like *f* and *g* are not in set A, they cannot be in A' because *f* and *g* are not in the universal set.

Example 4

Consider the sets:

$A = \{ \text{red, green, blue} \}$ $B = \{ \text{red, yellow, orange} \}$ $C = \{ \text{red, orange, yellow, green, blue, purple} \}$

a. Find $A \cup B$.
b. Find $A \cap B$.
c. Find $A' \cap C$.

Solution

a. The union contains all the elements in either set:

$$A \cup B = \{\text{ red, green, blue, yellow, orange }\}.$$

b. The intersection contains all the elements in both sets:

$$A \cap B = \{\text{ red }\}.$$

c. Here we're looking for all the elements that *are not* in set A but *are* in set C:

$$A' \cap C = \{\text{ orange, yellow, purple }\}.$$

Try it Now 1

Using the sets from the previous example, find $A \cup C$ and $B' \cap A$

Answer

$A \cup C = \{\text{ red, orange, yellow, green, blue purple }\}$

$B' \cap A = \{\text{ green, blue }\}$

As we saw earlier with the expression $A' \cap C$, set operations can be grouped together. Grouping symbols can be used with sets like they are with arithmetic - to force an order of operations. When there are multiple set operations to perform, they are performed in the following order:

1. First, perform any operation within parentheses, ( ).
2. Then, find the complement.
3. Next, perform the union $\cup$ and the intersection $\cap$ in order from left to right.

✓ Example 5

Suppose

$U = \{1, 2, 3, 4, 5, 6, 7, 8\} \quad A = \{2, 4, 7\} \quad B = \{1, 2, 3, 8\}$

a. Find $(A \cup B)'$.
b. Find $A' \cup B'$.

Solution

a. We start with the grouping symbols and find the union of set A and set B: $A \cup B = \{1, 2, 3, 4, 7, 8\}$.

Now find the complement of that result with reference to the universal set: $(A \cup B)' = \{5, 6\}$.

b. We start by finding the complements of sets A and B: $A' = \{1, 3, 5, 6, 8\}$ and $B' = \{4, 5, 6, 7\}$.

Now, union the two results: $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8\}$.

✓ Example 6

Suppose

$H = \{\text{ cat, dog, rabbit, mouse }\} \quad F = \{\text{ dog, cow, duck, pig, rabbit }\} \quad W = \{\text{ duck, rabbit, deer, frog, mouse }\}$

a. Find $(H \cap F) \cup W$
b. Find $H \cap (F \cup W)$
c. Find $(H \cap F)' \cap W$

Solution

a. We start with the intersection: $H \cap F = \{\text{ dog, rabbit }\}$.

Now we union that result with W : $(H \cap F) \cup W = \{\text{dog, rabbit, duck, deer, frog, mouse }\}$.

b. We start with the union: $F \cup W = \{\text{dog, cow, duck, pig, rabbit, deer, frog, mouse}\}$.

Now we intersect that result with H : $H \cap (F \cup W) = \{\text{ dog, rabbit, mouse }\}$.

c. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$.

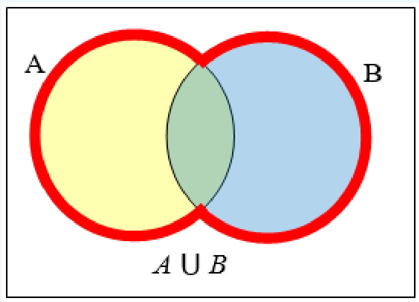
Now we want to find the elements of W that are not in $\mathrm{H} \cap F$: $(H \cap F)' \cap W = \{\text{ duck, deer, frog, mouse }\}$.

Sometimes it is useful to represent set operations using Venn diagrams when the elements of the sets are unknown or the number of elements in the sets is too large. Basic Venn diagrams can illustrate the interaction among two or three sets.

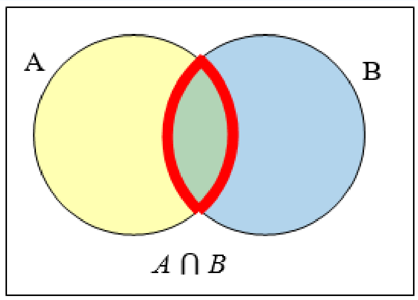
Example 7

Use Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A' \cap B$.

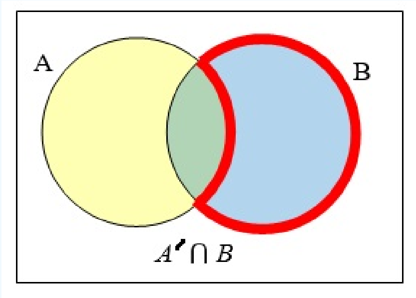
$A \cup B$ contains all elements in either set (or both.)



$A \cap B$ contains only those elements in both sets - in the overlap of the circles.



$A' \cap B$ contains those elements that *are not in* set A but *are* in set B.

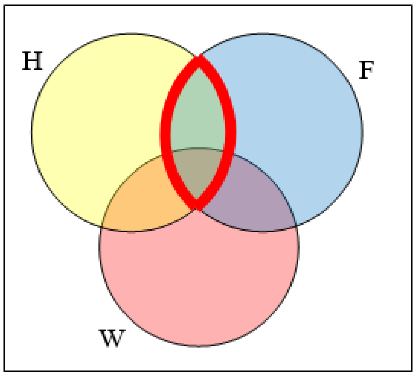


Example 8

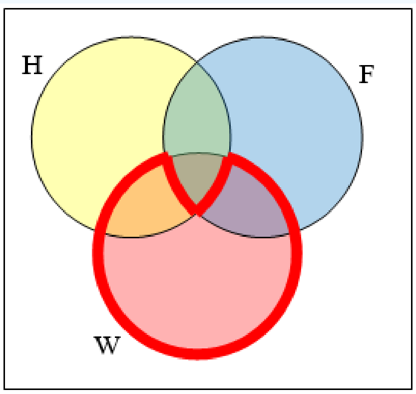
Use a Venn diagram to illustrate $(H \cap F)' \cap W$.

Solution

We'll start by identifying everything in the set $H \cap F$.



Now, $(H \cap F)' \cap W$ will contain everything not in the region shown above but that is in set W.

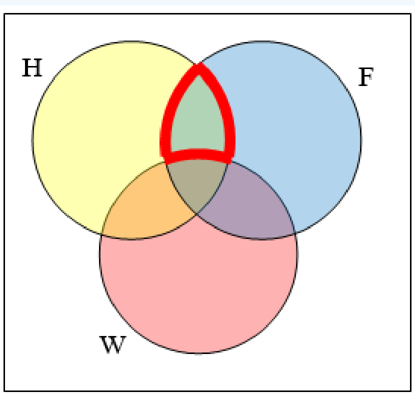


Example 9

Write an expression to represent the outlined part of the Venn diagram shown.

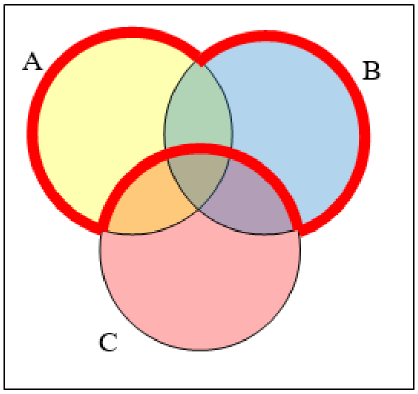
Solution

The elements in the outlined set are in sets H and F, but are not in set W. So we could represent this set as $(H \cap F) \cap W'$.



Try it Now 2

Write an expression to represent the outlined portion of the Venn diagram shown:



Answer

$(A \cup B) \cap C'$

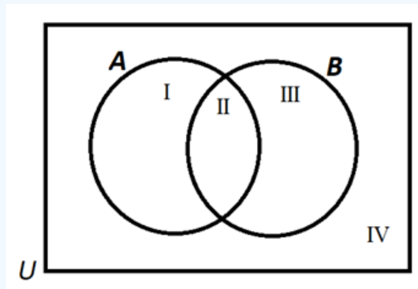
1.3: Applications of Sets

A Venn diagram is a useful tool in showing the relationship between the actual elements of sets.

✓ Example 1

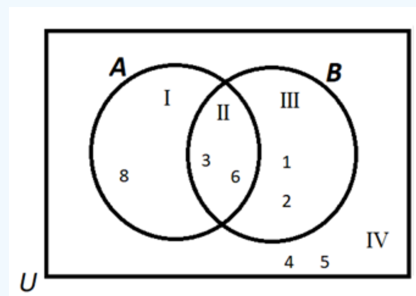
Consider the universal set $U = \{1, 2, 3, 4, 5, 6, 8\}$.

Place the elements in sets $A = \{3, 6, 8\}$ and $B = \{1, 2, 6, 3\}$ in the Venn diagram.



Solution

All elements in the universal set U must appear in one of the 4 regions labeled I, II, III, and IV. We must decide where each element 1, 2, 3, 4, 5, 6, and 8 go.



- The element 1 appears in set B but not in set A. It should be placed in III.
- The element 2 appears in set B but not in set A. It should be placed in III.
- The element 3 appears in both set A and in set B. It should be placed in II.
- The elements 4 and 5 appear in neither set A nor set B. They should be placed in IV.
- The element 6 appears in both set A and in set B. It should be placed in II.
- The element 8 appears in set A but not in set B. It should be placed in I.

The completed Venn diagram is shown at the right.

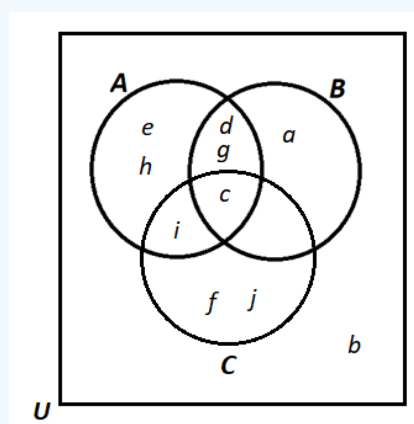
✓ Example 2

Construct a Venn diagram illustrating the following sets:

$U = \{a, b, c, d, e, f, g, h, i, j\}$ $A = \{c, d, e, g, h, i\}$ $B = \{a, c, d, g\}$ $C = \{c, f, i, j\}$

Solution

All elements in the universal set U must appear in one of 8 regions of a three-set Venn diagram.



- The element *a* belongs to set B but not to either of the other sets.
- The element *b* does not belong to any of the 3 sets but it is part of the universal set.
- The element *c* belongs to all three sets.
- The element *d* belongs to set A and set B, but not to set C.
- The element *e* belongs to set A but not to either of the other sets.
- The element *f* belongs to set C but not to either of the other sets.
- The element *g* belongs to set A and set B, but not to set C.
- The element *h* belongs to set A but not to either of the other sets.
- The element *i* belongs to set A and set C, but not to set B.
- The element *j* belongs to set C but not to either of the other sets.

The completed Venn diagram is shown at the right.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in application problems involving surveys.

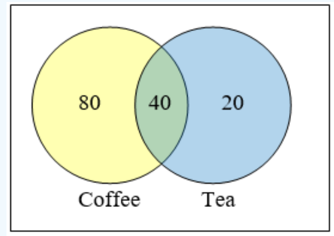
Example 3

A survey asks 200 people *What beverage do you drink in the morning?*, and offers these choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Solution



This question can most easily be answered by creating a Venn diagram. We can see that we can find the number of people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200:

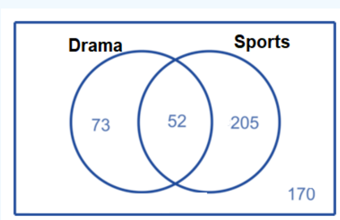
So we have $200-20-80-40=60$ people drink neither.

Try it Now 1

At a school of 500 students, there are 125 students in the drama club, 257 students who play sports, and 52 students who are in the drama club and play sports. Use a Venn diagram to illustrate this information. Then, determine how many students

a. play sports but are not in the drama club
b. play sports or are in the drama club or both

Answer



a. 205
b. 330

Example 4

A survey asks *Which online services have you used in the last month?*

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Solution

Let T be the set of all people who have used Twitter and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply $70\% + 40\%$, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we add the cardinality of F and the cardinality of T, and then subtract those in intersection that we've counted twice.

In symbols,

$n(F \cup T) = n(F) + n(T) - n(F \cap T)$

$n(F \cup T) = 70\% + 40\% - 20\%$

$n(F \cup T) = 90\%$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)'$. Since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)'$ must be the other 10%.

The previous example illustrates two important properties.

Cardinality properties

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A') = n(U) - n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Try it Now 2

In a group of 120 students, 35 were enrolled in art class and 57 were enrolled in dance class. There were 12 students who were enrolled in both classes. How many students were enrolled in neither of the classes?

Answer

40

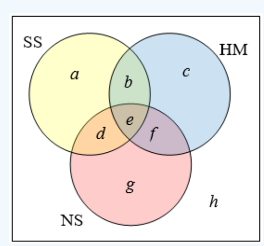
Example 5

Fifty students were surveyed and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are taking only a SS course?

Solution



It might help to look at a Venn diagram and complete the cardinality for each of the 8 regions of the diagram. The question here is how many elements are in region a.

From the given data, we know that there are 3 students in region e and 7 students in region h.

Because 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. Since we know there are 3 students in region e, there must be $7 - 3 = 4$ students in region d.

Similarly, since there are 10 students taking HM and NS, which includes regions e and f, there must be $10 - 3 = 7$ students in region f.

Since 9 students were taking SS and HM, there must be $9-3=6$ students in region b.

Now, we know that 21 students were taking a SS course. This includes students from regions a, b, d, and e. Since we know the number of students in all but region a, we can determine that $21-6-4-3=8$ students are in region a.

There are 8 students are taking only a SS course.

Try it Now 3

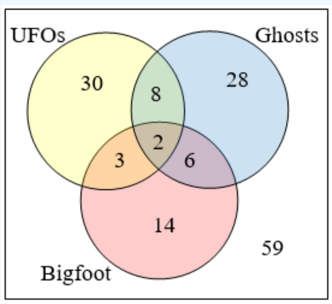
One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in none of these things?

Answer

Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. Continue filling in all the regions. Once the cardinality for each region in the circles is completed, add the cardinalities to get 91 people in the union of all three sets. This leaves $150-91=59$ people who believe in none.



1.4: Review Exercises

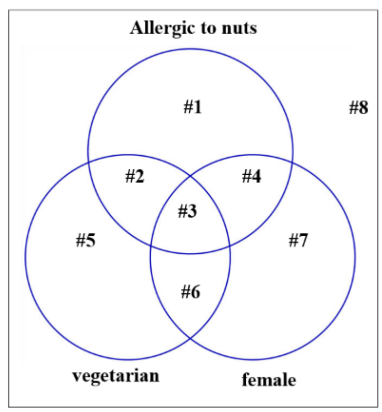
1. Set A is the set of letters in the word *mississippi*. Present set A in roster form and give $n(A)$.

2. Write a verbal description of the set $V=\{a, e, i, o, u\}$. Then, find $n(V)$.

3. Present set B in roster notation: $B=\{x \mid x \in \mathbb{N} \text{ and } 3 \leq x<10\}$. Then, find $n(B)$.

4. Consider sets $A=\{1,2,3,5\}$, $B=\{1,2, a\}$, $C=\{a, b, c, d\}$ and $D=\{1, a, 2\}$.
   a. Determine if the following statements are true or false.
      i. $b \in C$
      ii. $1 \notin A$
      iii. $4 \notin A$
      iv. $B \subseteq D$
      v. $B \subset D$
      vi. $B \subset A$
      vii. $B=D$
      viii. $A \cong C$
      ix. $A=C$
   b. Make up your own set E so that set E is equivalent but not equal to set B.
   c. List all subsets of set B. How many of these are proper subsets of set B?
   d. How many subsets does set C have?

5. The Venn diagram represents the sets of guests at a dinner party.
   a. Lucy is allergic to nuts but not vegetarian. In which region does Lucy belong in this Venn diagram?
   b. Darren is a vegetarian who is not allergic to nuts. In which region would Darren belong in this Venn diagram?
   c. Describe a guest who belongs in region #6.



6. Draw and label a 3-set Venn diagram so that it shows how these sets are related:

   $U=\{1,2,3,4,5,6, a, b, c\}$

   $A=\{1,3, a, b\}$

   $B=\{1,2, a\}$

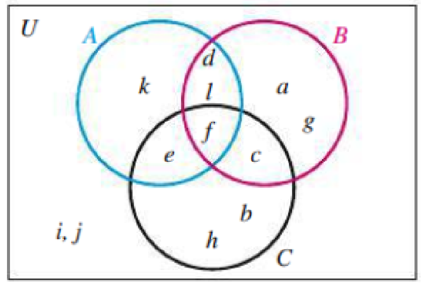
   $C=\{3,4,5, b\}$

7. For the universal set $U=\{1,2,3,4,5,6,7,8,9\}$, $A=\{3,4,5,7\}$, $B=\{1,3,4,5,8\}$, and $C=\{6,7,9\}$, find each result below. *Be sure to show the steps as needed to justify how you reached the answer.*
   a. $A \cap B$
   b. $B \cup C$
   c. $A \cap(B \cup C)$
   d. $A^{\prime} \cap B^{\prime}$
   e. $(C \cup A)^{\prime}$
   f. $A \cap B^{\prime}$

g. $n(C')$ *Note this asks for cardinality.*
h. $n(A \cap B \cap C)$ *Note this asks for cardinality.*

8. Use the Venn diagram to find the results of the set operations.

   a. B'
   b. $A \cup B \cup C$
   c. $A \cup B$
   d. $A \cap C'$
   e. $(A \cup B)'$



9. Among 100 students surveyed, 60 are taking a math course, 70 are taking psychology, and 40 are taking both math and psychology.

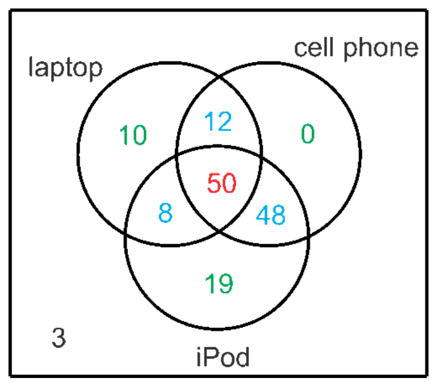
   a. Use the information to enter numbers a Venn diagram so it shows the appropriate cardinality in each of the four regions.
   b. How many students are taking math or psychology?
   c. How many students are taking math but not psychology?
   d. How many students are taking neither math nor psychology?
   e. How many students are taking exactly one course?

10. Use the cardinality principle to solve:

    a. If $n(A) = 16$, $n(B) = 22$, and $n(A \cup B) = 30$, find $n(A \cap B)$.
    b. If $n(B) = 12$, $n(A \cup B) = 30$ and $n(A \cap B) = 10$, find $n(A)$.

11. This Venn diagram shows the results of a survey that asked if students owned a laptop, cell phone, and iPod. Use the diagram to answer the questions.

    a. How many students own an iPod?
    b. How many students do not own a lap top?
    c. How many students own *only* a laptop?
    d. How many students own an iPod and a laptop?
    e. How many students own an iPod or a laptop, but no cell phone?



12. Fifty (50) moviegoers were asked whether they had ever seen *The Matrix* (M), *Star Wars* (SW), and *Lord of the Rings* (LOTR). The results are below. Use the given information to show the appropriate cardinalities in a 3-set Venn diagram. Be sure to show cardinalities for all eight regions.

    - 18 had seen The Matrix
    - 20 had seen Lord of the Rings
    - 24 had seen Star Wars
    - 14 had seen Lord of the Rings and Star Wars
    - 12 had seen The Matrix and Lord of the Rings
    - 10 had seen The Matrix and Star Wars
    - 6 had seen all three

CHAPTER OVERVIEW

2: Logic

2.1: Introduction to Formal Logic

In our everyday life we constantly evaluate statements as being true or false. The truth of these statements help us determine whether arguments used to persuade us to vote a certain way or to buy a certain product are valid or invalid. We should always create our arguments as persuasively and logically as we can. All this we do using informal (intuitive) and formal logical thinking. We are surrounded and guided by apps and algorithms written by use of formal logic as any programming code is an exercise in use of formal logic. Having a well-developed intuition and an ability to apply formal logical analysis to an argument are equally important for a fulfilling successful life.

Logic

Logic is the study of reasoning.

This chapter will look at the foundations of formal logic and apply them to determine whether an argument is valid and sound. This section, in particular, will examine statements and logical connectors that are the building blocks of arguments. Just as arithmetic operates with numbers and set theory operates with sets, formal logic operates with *statements*.

Statements

A **statement** is a declarative sentence that can be objectively determined to be either true or false but not both at the same time.

A *simple statement* conveys only one idea. A *compound statement* conveys two or more ideas. Phrases, questions, and commands can never be statements in logic because it would be impossible to determine whether the ideas are true or false.

Example 1

Determine whether the following are statements. For each statement, say whether it is simple or compound and whether it is true or false.

a. PGCC is a four-year college.
b. Show me the money.
c. The Mississippi River flows into the Gulf of Mexico.
d. I like ice cream and I like cookies.
e. If $m = 6$, then $2m = 12$.
f. $9 + 6 = 16$
g. Where are you going?

Solution

a. This is a statement because its truth value can be determined. It is false that PGCC is a four-year college. The statement expresses only one idea so this is a simple statement that is false.
b. This is not a statement. Rather, this is a command which has no truth value that can be determined.
c. This is a statement because its truth value can be determined. It is true that the Mississippi River flows in the Gulf of Mexico. The statement expresses only one idea so this is a simple statement that is true.
d. This is a statement because its truth value can be determined -- although its truth value depends on the person referred to by "I." The statement express two ideas so this is a compound statement.
e. This is a statement because its truth value can be determined. By multiplying both sides of the equation $m = 6$ by 2, we see that $2m = 12$ regardless of the value of m. There are two ideas expressed in the statement (the *if* part and the *then* part), so this is a compound statement that is true.
f. This is a statement because its truth value can be determined. It is false that that $9 + 6 = 16$. The statement expresses only one idea so this is a simple statement that is false.
g. This is not a statement. Questions are never statements because a truth value cannot be determined.

Statements may be negated or combined with connector words like "and", "or", "if", and "then." Let's take a closer look at how negations and logical connectors are used with simple statements to create more complex statements. In the definitions that follow, we assume that p and q represent two simple statements.

One way to change a statement is to use its *negation*, or opposite meaning. We often use the word "not" to negate a statement.

Negation: $\sim$

If p is a statement, the **negation** of p is another statement that is exactly the opposite of p. The symbol for negation is $\sim$.

The negation of a statement p is notated $\sim p$ and means "not p."

A statement p and its negation $\sim p$ will always have opposite truth values. That is, if statement p is true, then $\sim p$ is false. If statement p is false, then $\sim p$ is true. It is impossible to have a situation in which a statement and its negation will have the same truth value.

Example 2

Write the negation of each statement in words.

a. I am reading a math book.
b. Math is fun!
c. The sky is not green.
d. Cars have wheels.

Solution

a. I am ***not*** reading a math book.
b. Math is ***not*** fun!
c. The sky is green. (or, The sky is ***not*** not green.)
d. Cars do ***not*** have wheels.

It is possible to use more than one negation in a statement. If you've ever said something like, "*I can't not go*," you are really saying you must go. It's a lot like multiplying two negative numbers which gives a positive result.

In the media and in ballot measures we often see multiple negations, and it can be confusing to figure out what a statement means.

Example 3

Read the statement to determine the result of the voting measure.

"*This voting measure repeals the ban on plastic bags.*"

Solution

The measure enables plastic bag usage. The ban stopped plastic bag usage, so to repeal the ban would allow it again. This statement has a double negation and is also not very good for the environment.

Try it Now 1

Write the negation of each statement in words.

a. I like getting up early in the morning.
b. I will not go to work.
c. Tom is unable to attend class.

Answer

a. I do not like getting up early in the morning.
b. I will go to work.
c. Tom is able to attend class. (or, Tom is not unable to attend class.)

When we use the word "and" between two statements, it connects them to create a new statement that is compound. For example, if you said, "*When you go to the store, please get eggs **and** cereal*," you would be expecting both items. For an *and* statement to be true, the connected statements must both be true. If even one statement is false (for instance, you get eggs but not cereal), the entire connected *and* statement is false.

When two statements are connected with the word "and," this new compound statement is called a *conjunction*.

Conjunction: $\wedge$

If p and q are statements, their **conjunction** is the statement "p *and* q." The symbol for the conjunction of two statements is $\wedge$.

The conjunction of statement p and statement q is notated $p \wedge q$.

For any statement of the form $p \wedge q$ to be true, both statement p and statement q must be true.

Example 4

Let p be the statement "*I have a penny*" and q be the statement "*I have a dime*."

$p \wedge q$ is the compound statement "*I have a penny **and** I have a dime*."

Example 5

Determine the truth value of each conjunction.

a. Annapolis is in Maryland and Baltimore is in Virginia.
b. 7 is an odd number and 10 is an even number.

Solution

a. p is the statement "*Annapolis is in Maryland*," which is true, while q is the statement "*Baltimore is in Virginia*," which is false. Because only one statement is true and the other statement is false, the truth value of this conjunction is false.
b. p is the statement "*7 is an odd number*," which is true, while q is the statement "*10 is an even number*," which is true. Because both statements are true, the truth value of this conjunction is true.

The word "or" between two statements similarly connects the statements to create a new statement that is compound. In this case, if you said, "*Please get eggs **or** cereal*," you would be expecting one or the other (but probably not both). For an *or* statement to be true, *at least one* of the statements must be true. That is, an *or* statement is true when one or both statements are true.

It should be pointed out that in the English language we often mean for *or* to be exclusive: one or the other, but not both. In math, however, *or* is usually ***inclusive***: one or the other, or both.

When two statements are connected with the word "or" this new compound statement is called a *disjunction*.

Disjunction: $\vee$

If p and q are statements, their **disjunction** is the statement "p *or* q." The symbol for the disjunction of two statements is $\vee$.

The disjunction of statement p and statement q is notated $p \vee q$.

For any statement of the form $p \vee q$ to be true, at least one of statement p and statement q must be true. The only time when a disjunction is false is when both of its statements are false.

Example 6

Let p be the statement "*Today is Tuesday*" and q be the statement "$1+1=2$."

$p \vee q$ is the compound statement "Today is Tuesday ***or*** $1+1=2$."

Example 7

Determine the truth value of each disjunction.

a. Annapolis is in Maryland or Baltimore is in Virginia.
b. 7 is an odd number or 10 is an even number.
c. There are 60 stars on the United States flag or a banana is red.

Solution

a. p is the statement "*Annapolis is in Maryland*," which is true, while q is the statement "*Baltimore is in Virginia*," which is false. Because at least one statement is true, the truth value of this disjunction is true.
b. p is the statement "*7 is an odd number*," which is true, while q is the statement "*10 is an even number*," which is true. Because at least one statement is true (both are true!), the truth value of this disjunction is true.
c. p is the statement "*There are 60 stars on the United States flag*," which is false, while q is the statement "*A banana is red*," which is also false. Because neither statement is true, the truth value of this disjunction is false.

Try it Now 2

Consider these simple statements:

p: 5 is an odd number. q: $6+4=12$

Translate these symbolic statements to words and tell whether the resulting statement is true or false.

a. $\sim p$
b. $\sim q$
c. $p \wedge q$
d. $p \vee q$

Answer

a. 5 is not an odd number. False.
b. $6+4 \neq 12$. True.
c. 5 is an odd number and $6+4=12$. False.
d. 5 is an odd number or $6+4=12$. True.

We often want to be able to *conditionally* do something. That is, we want to be able to say "*If this thing is true, then do X.*" It's like when we leave our house in the morning -- "*If it's cold outside, then I will wear a coat.*"

A *conditional statement* connects two statements using *if ... then*.

Another example of a conditional statement is "***If** it is raining, **then** we'll go to the mall.*" The statement "*If it is raining,*" may be either true or false for any given day. If the condition is true, then we will follow the course of action and go to the mall. If the condition is false though, we haven't said anything about what we will or won't do. Truth values of conditional statements will be discussed in a later section.

Conditional: $\rightarrow$

A **conditional statement** is a compound statement of the form "If p, then q." Often, we say this as "p *implies* q." The symbol used to indicate a conditional statement is $\rightarrow$.

Statement p is the "*if* part" and is called the *antecedent*. Statement q is the "*then* part" and is called the consequent.

The conditional statement "If p, then q" is notated $p \rightarrow q$.

Example 8

Let p be the statement "*You are hungry*" and q be the statement "*You are cranky*."

In this case $p \rightarrow q$ is the compound statement "***If** you are hungry, **then** you are cranky.*"

Here, "*You are hungry*" is the antecedent, and "*You are cranky*" is the consequent. When the condition of a person being hungry has been met, it implies that the person will be be cranky.

Example 9

Let p be the statement "*It rains*" and q be the statement "*The game is cancelled.*"

Write each statement using correct symbols.

a. If it rains, then the game is cancelled.
b. If the game is not cancelled, then it doesn't rain.
c. The game is cancelled if it rains.
d. It is not the case that if it rains, then the game is cancelled.

Solution

a. The antecedent is "*It rains*" because this statement is the *if* part of the conditional. The consequent is "*The game is cancelled*" because this statement is the *then* part of the conditional. In symbols, the conditional statement is $p \rightarrow q$.
b. The antecedent is "*The game is not cancelled*" because this statement is the *if* part of the conditional. This is the negation of statement q. The consequent is "*It doesn't rain*" because this statement is the *then* part of the conditional. This is the negation of statement p. In symbols, the conditional statement is $\sim q \rightarrow \sim p$.
c. This statement is the same as the statement in part a with the *if* and *then* parts written in different orders. The antecedent is "*It rains*" because this statement is the *if* part of the conditional. The consequent is "*The game is cancelled*" because this statement is the *then* part of the conditional. In symbols, the conditional statement is $p \rightarrow q$.
d. Note that in this case it is the entire "*if*…*then*" statement that is being negated, rather than just one or both of its components. This statement is the negation of the conditional statement in part a. In symbols, this is written $\sim (p \rightarrow q)$.

As noted above for a conditional statement involving two statements, there are two roles: one statement is the antecedent and one statement is the consequent. In general, the antecedent and the consequent *cannot* be interchanged. To illustrate why this is true, consider these two conditional statements:

- Conditional 1: *If today is Saturday, then it is the weekend.*
- Conditional 2: *If it is the weekend, then today is Saturday*.

Conditional 1 is always a true statement. However, Conditional 2 may or may not be true because today may be Sunday. Interchanging the *if* part and the *then* part makes a big difference in whether the resulting conditional statement is true or not. '*Today is Saturday*" implies "*It is the weekend*," but "*It is the weekend*" does not imply "*It is Saturday*."

As we just saw, $p \rightarrow q$ does not always mean that $q \rightarrow p$. However, in certain instances it may happen that both $p \rightarrow q$ and $q \rightarrow p$. To illustrate this with an example, consider the statements "*Today is January 1*" and "*Today is New Year's Day*." If we form conditional statements, we have

- Conditional 1: *If today is January 1, then today is New Year's Day.*
- Conditional 2: *If today is New Year's Day, then today is January 1.*

It is possible to interchange the simple statements in the *if* and *then* parts because each simple statement implies the other simple statement. This type of powerful situation is represented using a *biconditional statement.*

When two statements form a biconditional in this way, we can express this more compactly using the phrase "...***if and only if***...". For the previous example, we could write "*Today is January 1 if and only if today is New Year's Day.*" A double-headed arrow $\leftrightarrow$ is used to indicate a biconditional statement instead of the one-headed arrow $\rightarrow$ used for a conditional statement.

Biconditional: $\leftrightarrow$

A **biconditional statement** is a statement of the form "p if and only if q." It is a conjunction of the two conditional statements "*If* p*, then* q" and "*If* q*, then* p." The symbol used to indicate a biconditional statement is $\leftrightarrow$.

The biconditional statement "p if and only q" is notated $p \leftrightarrow q$.

Example 10

p is the statement "*I eat*" and q is the statement "*I am hungry*."

In this case $p \leftrightarrow q$ is the compound statement "*I eat if and only if I am hungry.*"

Here, "*I eat if and only if I am hungry"* is a short way of saying both "*If I eat, then I am hungry*" and "*If I am hungry, then I eat*."

Example 11

Consider these simple statements:

p: The sun is shining. q: It is raining. r: The grass is green.

Translate these statements to symbolic form using p, q, r, $\sim$, $\rightarrow$, and $\leftrightarrow$.

a. It is not raining.
b. If it is raining, then the sun is not shining.
c. It is raining and the grass is green.
d. It is false that the sun is shining or it is raining.
e. The grass is green, if it is raining and the sun is shining.
f. The sun is shining or it is raining.
g. The grass is green if and only if it is raining.

Solution

a. $\sim q$. This is the negation of statement q.
b. $q \rightarrow \sim p$. This is a conditional statement where statement q is the antecedent and the negation of statement p is the consequent.
c. $q \wedge r$. This is a conjunction with statement q and statement r.
d. $\sim (p \vee q)$. This is the negation of the disjunction of statement p and statement q.
e. $(q \wedge p) \rightarrow r$. This is a conditional statement where the antecedent is the conjunction of statement q and statement p and the consequent is statement r. We must use grouping symbols to show the antecedent is itself a compound statement.
f. $p \vee q$. This is a disjunction with statement p and statement q.
g. $r \leftrightarrow q$. This is a biconditional using statement r and statement q.

Try it Now 3

Consider these simple statements:

p: Tom plays hard. q: Tom is a guitar player. r: Tom's commute to work is long. s: Tom gets bored in the car.

Translate these statements to symbolic form using p, q, r, s, $\sim$, $\rightarrow$, and $\leftrightarrow$.

a. Tom plays hard or Tom is a guitar player.
b. Tom's commute to work is not long.
c. It is false that Tom's commute is long and Tom gets bored in the car.
d. If Tom's commute to work is long, then Tom gets bored in the car.
e. If Tom gets bored in the car and the commute to work is long, then Tom does not play hard.
f. Tom is not a guitar player if and only Tom does not play hard.

Answer

a. $p \vee q$
b. $\sim r$
c. $\sim(r \wedge s)$
d. $r \rightarrow s$
e. $(s \wedge r) \rightarrow \sim p$
f. $\sim q \leftrightarrow \sim p$

Example 12

Consider these simple statements:

a: Roses are red. b: The sky is blue. c: Turtles are green.

Translate these symbolic statements to words.

a. $a \wedge b$
b. $\sim c \vee b$
c. $\sim(c \vee b)$
d. $a \wedge(b \vee c)$
e. $b \rightarrow \sim c$
f. $\sim b \rightarrow(\sim a \wedge \sim c)$
g. $a \leftrightarrow \sim b$

Solution

a. Roses are red and the sky is blue.
b. Turtles are not green or the sky is blue.
c. It is false that turtles are green or the sky is blue.
d. Roses are red, and the sky is blue or turtles are green.
e. If the sky is blue, then turtles are not green.
f. If the the sky is not blue, then roses are not red and turtles are not green.
g. Roses are red if and only if the sky is not blue.

Try it Now 4

Consider these simple statements:

p: I go to class every day. q: I do my homework. r: I get a good grade in this class.

Translate these symbolic statements to words.

a. $p \wedge q$
b. $q \vee \sim r$
c. $\sim(p \vee r)$
d. $\sim q \rightarrow \sim r$
e. $p \leftrightarrow r$

Answer

a. I go to class everyday and I do my homework.
b. I do my homework or I do not get a good grade in this class.
c. It is false that I go to class every day or I get a good grade in this class.
d. If I don't do my homework, then I don't get a good grade in this class.
e. I go to class every day if and only if I get a good grade in this class.

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2.2: Introduction to Truth Tables

To evaluate a logic statement, we first must learn how to evaluate results of the basic logic operations discussed in the previous section – *negation*, *conjunction*, *disjunction*, *conditional* and *biconditional.* Then, we will learn to use the order of operations for connectors in more complex statements.

The situation in evaluating logic statement is similar to simplifying an arithmetic statement, where one first learns how to perform addition, subtraction, multiplication, and division and then to evaluate any arithmetic expression using the proper order of operations.

According to our previous definition, a *statement* in logic is either true or false, but never both. This means that a simple statement p can only have two values: 'True' noted as $p = T$, or 'False' noted $p = F$. To evaluate basic logic operations, we need to determine truth value outcomes for all possible truth values for the simple statements involved in the operation.

Let us consider how *negation* works.

Negation

Recall from the previous section of this chapter that the definition of the **negation** of statement p is another statement that is exactly the opposite of statement p. The negation of p is "not p."The symbol used to indicate the negation is $\sim$, and the negation of statement p is written $\sim p$.

Suppose we let p represent the statement "*I like ice cream*." Then, $\sim p$ represents the statement "*I do not like ice cream*" or "*It is not true that I like ice cream*."

Statement p can be true or statement p can be false as presented in the first column of the table:

p	$\sim p$
T	
F	

Obviously, when statement p is true, statement $\sim p$ is false. When statement p is false, statement $\sim p$ is true. A statement and its negation will always have opposite truth values. In short, the operation of negation works in following way:

- $\sim T = F$
- $\sim F = T$.

The truth values for negation can also be summarized in a type of table called a *truth table*. A **truth table** is a table that shows the resulting truth value of a compound statement for all possible truth values of the simple statements. Because many logic statements can get tricky to think about, we often create a truth table to keep track of what truth values for the simple statements make a compound statement true or false.

Truth Table for Negation $\sim p$

p	$\sim p$
T	F
F	T

Next, we review the logical connector '*and*' and build a truth table for the *conjunction* of two simple statements.

Conjunction

If p and q are simple statements, their **conjunction** is “p and q” noted as $p \wedge q$.

Because each of statements p and q can be true or false, we must consider all possible combinations of their truth values. There are four of these possible combinations:

- p and q are both true,
- p is be true but q is false
- p is false but q is true,

- p and q are both false.

These four combinations are presented as the first two columns in the following table:

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

Recall from the previous section of this chapter that for the conjunction of two simple statements to be true, both simple statements must be true. To illustrate how the outcome of a conjunction is determined, let us consider the following example:

Example 1

A company is offering a job you are interested in. To apply, an applicant must have an associates degree in information technology (IT) ***and*** at least two years of experience in the IT field. These application requirements can be translated into symbolic form:

p: must have associates degree in IT q: must have at least two years of experience in IT field

Then, the requirement to apply is $p \wedge q$. That is, to qualify for the job, $p \wedge q$ must be true.

The only way a candidate qualifies for the job $(p \wedge q = T)$ is to have both an associates degree $(p = T)$ and at least two years of experience $(q = T)$. In all other cases, a candidate does not qualify.

That is, when a candidate

- has an associates degree and has at least two years of experience, they qualify: $T \wedge T = T$
- has an associates degree and does not have at least two years of experience, they do not qualify: $T \wedge F = F$
- does not have an associates degree and has at least two years of experience, they do not qualify: $F \wedge T = F$
- does not have an associates degree and does not have at least two years of experience, they do not qualify: $F \wedge F = F$.

These results can be presented as the truth table for conjunction:

Truth Table for Conjunction $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Next, we review the logical connector '*or*' and build a truth table for the *disjunction* of two simple statements.

Disjunction

If p and q are simple statements, their **disjunction** is "p or q" noted as $p \vee q$.

Just as with a conjunction, each of p and q can be either true or false. We can present all four possible combinations of truth values in the first two columns of a truth table:

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

Recall from the previous section of this chapter that for the disjunction of two simple statements to be true, at least one of the simple statements must be true. The only time that a disjunction of two simple statements is false is when both simple statements are false. To illustrate how the outcome of a disjunction is determined, let us consider the following example:

Example 2

A company is offering a job you are interested in. To apply, an applicant must have an associates degree in information technology (IT) ***or*** at least two years of experience in the IT field. These application requirements can be translated into symbolic form:

p: must have associates degree in IT q: must have at least two years of experience in IT field

Then, the requirement to apply is $p \vee q$. That is, to qualify for the job, $p \vee q$ must be true.

Obviously, any candidate who has either an associates degree $(p = T)$ or two years of experience $(q = T)$ qualifies for the job $(p \vee q = T)$. Only a person who has neither the degree $(p = F)$ nor the experience $(q = F)$ will not qualify $(p \vee q = F)$.

That is, when a candidate

- has an associates degree and has at least two years of experience, they qualify: $T \vee T = T$
- has an associates degree and does not have at least two years of experience, they still qualify: $T \vee F = T$
- does not have an associates degree and has at least two years of experience, they still qualify: $F \vee T = T$
- does not have an associates degree and does not have at least two years of experience, they do not qualify: $F \vee F = F$.

These results can be presented as the truth table for disjunction:

Truth Table for Disjunction $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Now, we review the logical connector '*if...then*' and build a truth table for a *conditional statement*.

Conditional

Recall that a **conditional statement** consists of two simple statements p and q joined as “If p, then q.” We write this as $p \rightarrow q$. The simple statement used in the "*if*" part is the *antecedent* and the simple statement used in the "*then*" part is called the *consequent*. Sometimes a conditional statement is called an *implication*, and we say that "p implies q.

Just as with conjunctions and disjunctions, each of statements p and q can be either true or false. We can present all four possible combinations of truth values in the first two columns of a truth table:

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

We must determine the truth value for each of the four outcomes in the table much the same way we did for the conjunction and disjunction of statements p and q. To illustrate how the outcome of a conditional statement is determined, let us consider an example.

Example 3

A parent makes this statement to a child: "*If you finish your homework, I will buy you ice cream."*

The two simple statements can be translated into symbolic form:

p: You *finish your homework.* q: *I will buy you ice cream.*

Then, the promise made by the parent can be written as $p \rightarrow q$.

Let's consider the outcome of the promise on a case-by-case basis for each of the four combinations of truth values for statement p and statement q. It may be helpful to think about the truth value of the conditional statement $p \rightarrow q$ in the following way: $p \rightarrow q$ will be true only when the parent's promise is not broken.

- If the child finishes the homework $(p = T)$ and the parent buys ice cream $(q = T)$, then statement $p \rightarrow q$ is true as the promise was fulfilled and not broken: $T \rightarrow T = T$.
- If the child finishes the homework $(p = T)$ and the parent does not buy ice cream $(q = F)$, then the parent's promise $p \rightarrow q$ was a lie. The parent broke the promise, and so the conditional statement is false: $T \rightarrow F = F$.
- If the child does not finish the homework $(p = F)$ and the parent buys ice cream $(q = T)$, then statement $p \rightarrow q$ is still true as again the parent did not break the promise. The parent's promise is in effect only when the child finishes the homework. The parent didn't say what would happen if the child failed to finish the homework. With the '*if*' part being false, we haven't said anything about what the parent will do so the promise was not broken: $F \rightarrow T = T$.
- If the child does not finish the homework $(p = F)$ and the parent does not buy ice cream $(q = F)$, then statement $p \rightarrow q$ is true as the parent did not break the promise because the child did not fulfill the required action to finish the homework: $F \rightarrow F = T$.

These results can be presented as the truth table for a conditional statement.

Truth Table for Conditional $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that a conditional statement is false only in one case -- when the antecedent is true and the consequent is false. In our example of the homework and ice cream promise, that is the only time the parent's promise is broken.

Finally, we review the logical connector '*if and only if*' and build a truth table for a *biconditional statement*.

Biconditional

A **biconditional statement** consists of two simple statements p and q joined as "p if and only if q." A biconditional statement is a conjunction of the two conditional statements "If p, then q" and "If q, then p." We write this as $p \leftrightarrow q$.

We can use this definition of a biconditional statement to construct the truth table for it. For each of the four possible combination of values for p and q, we evaluate $p \rightarrow q$ in the third column and $q \rightarrow p$ in the fourth column using the truth table for the conditional statement we created previously. Next, we evaluate the conjunction using the truth values in the third and fourth columns to determine the truth value of $(p \rightarrow q) \wedge (q \rightarrow p)$ in the fifth column using the truth table for the conjunction.

Truth Table for Biconditional $p \leftrightarrow q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$
T	T	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$
T	F	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$
F	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$
F	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$

As you can see from the truth table, a biconditional statement is only true when both simple statements have the same truth values (both p and q are true or both p and q are false.) When the simple statements in a biconditional have different truth values, a biconditional is false.

Now that we have explored truth tables for five basic operations used to form compound logical statements, here is a summary chart of truth values for easy reference.

Summary of Truth Table Results

Operation	**Notation**	**Summary of truth values**
Negation	$\sim p$	The opposite truth value of p
Conjunction	$p \wedge q$	True only when both p and q are true
Disjunction	$p \vee q$	False only when both p and q are false
Conditional	$p \rightarrow q$	False only when p is true and q is false
Biconditional	$p \leftrightarrow q$	True only when both p and q are true or are both are false

Example 4

Consider the following compound statements. Use appropriate entries from truth tables to determine whether each statement is true or false.

a. There are 50 states in the United States or Santa Claus is vice president.
b. There are 50 states in the United States and Santa Claus is vice president.
c. If there are 50 states in the United States, then Santa Claus is vice president.
d. If Santa Claus is vice president, then there are 50 states in the United States.
e. There are 50 states in the United States if and only if Santa Claus is vice president.

Solution

First, identify and determine the truth value for each simple statement:

There are 50 states in the United States. (True) *Santa Claus is vice president.* (False)

a. True. This statement can be represented as $T \vee F$ because the word "*or*" connects a true statement and a false statement. According to the truth table for a disjunction, $T \vee F = T$.
b. False. This statement can be represented as $T \wedge F$ because the word "*and*" connects a true statement and a false statement. According to the truth table for a conjunction, $T \wedge F = F$.
c. False. This statement can be represented as $T \rightarrow F$ because "*if ... then...*" connects a true antecedent statement to a false consequent statement. According to the truth table for a conditional, $T \rightarrow F = F$. This is a broken promise situation.
d. True. This statement can be represented as $F \rightarrow T$ because "*if ... then...*" connects a false antecedent statement to a true consequent statement. According to the truth table for a conditional, $F \rightarrow T = T$.
e. False. This statement can be represented as $T \leftrightarrow F$ because "*...if and only if...*" connects a true statement and a false statement. According to the truth table for a biconditional, $T \leftrightarrow F = F$ because the two simple statements have different truth values.

Try it Now 1

Consider the following compound statements. Use appropriate entries from truth tables to determine whether each statement is true or false.

a. 8 is odd and 6 is even.
b. 8 is odd or 6 is even.
c. If 8 is odd, then 6 is even.
d. If 6 is even, then 8 is odd.
e. 8 is odd if and only if 6 is even.

Answer

a. False because $(F \wedge T = F)$
b. True because $(F \vee T = T)$
c. True because $(F \rightarrow T = T)$
d. False because $(T \rightarrow F = F)$
e. False because $(F \leftrightarrow T = F)$

Compound statements in logic can be quite complex and can have more than one operation or connector. It is important to follow the correct order of performing operations when evaluating compound statements for truth values just as we follow a prescribed order of operating when evaluating an expression in arithmetic or algebra.

To evaluate any compound statement that includes grouping symbols or more than one connective we need to adhere to the following order of operations:

1. Any expression inside parentheses
2. Negation $(\sim)$
3. Conjunction or disjunction $(\wedge, \vee)$
4. Conditional $(\rightarrow)$
5. Biconditional $(\leftrightarrow)$

Let us now practice constructing truth tables for various compound statements that are slightly more complex.

Example 5

Create a truth table for $\sim p \rightarrow \sim q$.

Solution

Recall that all four possible combinations of truth values for p and q can be presented in the first two columns of the truth table. Then, following the order of operations, we evaluate $\sim p$ in the third column and $\sim q$ in the fourth column using the truth table for negation determined in the beginning of this section. Finally, we evaluate the conditional $\sim p \rightarrow \sim q$ using truth values of $\sim p$ found in the third column and truth values of $\sim q$ found in the fourth column, applying the rules for a conditional connector.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	F	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
T	F	F	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
F	T	T	F	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$
F	F	T	T	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$

Example 6

Create a truth table for the statement $p \vee \sim q$.

Solution

All possible combinations of truth values for simple statements p and q are presented in the first two columns of the table. Next, we evaluate $\sim q$ in the third column using the truth values for statement q in the second column. Now, we can use the first and third columns to find the truth values of $p \vee \sim q$.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$
T	F	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$
F	T	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$
F	F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$

The truth table shows that $p \vee \sim q$ is true in three cases and false in one case.

Try it Now 2

Create a truth table for the statement $\sim p \wedge\ q$.

Answer

p	q	$\sim p$	$\sim p \wedge\ q$
T	T	F	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$
T	F	F	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$
F	T	T	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$
F	F	T	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$

In the previous examples, each compound statement was formed using two simple statements, p and q. In these cases, there were only $2 \times 2 = 4$ possible combinations of truth values for the two statements. Sometimes, however, compound statements have more than two simple statements. For example, a compound statement might include three simple statements p, q, and r. When there are three simple statements, then there are $2 \times 2 \times 2 = 8$ possible combinations of truth values for the three statements. The next example is an illustration.

Example 7

Create a truth table for the statement $p \wedge \sim (q \vee r)$.

Solution

The eight combinations of truth values for three simple statements obviously can be listed in any order, but the standard way is presented in the first two columns of the truth table. This list of combinations is created by first letting p = T and then creating the standard four combinations of truth values for q and r. Then, the rest of the list of combinations is created by letting p = F and repeating the combinations of truth values for q and r.

Following the order of operations rules, we first use the rules for disjunction to evaluate the statement in parentheses $(q \vee r)$ using the truth values for q and r in the second and third columns. These results for $(q \vee r)$ are placed in the fourth column. Next, we negate the results in this fourth column and write the results for $\sim (q \vee r)$ in the fifth column. The last step is to evaluate $p \wedge \sim (q \vee r)$, the conjunction between truth values in the first and fifth columns.

p	q	r	$q \vee r$	$\sim (q \vee r)$	$p \wedge \sim (q \vee r)$
T	T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$
T	T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$
T	F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$
T	F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$
F	T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$
F	T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$
F	F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$
F	F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$

It turns out that this complex expression is true in only one case -- when p is true, q is false, and r is false.

Example 8

Create a truth table for the statement $(p \vee q) \leftrightarrow \sim r$.

Solution

There are 3 simple statements so start by listing all the possible truth value combinations for p, q, and r in the first three columns. After creating the 8 combinations, use the truth values for p and q to write the results for $p \vee q$ in the fourth column. In the fifth column, write the truth values for $\sim r$, the negation of the truth values in the third column. Finally, find the truth values of $(p \vee q) \leftrightarrow \sim r$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

p	q	r	$p \vee q$	$\sim r$	$(p \vee q) \leftrightarrow \sim r$
T	T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \leftrightarrow \mathrm{F} = \mathrm{F}$
T	T	F	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{T} \leftrightarrow \mathrm{T} = \mathrm{T}$
T	F	T	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \leftrightarrow \mathrm{F} = \mathrm{F}$
T	F	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{T} \leftrightarrow \mathrm{T} = \mathrm{T}$
F	T	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \leftrightarrow \mathrm{F} = \mathrm{F}$
F	T	F	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{T} \leftrightarrow \mathrm{T} = \mathrm{T}$
F	F	T	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{F} \leftrightarrow \mathrm{F} = \mathrm{T}$
F	F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{F} \leftrightarrow \mathrm{T} = \mathrm{F}$

Try it Now 3

Create a truth table for the statement $(p \wedge \sim q) \rightarrow r$.

Answer

p	q	r	$\sim q$	$p \wedge \sim q$	r	$(p \wedge \sim q) \rightarrow r$
T	T	T	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
T	T	F	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
T	F	T	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	T	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$
T	F	F	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	F	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$
F	T	T	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
F	T	F	$\sim \mathrm{T} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
F	F	T	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
F	F	F	$\sim \mathrm{F} = \mathrm{T}$	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$

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2.3: Equivalent Logical Statements

In this section we will analyze whether or not a statement is always true, always false, or sometimes true and sometimes false. We will also analyze two or more statements to determine whether they mean exactly the same thing.

Tautologies and Self-Contradictions

Consider the statements below.

- *I am right or I am wrong.*
- *A number is odd or a number is not odd.*
- *It is snowing or it is not snowing.*
- *This apple is red or this apple is not red.*

What do you notice about each statement?

Each statement is the disjunction of a statement and its negation. That is, each compound statement is formed by joining two simple statements that are opposite in meaning with the word '*or*.'

Let's explore the first statement: *I am right or I am wrong*. We intuitively see that this statement will always be true, regardless of truth values of the simple statements that form it. Our intuition can be easy confirmed by constructing the truth table for the statement. Let p represent simple statement "*I am right.*" Then, $\sim p$ represents "*I am wrong*". The compound statement in symbolic form is represented by $p \vee \sim p$. Recall the results of a truth table for a disjunction: the disjunction of two simple statements is always true unless both of the statements in the disjunction are false. The truth table is shown below.

p	$\sim p$	$p \vee \sim p$
T	F	$\mathrm{T} \vee \mathrm{F}=\mathrm{T}$
F	T	$\mathrm{F} \vee \mathrm{T}=\mathrm{T}$

The last column of the truth table shows that $p \vee \sim p$ is always true regardless of the truth values of the individual statements. This circumstance is called a *tautology*.

Tautology

A **tautology** is compound statement that is always true regardless of the truth values of the individual statements.

Sometimes a compound statement may always be false regardless of the truth value of the individual statements. Let's change the previous statement from a disjunction to a conjunction and join two simple statements opposite in meaning with the word '*and*'. We will create and analyze a truth table for the statement "*I am right and I am wrong.*"

Let p represent simple statement "*I am right.*" Then, $\sim p$ represents "*I am wrong*". The compound statement in symbolic form is represented by $p \wedge \sim p$. Recall that a conjunction is true only in one case -- when both simple statements are true. Otherwise, a conjunction is false. The truth table is shown below.

p	$\sim p$	$p \wedge \sim p$
T	F	$\mathrm{T} \wedge \mathrm{F}=\mathrm{F}$
F	T	$\mathrm{F} \wedge \mathrm{T}=\mathrm{F}$

The last column of the truth table shows that $p \wedge \sim p$ is always false regardless of the truth values of the individual statements. This illustrates another special type of statement known as a *self-contradiction.*

Self-contradiction

A **self-contradiction** is compound statement that is always false regardless of the truth values of the individual statements.

> **Example 1**
>
> Classify the statement as a tautology, a self–contradiction, or neither: $(p \wedge q) \rightarrow p$.
>
> **Solution**
>
> We need to construct and analyze a truth table. Recall the only case where a conditional statement is false is when the antecedent is true but the consequent is false (a broken promise.)

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	$\mathrm{T} \wedge \mathrm{T}=\mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{T}=\mathrm{T}$
T	F	$\mathrm{T} \wedge \mathrm{F}=\mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{T}=\mathrm{T}$
F	T	$\mathrm{F} \wedge \mathrm{T}=\mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F}=\mathrm{T}$
F	F	$\mathrm{F} \wedge \mathrm{F}=\mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F}=\mathrm{T}$

> The last column shows that the compound statement $(p \wedge q) \rightarrow p$ is always true no matter what truth values are used for p and q. Therefore, this statement is a tautology.

> **Example 2**
>
> Classify the statement as a tautology, a self–contradiction, or neither: $\sim(p \rightarrow q) \vee(q \rightarrow p)$.
>
> **Solution**
>
> Construct a truth table for this compound statement.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$q \rightarrow p$	$\sim(p \rightarrow q) \vee(q \rightarrow p)$
T	T	$\mathrm{T} \rightarrow \mathrm{T}=\mathrm{T}$	F	$\mathrm{T} \rightarrow \mathrm{T}=\mathrm{T}$	$\mathrm{F} \vee \mathrm{T}=\mathrm{T}$
T	F	$\mathrm{T} \rightarrow \mathrm{F}=\mathrm{F}$	T	$\mathrm{F} \rightarrow \mathrm{T}=\mathrm{T}$	$\mathrm{T} \vee \mathrm{T}=\mathrm{T}$
F	T	$\mathrm{F} \rightarrow \mathrm{T}=\mathrm{T}$	F	$\mathrm{T} \rightarrow \mathrm{F}=\mathrm{F}$	$\mathrm{F} \vee \mathrm{F}=\mathrm{F}$
F	F	$\mathrm{F} \rightarrow \mathrm{F}=\mathrm{T}$	F	$\mathrm{F} \rightarrow \mathrm{F}=\mathrm{T}$	$\mathrm{F} \vee \mathrm{T}=\mathrm{T}$

> The last column shows that the compound statement $\sim(p \rightarrow q) \vee(q \rightarrow p)$ is sometimes true and sometimes false. Therefore, this statement is a neither a tautology nor a self-contradiction.

> **Try it Now 1**
>
> Classify each statement as a tautology, a self–contradiction, or neither:
>
> a. $(p \vee q) \vee(\sim p \wedge \sim q)$
> b. $(p \wedge q) \wedge(\sim p \vee \sim q)$
> c. $(\sim p \rightarrow q) \leftrightarrow(p \wedge q)$
>
> **Answer**
>
> a. tautology
> b. self-contradiction
> c. tautology

Logically Equivalent Statements

On many occasions it is important to determine whether statements that are worded differently have the same meaning or not. To determine whether statements have exactly the same meaning, we construct truth tables and then compare the results.

Let's explore this idea with three similar, but differently worded, statements:

A. *I leave now or I am late.*
B. *If I do not leave now, then I am late.*
C. *If I leave now, then I am not late.*

You may have an opinion about which, if any, of these statements have the same meaning, but only tools of formal logic allow us to make a sure conclusion about whether they mean the same.

Here, we let p represent the simple statement "*I leave now*" and q represent "*I am late*." The three compound statements can be presented symbolically as

A. $p \vee q$
B. $\sim p \rightarrow q$
C. $p \rightarrow \sim q$

We construct truth tables for each of the three compound statements.

Truth Tables for $p \vee q$, $\sim p \rightarrow q$, and $p \rightarrow \sim q$

A: $p \vee q$

p	q	$p \vee q$
T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$
T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$
F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$
F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$

B: $\sim p \rightarrow q$

p	q	$\sim p$	$\sim p \rightarrow q$
T	T	F	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
T	F	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
F	T	T	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$
F	F	T	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$

C: $p \rightarrow \sim q$

p	q	$\sim q$	$p \rightarrow \sim q$
T	T	F	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$
T	F	T	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$
F	T	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
F	F	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$

Comparing the last columns of the three truth tables, we see that Statement A and Statement B have exactly the same truth values for all possible truth values of simple statements. This means that statements A and B are *logically equivalent*. They have exactly the same meaning.

However, Statement C is not logically equivalent to Statements A and B. Statement C sometimes means something different than Statements A and B.

Here, we say $p \vee q$ is logically equivalent to $\sim p \rightarrow q$. To illustrate how this works verbally, "*You use an umbrella or you'll get soaked*" means the same as "*If you don't use an umbrella, then you'll get soaked.*"

Logically Equivalent

Two compound statements are **logically equivalent** if and only if the statements have the same truth values for all possible combinations of truth values for the simple statements that form them.

The symbol commonly used to show two statements are logically equivalent is $\Leftrightarrow$. This symbol $\equiv$ may also be used.

Example 3

Decide if the statements $\sim (p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent.

Solution

We construct and compare the truth tables for each statement.

Truth Tables for $\sim (p \vee q)$ and $\sim p \wedge \sim q$

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$	F
T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$	F
F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$	F
F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$	T

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$
T	F	F	T	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$
F	T	T	F	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$
F	F	T	T	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$

Since both statements have the same resulting truth values in the last columns of their truth tables, the statements are logically equivalent and we can write

$$\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

This equivalence can be described verbally as "*The negation of a disjunction is equivalent to the conjunction of the negations*" and is one of *DeMorgan's Laws of Logic*.

Try it Now 2

Construct truth tables to show $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$. Then, describe the equivalence in words.

Answer

Truth Tables for $\sim(p \wedge q)$ and $\sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	F
T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{F}$	T
F	T	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$	T
F	F	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$	T

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$
T	F	F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$
F	T	T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$
F	F	T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$

Both statements have the same resulting truth values in the last columns of their truth tables, so $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$. This equivalence can be verbally described as "*The negation of a conjunction is equivalent to the disjunction of the negations*" and is the second of *DeMorgan's Laws of Logic*.

We have now proven both of DeMorgan's Laws of Logic.

DeMorgan's Laws of Logic

$$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$

Example 4

Use DeMorgan's Laws to write a statement equivalent to "*It is false that I am taking geometry or I am taking biology*."

Solution

First, write the statement in symbolic form. Let p represent the statement "*I take geometry*" and q represent the statement "*I take biology*." So, the statement is presented as $\sim(p \vee q)$.

Using the first of DeMorgan Laws, the negation of a disjunction is the conjunction of the negations. Or, $\sim(p \vee q)$ is equivalent to $\sim p \wedge \sim q$.

The statement $\sim p \wedge \sim q$ can be translated to words as "*I am not taking geometry and I am not taking biology.*

Try it Now 3

Use one of DeMorgan's Laws to write a statement equivalent to "*The dog is small and the dog barks loudly.*"

Answer

It is false that the dog is small or it barks loudly.

Several other useful equivalencies can be proven using truth tables. One relates a conditional statement and a disjunction, and another relates the negation of a conditional statement and a conjunction. They are presented here.

Equivalencies to a Conditional and the Negation of a Conditional

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$

$$\sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$$

Example 5

Write the following statement as a disjunction: *If I am tired, then I am sleepy.*

Solution

The given statement is a conditional statement and can be written as $p \rightarrow q$ where p is the statement "*I am tired*" and q is the statement "*I am sleepy*."

Using the equivalency $p \rightarrow q \Leftrightarrow \sim p \vee q$, the statement can be rewritten as "*I am not tired or I am sleepy*".

Try it Now 4

Write the negation of the statement "*If I am tired, then I am sleepy*" as a conjunction using the equivalency $\sim (p \rightarrow q) \Leftrightarrow p \wedge \sim q$.

Answer

I am tired but I am not sleepy.

You may also say... I am tired and I am not sleepy.

Variations of the Conditional Statement

Variations of conditional statements play a big role in mathematical proofs as well as in writing various IF – ELSE logical flow operators in programming languages. The variations of a conditional statement are the *converse* of the conditional, the *inverse* of the conditional, and the *contrapositive* of the conditional.

Variations of the Conditional Statement

Name	Symbolic Form	In Words
Conditional	$p \rightarrow q$	If p, then q.
Converse	$q \rightarrow p$	If q, then p.
Inverse	$\sim p \rightarrow \sim q$	If $\sim p$, then $\sim q$.
Contrapositive	$\sim q \rightarrow \sim p$	If $\sim q$, then $\sim p$.

Example 6

Write the converse, inverse, and contrapositive to the conditional statement "*If I work diligently, then I will gain understanding.*"

Solution

Let p represent "*I work diligently*" and q represent "*I gain understanding.*"

The converse $q \rightarrow p$ is "*If I gain understanding, then I work diligently.*"

The inverse $\sim p \rightarrow \sim q$ is "*If I do not work diligently then I will not gain understanding.*"

The contrapositive $\sim q \rightarrow \sim p$ is "*If I do not gain understanding, then I will not work diligently*."

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 7

Write the converse, inverse, and contrapositive to the conditional statement “*If I teach third grade, then I am an elementary teacher*”.

Solution

Let p represent “*I teach third grade*” and q represent “*I am an elementary teacher.*”

The converse $q \rightarrow p$ is “*If I am an elementary teacher, then I teach third grade.*”

The inverse $\sim p \rightarrow \sim q$ is “*If I don't teach third grade, then I am not an elementary teacher.*”

The contrapositive $\sim q \rightarrow \sim p$ is “*If I am not an elementary teacher, then I don't teach third grade*."

Try it Now 5

For each conditional statement, write the indicated statement related to the conditional.

a. If it is raining, then I wear a raincoat. (inverse)
b. If I confess, then I am guilty. (contrapositive)
c. If I drive a truck, then I need a driver’s license. (converse)
d. If I make less than 60%, then I won’t pass the test. (contrapositive)

Answer

a. If it is not raining, then I don't wear a raincoat.
b. If I am not guilty, then I won't confess.
c. If I need a driver's licenses, then I drive a truck.
d. If I passed the test, then I didn't make less than 60%.

Equivalencies Involving the Conditional Statement and Its Variations

Once looking at a conditional statement and its variations, a natural question to ask is whether or not any of these statements mean the same thing. In other words, are any of the four statements - the conditional, its converse, its inverse, and its contrapositive - logically equivalent? If any of the statements are logically equivalent, then we can substitute one for the other without changing the meaning. Sometimes it can be helpful to make a complicated statement more clear by rewriting it using an logically equivalent variation.

Looking back at Example 7, you can see that the conditional statement "*If I I teach third grade, then I am an elementary teacher*" does not mean the same as its converse "*If I am elementary teacher, then I teach third grade*." They are not logically equivalent because it is possible for the conditional statement to be true while at the same time the converse to be be false. It is possible that someone is an elementary teacher but not teach third grade. That is, the person might teach second grade. This example shows us that a conditional and its converse are not necessarily logically equivalent.

However, looking at truth tables below, we can see that the original conditional statement and its contrapositive are logically equivalent. We can also see that its converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalencies Involving a Conditional Statement and Its Variations

A conditional statement and its contrapositive are logically equivalent: $p \rightarrow q \quad \Leftrightarrow \quad \sim q \rightarrow \sim p$.

The converse and inverse of a conditional statement are logically equivalent: $q \rightarrow p \quad \Leftrightarrow \quad \sim p \rightarrow \sim q$.

In other words, the original statement and the contrapositive must agree with each other. They must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other. They must both be true, or they must both be false.

Also, recall that we have already proven that the conditional $p \rightarrow q$ can be rewritten as disjunction $\sim p \vee q$. Therefore, the following three statements are logically equivalent:

$$p \rightarrow q \quad \Leftrightarrow \quad \sim q \rightarrow \sim p \quad \Leftrightarrow \quad \sim p \vee q$$

We now will use the equivalencies discussed in this section to solve problems.

Example 8

Suppose the following statement is true: *If it rains today, then I will scream.*

Which of the following must also be true?

a. If I do not scream, then it is not raining.
b. If I scream, then it is raining today.
c. If it does not rain today, then I will not scream.

Solution

We let p represent "*It rains*" and q represent "*I scream*." The original conditional statement takes the form of $p \rightarrow q$.

a. "*If I do not scream, then it is not raining*" can be translated to $\sim q \rightarrow \sim p$. This is the contrapositive and is equivalent to the conditional. So, if the original conditional statement is true, so is this statement.
b. "*If I scream, then it is raining today*" can be translated to $q \rightarrow p$. This is the converse and is not equivalent to the conditional. So, if the conditional statement is true, we cannot say that this statement is also necessarily true. It may or may not be true.
c. "*If it does not rain today, then I will not scream*" can be translated to $\sim p \rightarrow \sim q$. This is the inverse and is not equivalent to the conditional. So, if the conditional statement is true, we cannot say that this statement is also necessarily true. It may or may not be true.

Example 9

Rewrite the statement "*If I'm not happy, then there isn't a puppy in the house*" as a simpler, but logically equivalent, conditional statement.

Solution

This conditional statement is awkward and difficult to think about since it contains the word "*not*" twice. Recall that its contrapositive is logically equivalent to a conditional statement. Rewriting the statement with its contrapositive perhaps will make the meaning more clear.

We let p represent "*I'm not happy*" and q represent "*There isn't a puppy in the house*." The original conditional statement takes the form of $p \rightarrow q$ and the contrapositive takes on the form $\sim q \rightarrow \sim p$.

A less cumbersome way to make this statement is "*If there is a puppy in the house, then I am happy.*"

Try it Now 6

Suppose the following statement is true: "*If you microwave salmon in the staff kitchen, then I will be mad at you.*"

If this statement is true, which of the following statements must also be true? Be sure to give a reason for your answer.

a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.

b. If I am not mad at you, then you didn’t microwave salmon in the staff kitchen.
c. If I am mad at you, then you microwaved salmon in the staff kitchen.
d. You didn't microwave salmon in the staff kitchen or I am mad at you.

Answer

Statements b and d must also be true. Statement b is the contrapositive of the original statement. Statement d is the original conditional statement rewritten as a disjunction: $p \rightarrow q \Leftrightarrow \sim p \vee q$.

2.4: Analyzing Symbolic Arguments

Our previous work with statements and truth tables allows us to analyze and to evaluate arguments using logic. Here, the word *argument* is not used in the sense that two or more people disagree with each other. Rather, *argument* is used in the sense of trying to make a convincing case that something must be true using ideas of logic.

The everyday notion of an argument is that it is used to convince us to believe something. An argument is formed with two components: a set of *premises* and a *conclusion*. The thing that we are being encouraged to believe is the **conclusion,** while the **premises** are the statements offered as supporting evidence for the conclusion that we want to make.

Here are some examples of arguments.

Example 1

Premise 1:	If I do not have to go to summer school, then I will get an internship.
Premise 2:	I have to go to summer school.
Conclusion:	I won't get an internship.

Example 2

Premise 1:	I studied or I failed the class.
Premise 2:	I did not fail the class.
Conclusion:	I studied.

To give an argument is to give some premises in support of a conclusion. But suppose that you are given an argument for some conclusion such as the in the previous examples. How can you tell whether that argument is a good argument or bad argument? Our goal in this section will be to analyze an argument to determine whether or not it is a "good" argument.

For an argument to be "good," the truth of an argument's premises must guarantee the conclusion occurs. If this is the case, we will say that the argument is *valid*. When the truth of an argument's premises fails to guarantee the conclusion, we will say that the argument is *invalid*.

Valid Argument

An argument is **valid** if its conclusion necessarily follows from the premises. Otherwise, the argument is **invalid.**

For the conclusion of an argument to necessarily follow from the premises, it means that the symbolic statement

$$[\text{ Premise 1 } \wedge \text{ Premise 2 }] \rightarrow \text{Conclusion}$$

is always true for all possible truth values of the simple statements involved. You might recall that these types of statements are known as *tautologies.* If at least one truth value of this symbolic statement is false, then the argument is invalid. Sometimes, this type of argument is called a *fallacy*.

It is important to note here that if an argument is valid, it only means that conclusion necessarily follows from the premises. It does not mean that conclusion is true.

Arguments can be analyzed using truth tables.

Analyzing Arguments Using Truth Tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 3

Consider the argument

Premise 1:	If you bought bread, then you went to the store.
Premise 2:	You bought bread.
Conclusion:	You went to the store.

Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let b represent "*you bought bread*" and s represent "*you went to the store*". Then, the argument becomes

Premise 1:	$b \rightarrow s$
Premise 2:	b
Conclusion:	s

To test the validity, we look at whether the combination of both premises implies the conclusion. Is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?

b	s	$b \rightarrow s$	$(b \rightarrow s) \wedge b$	$[(b \rightarrow s) \wedge b] \rightarrow s$
T	T	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$
T	F	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
F	T	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
F	F	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$

The last column of the truth table shows that $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true. This is a tautology, and so the argument is valid. Based on these premises, it is correct to conclude "*You went to the store.*"

The next two examples revisit the arguments presented in Example 1 and Example 2. Let's determine whether the arguments presented there were examples of valid or invalid reasoning.

Example 4

Determine whether the argument is valid or invalid:

Premise 1:	If I do not have to go to summer school, then I will get an internship.
Premise 2:	I have to go to summer school.
Conclusion:	I won't get an internship.

Solution

We'll let p represent "*I go to summer school*" and q represent "*I will get an internship*". In symbolic form, the argument becomes

Premise 1:	$\sim p \rightarrow q$
Premise 2:	p
Conclusion:	$\sim q$

Form a truth table to find the truth values for $[(\sim p \rightarrow q) \wedge p] \rightarrow \sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow q$	$(\sim p \rightarrow q) \wedge p$	$[(\sim p \rightarrow q) \wedge p] \rightarrow \sim q$
T	T	F	F	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$
T	F	F	T	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$
F	T	T	F	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
F	F	T	T	$\mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$	$\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$

One truth value for $[(\sim p \rightarrow q) \wedge p] \rightarrow \sim q$ in the last column of the table is False, while the remaining truth values are True. This is not a tautology, and so the argument is invalid. Based on these premises, it is not correct to conclude "*I won't get an internship.*"

Example 5

Determine whether the argument is valid or invalid:

$$\begin{array}{ll} \text{Premise 1:} & \text{I studied or I failed the class.} \\ \text{Premise 2:} & \text{I did not fail the class.} \\ \text{Conclusion:} & \text{I studied.} \end{array}$$

Solution

Let $s =$ *I studied and* $f =$ *I failed the class.*

The premises and conclusion can be stated as:

$$\begin{array}{ll} \text{Premise 1:} & s \vee f \\ \text{Premise 2:} & \sim f \\ \text{Conclusion:} & s \end{array}$$

Form a truth table to find the truth values for $[(s \vee f) \wedge \sim f] \rightarrow s$.

s	f	$s \vee f$	$\sim f$	$(s \vee f) \wedge \sim f$	$[(s \vee f) \wedge \sim f] \rightarrow s$
T	T	$\mathrm{T} \vee \mathrm{T} = \mathrm{T}$	F	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{T} = \mathrm{T}$
T	F	$\mathrm{T} \vee \mathrm{F} = \mathrm{T}$	T	$\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$	$\mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$
F	T	$\mathrm{F} \vee \mathrm{T} = \mathrm{T}$	F	$\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$
F	F	$\mathrm{F} \vee \mathrm{F} = \mathrm{F}$	T	$\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$	$\mathrm{F} \rightarrow \mathrm{F} = \mathrm{T}$

All truth values for $[(s \vee f) \wedge \sim f] \rightarrow s$ in the last column of the table are True. This is a tautology, and so the argument is valid. Based on these premises, it is correct to conclude "*I studied.*"

Try it Now 1

Determine whether the argument is valid:

$$\begin{array}{ll} \text{Premise 1:} & \text{If I have a shovel, I can dig a hole.} \\ \text{Premise 2:} & \text{I dug a hole.} \\ \text{Conclusion:} & \text{Therefore, I had a shovel.} \end{array}$$

Answer

Let $S =$ *have a shovel* and $D =$ *dig a hole*. Premise 1 is equivalent to $S \rightarrow D$. Premise 2 is D. The conclusion is S. We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$.

S	D	$S \rightarrow D$	$(S \rightarrow D) \wedge D$	$[(S \rightarrow D) \wedge D] \rightarrow S$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

This is not a tautology, and so this is an invalid argument. It is not correct to conclude "I had a shovel."

The final example follows the same process of using a truth table to analyze an argument for validity. However, this argument contains 3 simple statements, and its truth table is a bit more complex.

Example 6

Premise 1:	If I go to the mall, then I'll buy new jeans.
Premise 2:	If I buy new jeans, I'll buy a shirt to go with it.
Conclusion:	If I go to the mall, I'll buy a shirt.

Solution

Let $m =$ *I go to the mall*, $j =$ *I buy jeans*, and $s =$ *I buy a shirt*.

The premises and conclusion can be stated as:

Premise 1:	$m \rightarrow j$
Premise 2:	$j \rightarrow s$
Conclusion:	$m \rightarrow s$

We can construct a truth table for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$. Try to recreate each step to verify how the truth table was constructed.

m	j	s	$m \rightarrow j$	$j \rightarrow s$	$(m \rightarrow j) \wedge (j \rightarrow s)$	$m \rightarrow s$	$[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the final column of the truth table, all truth values for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$ are True. This is a tautology, and so the argument is valid. Based on these premises, it is correct to conclude "*If I go to the mall, I'll buy a shirt.*"

2.5: Review Exercises

1. Determine whether the following are statements for the purposes of logic. For each statement, tell whether it is simple or compound.

a. Shut the door.
b. Houston is a city in Texas.
c. There are 7 hours in a day and there are 24 days in a week.
d. $2+3=6$
e. What time is it?
f. If Darren passes the course, then he will graduate.

2. Translate each statement from symbolic notation into a word sentence, where a represents “***Elvis is alive***” and let g represents “***Elvis gained weight.***”

a. $\sim g$
b. $a \vee g$
c. $\sim(a \wedge g)$
d. $g \rightarrow \sim a$
e. $a \leftrightarrow \sim g$

3. Use simple statements p, q, and r to translate each word sentence into symbolic notation.

p: "*The lasagna is hot.*" q: "*The breadsticks are cold.*" r: "*The pizza will be delivered.*"

a. The pizza will not be delivered.
b. The lasagna is hot or the breadsticks are cold.
c. If the breadsticks are cold, then the pizza will be delivered.
d. If the pizza won't be delivered, then the lasagna is hot and the breadsticks are cold.
e. The breadsticks aren't cold if and only if the lasagna isn't hot.
f. It is not true that if the pizza is delivered, then the breadsticks are cold.

4. Use simple statements p, q, r, and s to translate each symbolic statement into a word sentence. Then, determine the truth value of the resulting statement.

p: *The moon is made of cheese.* q: *We live on Earth.* r: *Earth has one moon.* s: *Earth is flat.*

a. $\sim p$
b. $p \vee q$
c. $q \wedge r$
d. $\sim(p \wedge r)$
e. $r \rightarrow p$
f. $p \rightarrow q$
g. $p \leftrightarrow s$

5. Complete the partial truth tables below to determine the truth value of the statements in the last column.

a.

p	q	r	$p \vee q$	$(\sim p \vee q)$	$\sim r$	$\sim(p \vee q) \rightarrow \sim r$
F	F	T				

b.

p	q	$\sim q$	$p \rightarrow q$	$\sim(p \rightarrow q)$	$p \wedge \sim q$	$\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
T	F					

6. Create a complete truth table for each compound statement. Then tell whether the statement is a *tautology*, *self-contradiction*, or *neither*.

a. $p \wedge \sim q$
b. $q \wedge \sim (\sim p \vee q)$
c. $p \rightarrow (\sim q \vee p)$
d. $(\sim p \vee \sim q) \leftrightarrow q$
e. $(p \vee q) \rightarrow \sim r$

7. Use the order of logical operations to determine the truth value of each compound statement.

a. $\sim q \rightarrow \sim p \vee q$ when p is true and q is false.
b. $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim r$ when p is true, q is false, and r is false.

8. Use a truth table to determine whether the two statements are logically equivalent.

$\sim (p \wedge q)$ and $\sim p \wedge \sim q$

9. Use one of DeMorgan's Laws to rewrite each statement.

a. It is not true that Tina likes Sprite or 7-Up.
b. It is not the case that you need a dated receipt and your credit card to return this item.
c. I am not going or she is not going.

10. Use one of DeMorgan's Laws to select the statement that is equivalent to **the negation of** "*Today is Monday and it isn't raining.*"

a. Today isn't Monday and it isn't raining.
b. Today isn't Monday or it isn't raining.
c. Today isn't Monday or it is raining.
d. Today isn't Monday and it is raining.

11. Rewrite each statement using the logical equivalency provided.

a. If you were talking, then you missed the instructions. *Use* $p \rightarrow q \equiv \sim p \vee q$.
b. It is not true that if Luke faces Vader, then Obi-Wan cannot interfere. *Use* $\sim (p \rightarrow q) \equiv p \wedge \sim q$.
c. If you don't look both ways before crossing the street, then you will get hit by a car. *Use* $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

12. Select the statement that is logically equivalent to "*If we get a pay raise, then we will be happy.*"

a. We don't get a pay raise or we are happy.
b. We get a pay raise and we are happy.
c. We get a pay raise and we aren't happy.
d. We don't get a pay raise or we aren't happy.

13. Select the statement that is logically equivalent to **the negation of** "*If you know the password, then you can get in.*"

a. You know the password and you can get in.
b. You don't know the password or you can get in.
c. You don't know the password and you can't get in.
d. You know the password and you can't get in.

14. Consider the conditional statement "*If you read a newspaper, then you will learn something.*"

a. Write its converse.
b. Write its inverse.
c. Write its contrapositive.

15. Consider the conditional statement "*If you are under age 17, then you cannot attend this movie.*"

a. Write its converse.
b. Write its inverse.
c. Write its contrapositive.

16. Select the statement that is logically equivalent to “*If you eat that day-old burrito, you will use lots of hot sauce*.”

a. If you didn’t use lots of hot sauce, then you didn’t eat that day-old burrito.
b. If you don’t eat that day-old burrito, then you won’t use lots of hot sauce.
c. If you used lots of hot sauce, then you ate that day-old burrito.
d. All three statements are equivalent to the given statement.

17. Select the statement that is logically equivalent to "*If I don't invest wisely, then I'll lose my money.*"

a. If I lose my money, then I didn't invest wisely.
b. If I don't lose my money, then I invested wisely.
c. If I invest wisely, then I won't lose my money.

18. Use a truth table to decide whether each argument is valid or invalid.

a. $\begin{array}{ll}\text{Premise 1:} & p \vee q \\ \text{Premise 2:} & p \\ \text{Conclusion:} & \sim q\end{array}$

b. $\begin{array}{ll}\text{Premise 1:} & p \rightarrow q \\ \text{Premise 2:} & \sim q \\ \text{Conclusion:} & \sim p\end{array}$

19. Rewrite each argument symbolically. Then, determine whether the argument is valid or invalid.

a. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.
b. Jamie scrubs the toilets or hoses down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie hoses down the garbage cans.
c. If I don’t change my oil regularly, then my engine dies. My engine died. Therefore, I didn’t change my oil regularly.

CHAPTER OVERVIEW

3: Probability

3.1: Basics of Probability

We see *probabilities* almost every day in our lives. When you pick up the newspaper or read the news on the internet, you most likely encounter probability. There is a 60% chance of rain today, or a poll shows that 52% of voters approve of the President's job performance. Probabilities are essential in sports, gaming, and gambling establishments, but probabilities are also used to make business decisions, figure out insurance premiums, and determine the price of raffle tickets. In its most general sense, probability provides us a way to measure the chance or likelihood that something will happen.

Terminology Used in Probability

Before discussing how to find probabilities, we need to familiarize ourselves with some basic terminology. When studying probability, we consider an **random experiment** to be an activity or an operation that gives a result that can be observed but not predicted ahead of time. If we roll a pair of dice, pick a card from a deck of playing cards, spin a spinner, or randomly select a person and observe their hair color, we are executing an experiment and observing the result.

Any possible result of conducting an experiment is called an **outcome.** For the experiment of flipping a coin, there are only two outcomes: *head* or *tail*. For the experiment of rolling a single die, there are 6 outcomes: *1, 2, 3, 4, 5*, or *6*. For an experiment, this collection of all possible outcomes is called the **sample space.**

An **event** is a collection of outcomes from an experiment. In some instances events contain only one outcome while at other times an event may contain more than one outcome. Consider the event of rolling a single dice. The event "*rolling a 3*" contains only the outcome $\{3\}$ while the event "*rolling an even number*" contains the outcomes $\{2,4,6\}$.

Terminology Associated with Probability

- A random **experiment** is an activity or operation with a result that cannot be predicted ahead of time.
- Any result from conducting an experiment is called an **outcome**.
- The **sample space** of an experiment is the set of all its possible outcomes.
- An **event** is a subset of the sample space and describes a collection of outcomes.

These terms are best illustrated with some examples.

Example 1

Consider an experiment of rolling a single die. When we roll it, only one outcome will occur, but we are unsure which outcome. There are six possible outcomes and so the sample space is $S=\{1,2,3,4,5,6\}$.

'"Rolling a 2" is an event that contains only one outcome $\{2\}$.

"Rolling a number greater than 2" is a another event that contains multiple outcomes $\{3,4,5,6\}$.

Example 2

Two pennies are tossed at the same time. Both pennies may land heads up (which we write as *HH*), or the first penny might land heads up and the second one tails up (which we write as *HT*), and so on. Write the sample space for the experiment and list the outcomes in the event "*getting at least one heads*."

Solution

The sample space for this experiment is $S=\{HH,HT,TH,TT\}$.

If we define event A as "*getting at least one heads,*" the outcomes in event A can be written as $A=\{HH,HT,TH\}$.

Try it Now 1

Gabe performs an experiment of flipping a coin and then rolling a regular six-sided die.

a. Give the sample space for how the coin and die could land.
b. Give the outcomes in event A: rolls an odd number.
c. Give the outcomes in the event B: gets tails and rolls an even number.

Answer

a. $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$
b. $\{H1, H3, H5, T1, T3, T5\}$
c. $\{T2, T4, T6\}$

Probability is one way to measure the chance or the likelihood that an event will occur. Probability is usually denoted in function notation by P, and the event is denoted by a capital letter such as A, B, C, *etc*. The mathematical notation that indicates the probability that event *A* happens is $P(A)$.

Probability

Probability is a numerical measure of the chance or the likelihood that an event will occur.

We will examine two types of probability in this chapter: *empirical probability* and *theoretical probability*. But, we will begin with theoretical probability and discuss the connection and differences between the two types later in this section.

Theoretical Probability

We turn to the concept of theoretical probability and relate it to the sample space, outcomes, and events. We define *theoretical probability* of an event as the ratio of the number of outcomes in the event to the number of outcomes in the sample space of an experiment. For this definition of probability, it is very important that the outcomes in the sample space must be equally likely to occur.

Theoretical Probability

A **theoretical probability** is based on a mathematical model where the number of outcomes in the event is compared with the number of outcomes in the sample space of an experiment. A formula for the theoretical probability of event E is

$$P(E) = \frac{\text{number of outcomes* in event E}}{\text{number of outcomes* in the sample space } S}$$

*where the outcomes are equally likely.

Let's apply this formula in some relatively simple examples.

Example 3

Consider the experiment of rolling a regular six-sided die. Find the probability of each event:

a. rolling a 5
b. rolling an even number
c. rolling a number greater than 4
d. rolling a 7
e. rolling a number less than 7

Solution

There are 6 possible equally likely outcomes in the sample space for this experiment: $S = \{1, 2, 3, 4, 5, 6\}$.

a. There is only 1 outcome in the event "*rolling a 5*": $\{5\}$

$$P(\text{rolling a 5}) = \frac{\text{number of ways to roll a 5}}{\text{number of outcomes in the sample space}} = \frac{1}{6}$$

b. There are 3 outcomes in the event "*rolling an even number*": $\{2, 4, 6\}$.

$$P(\text{rolling an even number}) = \frac{\text{number of ways to roll an even number}}{\text{number of outcomes in the sample space}} = \frac{3}{6} = \frac{1}{2}$$

c. There are 2 outcomes in the event "*rolling a number greater than 4*": $\{5, 6\}$.

$$P(\text{rolling a number greater than 4}) = \frac{\text{number of ways to roll a number greater than 4}}{\text{number of outcomes in the sample space}} = \frac{2}{6} = \frac{1}{3}$$

d. There are no outcomes in the event "*rolling a 7*": $\{\ \ \}$.

$$P(\text{rolling a 7}) = \frac{\text{number of ways to roll a 7}}{\text{number of outcomes in the sample space}} = \frac{0}{6} = 0$$

e. There are 6 outcomes in the event "*rolling a number less than 7*": $\{1, 2, 3, 4, 5, 6\}$

$$P(\text{rolling a number less than 7}) = \frac{\text{number of ways to roll a number less than 7}}{\text{number of outcomes in the sample space}} = \frac{6}{6} = 1$$

The previous example illustrates some important properties about values that can be legitimate probabilities.

- The number of outcomes in an event can never be lower than 0. So, the smallest a probability can be is 0. If the probability of an event is 0 (such as the probability of rolling a 7 in Example 3), we say that event is *impossible*.
- Furthermore, the number of outcomes in an event can never be more than the number of outcomes in the sample space. Therefore, the largest a probability can be is 1. If the probability of an event is 1 (such as the probability of rolling a number less than 7 in Example 3), we say that event is *certain*.
- The previous two facts tell us that the probability of any event must always fall between 0 and 1, inclusive. In the course of this chapter, if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should recheck your work.

Important Probability Properties

An event that cannot occur has a probability of 0. This event is **impossible.**

An event that must occur has a probability of 1. This event is **certain**.

The probability of any event must be between 0 and 1, inclusive. That is, $0 \leq P(E) \leq 1$.

Example 4

Recall the experiment in Example 2: Two pennies are tossed simultaneously and how they land is recorded. Find the probability getting each result:

a. exactly two heads
b. exactly one head
c. at least one head
d. more than 2 heads

Solution

Recall the the sample space for this experiment is $S = \{HH, HT, TH, TT\}$.

a. The event "*exactly two heads*" occurs in only 1 outcome $\{HH\}$: $P(\text{exactly two heads}) = \frac{1}{4}$
b. The event "*exactly one head*" occurs in 2 outcomes, $\{HT, TH\}$: $P(\text{exactly one head}) = \frac{2}{4} = \frac{1}{2}$
c. The event "*at least one head*" occurs in 3 outcomes, $\{HH, HT, TH\}$: $P(\text{at least 1 head}) = \frac{3}{4}$
d. The event "*more than 2 heads*" occurs in 0 outcomes, $\{\ \ \}$: $P(\text{more than 2 heads}) = \frac{0}{4} = 0$

Try it Now 2

Gabe performs an experiment of flipping a coin and then rolling a regular six-sided die. (See *Try it Now 1*.) Find the probability of each event:

a. Gabe rolls an odd number.
b. Gabe gets tails and rolls an even number.
c. Gabe rolls a number than is less than 10.

Answer

a. $\frac{6}{12}=\frac{1}{2}$
b. $\frac{3}{12}=\frac{1}{4}$
c. $\frac{12}{12}=1$

Example 5

A small bookcase contains five math books, three English books, and seven science books - all the same size. One book is chosen at random. What is the probability that a math book is chosen?

Solution

There are 15 possible equally likely books that could be selected, so the number of possible outcomes in the sample space is 15. Of these 15 outcomes, 5 are in the event "*math book*," so the probability that the book will be a math book is $\frac{5}{15}=\frac{1}{3}$.

Try it Now 3

Let's say you have a container of 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Answer

There are 20 possible cherries that could be picked, so the number of possible outcomes in the sample space is 20. Of these 20 possible outcomes, 14 are in the event "sweet," so the probability that the cherry will be sweet is $\frac{14}{20}=\frac{7}{10}$.

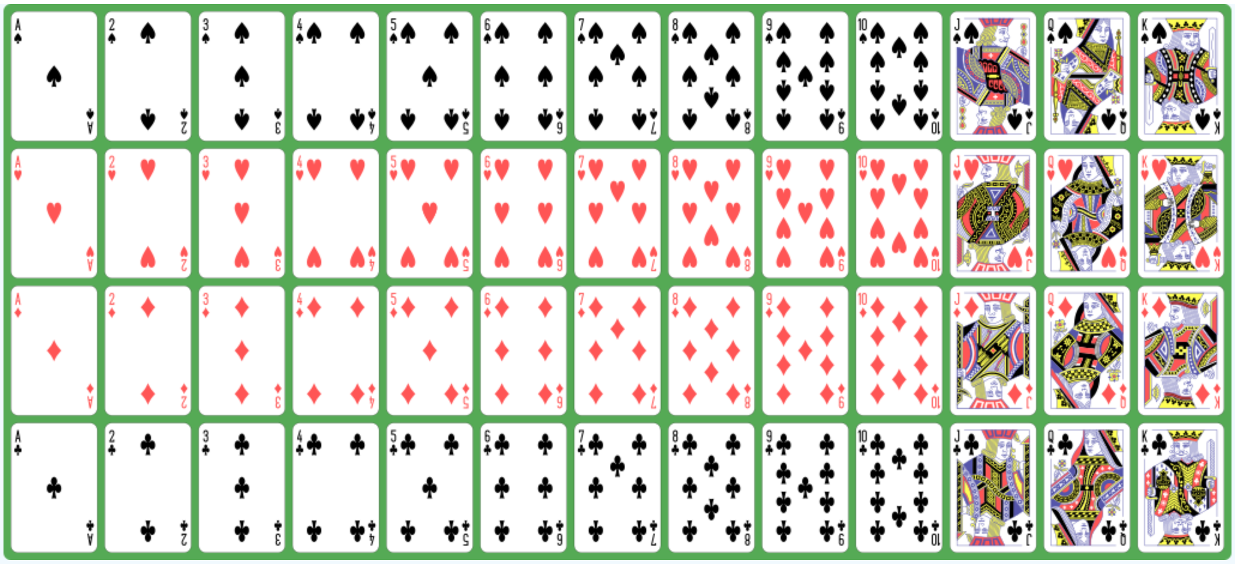
Example 6

Find the probability of drawing each of these cards on the first draw from a standard deck of playing cards:

a. a red card
b. a club
c. a King
d. a King of clubs

Solution

To assist in analyzing the outcomes and sample space, here is a diagram of a deck of cards. Counting the cards, there are 52 equally likely outcomes in the sample space.



[1] ""English pattern" playing cards deck laid out, in SVG vector format"(opens in new window) by Dmitry Fomin(opens in new window) is in the Public Domain, CC0(opens in new window).

a. There are 26 outcomes in the event "*drawing a red card.*" (13 hearts, 13 diamonds). $P(\text{red card} = \frac{26}{52} = \frac{1}{2}$.
b. There are 13 outcomes in the event "d*rawing a club.*" $P(\text{club} = \frac{13}{52} = \frac{1}{4}$.
c. There are 4 outcomes in the event "*drawing a King.*" $P(\text{King}) = \frac{4}{52} = \frac{1}{12}$.
d. There is only 1 outcome in the event "*drawing a King of clubs.*" $P(\text{King of clubs}) = \frac{1}{52}$

Try it Now 4

Find the probability of drawing each of these cards on the first draw from a standard deck of playing cards:

a. a 3 card
b. a red 3 card
c. a 3 of diamonds
d. a red face card (Jack, Queen, or King)

Answer

a. $\frac{4}{52} = \frac{1}{13}$
b. $\frac{2}{52} = \frac{1}{26}$
c. $\frac{1}{52}$
d. $\frac{12}{52} = \frac{3}{13}$

Empirical Probability

Up to this point, we have been finding probability theoretically by counting the equally likely outcomes in the sample space and in the event. For example, the theoretical probability of a coin '*landing heads up*' is $\frac{1}{2}$ because there are two equally likely outcomes in the sample space and only one of those outcomes is in the event '*landing heads up.*'

However, if you flip a coin 10 times, you may very well observe that the coin lands heads up 40%, 60%, or even 70% of the time! This discrepancy is perfectly natural and expected when conducting experiments, and it is important to recognize it. This brings us to the difference between *theoretical probability* and *empirical probability*.

- *Theoretical probability* is the calculated chance that an event occurs if the same experiment were completed an infinite number of times. We sometimes use our knowledge of sets and set operations when finding theoretical probabilities.
- *Empirical probability* is the calculated chance that an event occurs based on how often the outcome has occurred in the past over a limited number of trials. Generally speaking, when we determine the chance of something happening in the future by

studying past results we are using empirical probability. Because empirical probability is determined by actual observations of an experiment it is also known as *experimental probability*.

Theoretical vs. Empirical Probability

- A **theoretical probability** is based on a mathematical model where all outcomes are equally likely to occur.
- An **empirical probability** is based on collected data and is the relative frequency of the event occurring.

Example 7

Lawrence is playing with a standard 52-card deck and wants to find the probability of selecting a Queen from the deck.

The *theoretical* probability that Lawrence pulls a single Queen is $\frac{4}{52} = \frac{1}{13} \approx 0.0769$ or about 7.69%.

If Lawrence decides to try it out 25 times and pulls a Queen at random 3 times in 25 trials of "pull a card, record it, put it back," the *empirical* probability of pulling a Queen is $\frac{3}{25} = 0.12$, or about 12%.

Example 8

A game uses a spinner like the one shown below. When spun, the arrow lands on either a white, yellow, blue, or red region. Let's describe how we could use empirical probability to investigate the chance of landing on the white region.

The regions do not appear to be the same size, so the four color outcomes are not equally likely. We cannot apply our theoretical probability formula here unless we are told the areas of the regions or the measures of the central angles in the circular spinner. Instead, we can spin the spinner many times and record the results. Assume these hypothetical results have occurred:

Hypothetical Results for a Spinner

Color	Number of Spins
White	18
Red	12
Blue	6
Yellow	4

Using the data in the table, the spinner landed on '*white*' 18 times out of a total of $18+12+6+4=40$ times. The empirical probability that the spinner lands on white is $\frac{18}{40} = \frac{9}{20} = 0.45 = 45\%$. This seems reasonable as the white region makes up slightly less than half the spinner.

Often times, we calculate empirical probabilities from data that have already been collected and organized for us.

Example 9

In a given week, a veterinarian recorded how many times she treated each type of animal. What is the probability that the next animal she treats is a cat?

Number of Each Type of Animal Treated

Animal	Number Treated
Dog	40
Cat	32

Animal	Number Treated
Bird	9
Rabbit	7
Iguana	2

Solution

The total number of trials (animals treated) is $40+32+9+7+2=90$. Of these, 32 were cats. The fraction of trials that resulted in the event "*cat*" gives the empirical probability: $P(\text{cat})=\frac{32}{90}=\frac{16}{45}\approx 0.3556 \text{ or } 35.56\%$.

Try it Now 5

A survey was conducted to determine the programs that students enrolled in MAT 1130 were taking. The results are shown in the table. What is the probability that a student in MAT 1130 is in the Computer Information Technology program?

Number of MAT 1130 Students per Program

Major	English	Art	Nursing	Culinary Arts & Hospitality	Computer Information Technology
Number of Students	3	8	2	12	10

Answer

$\frac{10}{35}=\frac{2}{7}\approx 0.2857 \text{ or } 28.57\%$

Example 10

The table shows the results of a survey where each person was asked 1) *Do you have cable TV?* and 2) *Did you go on vacation in the past year?*.

Results of Cable TV and Vacation Survey

	Took a Vacation	**No Vacation**	**Total**
Has Cable TV	97	38	135
Doesn't Have Cable TV	14	17	31
Total	111	55	166

Use the table to find the probability of each event:

a. $P(\text{a person takes a vacation})$
b. $P(\text{a person does not have cable TV})$
c. $P(\text{a person takes a vacation and has cable TV})$
d. $P(\text{a person takes a vacation or has cable TV })$

Solution

There are 166 people surveyed so the number of trials of this experiment is 166. We need to determine the number of outcomes in each event and then find its relative frequency.

a. The total in the first column shows 111 people took a vacation last year:

$$P(\text{a person takes a vacation})=\frac{111}{166}\approx 0.6687 \text{ or } 66.87\%.$$

b. The total in the second row shows 31 people do not have cable TV:

$$P(\text{a person does not have cable TV})=\frac{31}{166}\approx 0.1867 \text{ or } 18.67\%.$$

c. There are 97 people who did *both* of these things:

$$P(\text{a person takes a vacation and has cable TV}) = \frac{97}{166} \approx 0.5843 \text{ or } 58.43\%.$$

d. There are $97+38+14=149$ people who did *one or the other or both* of these things:

$$P(\text{a person takes a vacation or has cable TV}) = \frac{149}{166} \approx 0.8976 \text{ or } 89.76\%.$$

Try it Now 6

Nate recorded the topping and type of crust for all the pizzas he sold yesterday.

Type of Crust and Toppings for Nate's Pizzas

	Pepperoni	Sausage	Veggie	Total
Thick Crust	10	15	5	30
Thin Crust	24	16	10	50
Total	34	31	15	80

Use the table to find the probability of each event the next time Nate sells a pizza.

a. $P(\text{sausage topping})$
b. $P(\text{sausage topping or thick crust})$
c. $P(\text{veggie topping and thin crust})$
d. $P(\text{topping contains meat })$

Answer

a. $\frac{31}{80} = 0.3875 \text{ or } 38.75\%$
b. $\frac{46}{80} = 0.575 \text{ or } 57.5\%$
c. $\frac{10}{80} = 0.125 \text{ or } 12.5\%$
d. $\frac{65}{80} = 0.8125 \text{ or } 81.25\%$

The Law of Large Numbers

Flipping a coin is often used to randomly make a decision when there are only two choices. For example, you may flip a coin to decide whether you have steak or fish for dinner. Or, a referee uses a coin flip to decide which football team receives the ball prior to kickoff. The reason why a coin flip seems fair in these circumstances is that most of us agree that the probability of getting heads (and tails) on a coin is $\frac{1}{2} = 0.5 = 50\%$. But what does this mean in practice?

Does that mean if we flip a coin twice we will get heads exactly once? If a coin is tossed 10 times, will we necessarily get heads five times? Most of intuitively know the answer is no. Indeed, if we flip a coin 10 times we might find that it lands on heads 7 times. So what does it mean to say that the probability of heads on a fair coin is $\frac{1}{2}$?

To investigate this question, consider the table showing results that may happen when a coin is toss several times. The top row shows the number times the coin has been tossed. The next row shows the number of heads that have occurred. The bottom row shows the empirical probability which is the ratio of the number of heads observed to the number trials.

Results of Tossing a Coin Many Times

Number of Trials	10	20	30	40	50
Number of Heads Observed	7	13	17	22	26
Empirical Probability of Heads	$\frac{7}{10} = 0.7$	$\frac{13}{20} = 0.65$	$\frac{17}{30} \approx 0.57$	$\frac{22}{40} = 0.55$	$\frac{26}{50} = 0.52$

Notice that as the number of trials increases, the empirical probability gets closer to 0.5, which is what we expect to happen theoretically. In fact, if we kept increasing the number of trials, we would find that the empirical probability would eventually be very, very close to $\frac{1}{2}=0.5$.

This relationship between empirical probability and theoretical probability can be summarized by the *Law of Large Numbers*. The probability of an event applies to a large number of trials, not a single or a few trials. We should not be surprised that the empirical probability calculated from only a few trials is different from the theoretical probability. It is only empirical probability calculated over the long-run that gives an accurate probability.

Law of Large Numbers

The **Law of Large Numbers** states that we can only expect the empirical probability of an event to approximate its true probability when the number of trials of the experiment is large.

Example 11

Stanley picks a card from a standard deck of cards and gets heart. He returns the card to the deck, picks a second time, and gets another heart. Stanley repeats this process a total of five times and gets a heart in four of those trials. Should Stanley conclude that the probability of selecting a heart is $\frac{4}{5}=0.80=80\%$?

Solution

No. Stanley's reasoning is incorrect. He should not make a conclusion about the true probability of selecting a heart based only on five trials. If Stanley wants to estimate the chance of selecting a heart using empirical probability, he should perform the experiment a lot more than five times.

3.2: Odds

When we talk about probability of getting tails on a coin, we generally think of the ratio of the one outcome in the event "*tails*" to the two outcomes in the sample space. So, we say the probability of getting tails is $\frac{1}{2}$ or 50%.

However, you also may have heard phrases such as "*fifty-fifty*" or "*1 to 1*" to describe the chance of getting tails. These phrases mean that the coin is as likely to come up tails as heads (each event occurring 50% of the time). This latter way to describe the chance of getting tails is what we refer to as the *odds*.

The odds of an event is a ratio that compares two sets of outcomes – those in the event and those not in the event. The **odds** in favor of event E is the ratio of the number of outcomes in event E to the number of outcomes that are not in the event in event E. Recall that we call this second set the *complement* of E. So, when we speak of odds in favor of E, we are giving the ratio $n(E)$ to $n(E')$.

For example, suppose we roll a regular six-sided die and define event E to be "*roll the number six*." There is only one way to roll a six, and there are five ways not to roll a six. This ratio is 1 to 5. So, the odds in favor of rolling the number six is $1:5$.

There are two types of odds, *odds in favor* and *odds against*, so it is always important that we are careful to distinguish between them. Casinos, race tracks and other types of gambling usually state the *odds against* an event happening. If the *odds in favor* of rolling a six on a die is $1:5$, then the *odds against* rolling a six on a die is the reciprocal ratio $5:1$.

Finding Odds from the Sample Space and from Probability

The formulas for *odds in favor* and *odds against* are shown below in two different ways. One way shows how to find odds if you know the number of outcomes in event E and in the event E'. The second way shows how to find odds if instead you know $P(E)$ and $P(E \text{ does not occur})$.

Odds

The **odds in favor** of event E is given by

$$\text{odds in favor of } E=\frac{\text{number of ways for E to occur}}{\text{number of ways for E to not occur}} \quad \text{OR} \quad \frac{P(E)}{P(E \text{ does not occur})}$$

Also, the **odds against** event E is given by

$$\text{odds against } E=\frac{\text{number of ways for E to not occur}}{\text{number of ways for E to occur}} \quad \text{OR} \quad \frac{P(E \text{ does not occur})}{P(E)}$$

It should be noted that odds always use whole numbers. You never include decimals or percents with odds. Also, we should always write odds in lowest terms using the greatest common divisor just as we would write a fraction. Let's explore odds using some examples.

Example 1

A single card is drawn from a well-shuffled deck of 52 cards. Find the odds in favor of "*drawing a red eight*."

Solution

There are two red eights in the deck -- which means there are $52-2=50$ cards that are not red eights. We can find the odds in favor of this event by forming the ratio

$$\text{odds in favor of a red eight}=\frac{\text{number of ways to get a red eight}}{\text{number of ways not to get a red eight}}=\frac{2}{50}=\frac{1}{25} \quad .$$

The odds in favor of getting a red eight is 1 to 25, or $1:25$.

Try it Now 1

A bag of 30 candies contain 12 that are peppermint. What are the odds in favor of selecting peppermint?

Answer

$2:3$

Example 2

Many roulette wheels have slots numbered 0, 00, and 1 through 36. The slots numbered 0 and 00 are green. The even numbered slots are red and the odd numbered slots are black. The game is played by spinning the wheel one direction and rolling a marble around the outer edge the other direction. Players bet on which slot the marble will fall into. What are the odds in favor of the marble landing in a red slot?

Solution

There are 38 slots in all. The slots 2, 4, 6, …, 36 are red so there are 18 red slots. The other 20 slots are not red.

We can find the odds in favor of the marble landing on red by forming the ratio

$$\text{odds in favor of red} = \frac{\text{number of ways to land on red}}{\text{number of ways not to land on red}} = \frac{18}{20} = \frac{9}{10} .$$

The odds in favor of the marble landing on red is 9 to 10, or $9:10$.

Try it Now 2

On a roulette wheel, what are the odds in favor of

a. the marble landing on green?
b. the marble landing on a slot numbered with two digits?

Answer

a. $1:18$
b. $14:5$

In the previous two examples, it was easy to count the outcomes in the event and its complement to form the odds. Sometimes, you may not know the number of outcomes in the event but you know the probability instead. This is where the second formula is useful.

Example 3

The probability of an event is $\frac{3}{8}$. What are the odds in favor of this event?

Solution

Here we know $P(E)=\frac{3}{8}$, so $P(E \text{ does not occur}) = 1-\frac{3}{8}=\frac{5}{8}$. We can find the odds in favor of the event by forming the ratio

$$\text{odds in favor of E} = \frac{P(E)}{P(E \text{ does not occur})} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{\cancel{8}} \times \frac{\cancel{8}}{5} = \frac{3}{5} .$$

The odds in favor of the event is 3 to 5, or $3:5$.

Example 4

The probability that it snows tomorrow is 60%. What are the odds that it snows tomorrow?

Solution

Here we know $P(\text{snow})=60\%$ or $\frac{60}{100}$. This means $P(\text{no snow})=1-\frac{60}{100}=\frac{40}{100}$. We can find the odds in favor of the event by forming the ratio

$$\text{odds in favor of snow}=\frac{P(\text{snow})}{P(\text{no snow})}=\frac{\frac{60}{100}}{\frac{40}{100}}=\frac{60}{\cancel{100}}\times\frac{\cancel{100}}{40}=\frac{60}{40}=\frac{3}{2}\quad.$$

The odds in favor of snow is 3 to 2, or $3:2$.

Try it Now 3

The probability that someone who takes a flu shot stills gets the flu is 12%. What are the odds in favor of getting the flu if you get the shot?

Answer

$3:22$

Example 5

Two fair dice are tossed and the sum is recorded. Find the odds against rolling a sum of nine.

Solution

The event "*roll a sum of nine*" contains 4 outcomes: $E=\{(3,6),(4,5),(5,4),(6,3)\}$.

There are $6\times 6=36$ ways to roll two dice and four ways to roll a sum of nine. That means there are 32 ways to roll a sum that is not nine.

$$\text{odds against rolling a sum of nine}=\frac{\text{number of ways not to roll a sum of nine}}{\text{number of ways to roll a sum of nine}}=\frac{32}{4}=\frac{8}{1}$$

The odds against rolling a sum of nine is 8 to 1, or $8:1$.

Try it Now 4

A bag contains 6 red marbles, 3 blue marbles, and 1 yellow marble.

a. What are the odds in favor of choosing a red marble?
b. What are the odds against choosing a blue marble?

Answer

a. $3:2$
b. $7:3$

Example 6

The probability that Stephan wins his race is $\frac{2}{15}$. What are the odds against Stephan winning his race?

Solution

Here we know $P(\text{Stephan wins})=\frac{2}{15}$, so $P(\text{Stephan doesn't win})=1-\frac{2}{15}=\frac{13}{15}$. We can find the odds against the event by forming the ratio

$$\text{odds against Stephan wining} = \frac{P(\text{Stephan doesn't win})}{P(\text{Stephan wins})} = \frac{\frac{13}{15}}{\frac{2}{15}} = \frac{13}{\cancel{15}} \times \frac{\cancel{15}}{2} = \frac{13}{2} \quad .$$

The odds against Stephan winning his race is 13 to 2, or $13 : 2$.

Try it Now 5

The probability winning a prize in a game is 0.15. What are the odds against winning a prize?

Answer

$17 : 3$

Finding Probabilities from Odds

Sometimes we may want to express the probability of an event happening based on the odds for the event. Saying that the odds in favor of an event are 3 to 5 means that the event happens three times for every five times it does not happen. If we add up the possibilities of both we get a sum of $3+5=8$. So the event happens about three out of every eight times. We would say the probability is $3/8$.

In other words, the sum of the odds gives the denominator of the probability. The value used in the numerator of the probability fraction depends on whether you are finding the probability of the event happening or the probability of the event not happening.

This relationship is summarized in the box.

Going from Odds to Probability

If the odds in favor of event E are $a : b$, then

$$P(E) = \frac{a}{a+b} \text{ and } P(E \text{ does not occur}) = \frac{b}{a+b} \, .$$

Example 7

A local little league baseball team is going to a tournament. The odds of the team winning the tournament are 3 to 7. Find the probability of the team winning the tournament.

Solution

$P(\text{winning}) = \dfrac{3}{3+7} = \dfrac{3}{10} = 0.3$, or 30%.

Example 8

The odds against Mathemagic winning the horse race derby are $5 : 9$. What is the probability that Mathemagic wins the race?

Solution

Be cautious. The odds are given as odds against winning, but the question asks for the probability of winning.

$P(\text{winning}) = \dfrac{9}{5+9} = \dfrac{9}{14} \approx 0.6429$, or 64.29%.

Try it Now 5

Jenny estimates her odds of earning an A on the next test as $7 : 18$. What is Jenny's probability of earning an A?

Answer

$\frac{7}{25} = 0.28$, or 28%.

3.3: Expected Value

In this section we look at *expectation* of a result that is determined by chance. Although the outcomes of an experiment is random and cannot be predicted on any one trial, we need a way to describe what should happen on average over the long run.

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions. It's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Let's begin with a very simple example of a game that Rena and Larson are playing. Rena rolls a regular six-sided die. If the die lands on an even number, Rena gets $1 from Larson. However, if the die lands on a odd number, Rena must give $1 to Larson. Either player could win $1 on each roll of the die. But, who would win money in the long run if Rena and Larson play this game many times?

Intuitively and based on our understanding of probabilities on a die, we expect Rena to win $1 about half the time and Larson to win $1 about half the time. We would expect Rena to break even, winning as much as she loses. Mathematically, we can determine Rena's expected gain or loss using the following procedure:

$$\begin{aligned}\text{Rena's expected gain or loss} &= \Big(\text{amount Rena wins} \times P(\text{Rena wins})\Big) + \Big(\text{amount Rena loses} \times P(\text{Rena loses})\Big)\\ &= \left(\$1 \times \frac{1}{2}\right) + \left(-\$1 \times \frac{1}{2}\right)\\ &= \$0.50 + (-\$0.50)\\ &= \$0\end{aligned}$$

Rena has an expected gain or loss of $0 as we predicted. The expected gain or loss is called the **expected value** and indicates that Rena should expect to break even if she plays the game many times. If Rena's expected value had been positive, she would have gained money after playing many times. If Rena's expected value had been negative, she would have lost money after playing many times. However, it is important to understand that this does not tell Rena what to expect on any one game -- just the long-term expectation.

As in the procedure above, we compute the expected value by multiplying the amount gained/lost for each outcome by the probability of that outcome. Then, we add the products.

Expected Value

Expected Value (EV) is the average gain or loss if an experiment or procedure with a numerical outcome is repeated many times.

$$EV = X_1 \cdot P_1 + X_2 \cdot P_2 + X_3 \cdot P_3 + \ldots + X_n \cdot P_n$$

where each X is the net amount gained or lost on each outcome of the experiment and P is the probability of that outcome.

Example 1

A company plans to invest in a particular construction project. There is a 35% chance that it will lose $30,000, a 40% chance that it will break even, and a 25% chance that it will make a profit of $55,000. How much can the company expect to make or lose on this project?

Solution

It is helpful to make a table that summarizes all the information before finding the expected value. Represent the the loss outcome with a negative number, the break even outcome with 0, and the profit outcome with a positive number.

Outcome (Amount of Profit or Loss)	$-\$30,000$	$\$0$	$\$55,000$
Probability	$35\% = 0.35$	$40\% = 0.40$	$25\% = 0.25$

Now use the formula for the expected value:

$$\begin{aligned} EV &= -\$30,000\,(0.35) + \$0\,(0.40) + \$55,000\,(0.25) \\ &= \$3,250 \end{aligned}$$

The company has an expectation, or average expected gain, of $3,250. If this company makes investments in projects like this one with these particular probabilities and amounts, in the long run it would make an average profit of $3,250 per project. However, there is still a 40% chance of not making any money and a 35% chance of losing $35,000.

Not all problems involving expectation are about games or money. The outcome can be anything that is numerical as the following problem shows.

Example 2

The organizers at a balloon festival know that if it's windy in town, only 9,000 people will show up for the festival. But if it's not windy, 30,000 people will show up. The probability of wind is 45%. What is the expectation for the number of people who attend?

Solution

There are two outcomes in this scenario. It is windy and it is not windy. The numerical values for those outcomes are 9,000 people and 30,000 people respectively. The probability of a windy day is 45% so the probability it is not windy is $100\% - 45\% = 55\%$. Organizing these values into a table, we have

Probabilities of Attendance

Outcome (attendance)	Windy $9,000$	Not Windy $30,000$
Probability	$45\% = 0.45$	$55\% = 0.55$

Now use the formula for the expected value:

$$\begin{aligned} EV &= 9,000\,(0.45) + 30,000\,(0.55) \\ &= 20,550 \end{aligned}$$

The organizers have an expectation that 20,550 people will attend this festival based on these attendance estimates and probabilities of wind.

Try it Now 1

A hot dog vendor knows that there is a 20% chance of selling a hot dog with no toppings, a 50% chance of a hot dog with 1 topping, and a 30% chance of selling a hot dog with 2 toppings. What is the expected number of toppings sold per hot dog?

Answer

1.1 toppings

Expectation is very important in analyzing games of chance. While the expected value won't tell you how much you will win or lose on any one play of the game, it will give you a sense of what should happen in the long run if you play the game many times. If the expected value is positive, then you more likely have an advantage overall. If the expected value is negative, then you more likely have an disadvantage overall. If the expected value of a game is 0, then we call it a *fair game*.

Fair Game

A game that has an expected value of zero is called a **fair game.**

In the following example, we will analyze a game to determine whether it is fair in terms of winnings.

Example 3

A carnival game consists of drawing one ball from a box containing two yellow balls, five red balls, and eight white balls. If the ball is yellow, you win \$5. If the ball is red, you win \$2. If the ball is white, you lose \$3.00.

Is this a fair game? If you play the game many times, how much would you expect to gain or lose per game?

Solution

It is helpful to make a table that summarizes all the information before finding the expected value.

Probabilities of Winning and Losing

Outcome (amount won or lost)	Yellow $\$5$	Red $\$2$	White $-\$3$
Probability	$\frac{2}{15}$	$\frac{5}{15}=\frac{1}{3}$	$\frac{8}{15}$

Now use the formula for the expected value:

$$EV=\$5\left(\frac{2}{15}\right)+\$2\left(\frac{1}{3}\right)+-\$3\left(\frac{8}{15}\right)\approx-\$0.27$$

Since the expected value is not zero this is not a fair game. The expected value is negative which indicates that a player can expect to lose about 27 cents per game on average when playing this game many times. That means that the carnival will make an average of \$0.27 for every game played.

Picking a ball from the bag is a random experiment so we cannot predict what will happen if we play the game once. We can predict what will happen only if we play the game many times.

Try it Now 2

A game consists of rolling a colored die with three green sides, two red sides, and one blue side. If you roll green, you earn 2 points. A roll of a red earns you 1 point. If you roll blue, you lose 8 points. Is this a fair game? If not, how many points can you expect to earn or lose per game?

Answer

This is a fair game because the expected value is 0 points.

Example 4

Valley View Elementary is raising money to buy tablets for the classrooms. The PTA sells 2,000 raffle tickets at \$3 each.

- First prize is a flat-screen TV worth \$500.
- Second prize is an android tablet worth \$375.
- Third prize is an e-reader worth \$200.
- Five \$25 gift certificates will also be awarded.

What are the expected winnings for a person who buys one ticket?

Solution

A total of 8 tickets are winners and the other 1,992 tickets are losers. The cost is \$3 per ticket so we must reduce all values won by \$3 to find the *net amount* won or lost.

Outcome (net amount won or lost)	First Prize $\$500-\$3=\$497$	Second Prize $\$375-\$3=\$372$	Third Prize $\$200-\$3=\$197$	Gift Certificates $\$25-\$3=\$22$	No Prize $0-\$3=-\$3$
Probability	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{5}{2000}=\frac{1}{400}$	$\frac{1992}{2000}=\frac{249}{250}$

Now use the formula for the expected value:

$$\begin{aligned} EV &= \$497\left(\frac{1}{2000}\right)+\$372\left(\frac{1}{2000}\right)+\$197\left(\frac{1}{2000}\right)+\$22\left(\frac{1}{400}\right)+\$-3\left(\frac{249}{250}\right) \\ &= -2.40 \end{aligned}$$

That means the expected winnings per ticket are \$0.60 - \$3 = -\$2.40.

We would expect to lose an average of \$2.40 for each ticket bought. This means that the school will earn an average of \$2.40 for each ticket bought for a profit of $\$2.40 \cdot 2,000 \text{ tickets } = \$4,800$.

✓ Example 5

A real estate investor buys a parcel of land for \$150,000. He estimates the probability that he can sell it for \$200,000 to be 0.40, the probability that he can sell it for \$160,000 to be 0.45 and the probability that he can sell it for \$125,000 to be 0.15. What is the expected profit for this purchase?

Solution

Find the net profit for each situation first: \$200,000 – \$150,000 = \$50,000 profit, \$160,000 - \$150,000 = \$10,000 profit, and \$125,000 - \$150,000 = -\$25,000 profit (loss).

Outcome (net profit or loss)	\$50,000	\$10,000	-\$25,000
Probability	0.40	0.45	0.15

Now use the formula for the expected value:

$$\begin{aligned} EV &= 50,000(0.40)+10,000(0.45)+(-25,000)(0.15) \\ &= \$20,750 \end{aligned}$$

The expected profit from the purchase is \$20,750.

Expected value is very common in making insurance decisions. Insurance companies take into account various factors to determine probabilities that policy holders will have an accident, fire, or death. They set prices on policies to make ensure the company will remain profitable even though they will occasionally pay out large monetary claims. In other words, even though the insurance companies expect some negative outcomes, it is important to their bottom line that their expected return remain positive.

Let's look at an illustration.

✓ Example 6

The cost of a \$50,000 life insurance policy is \$150 per year for a person who is 21-years old. Assume the probability that a person will die at age 21 is 0.001. What is the company's expected profit on a policy of this type? How much can the company expect to earn in profit if it sells 10,000 of these policies to 21-year olds?

Solution

There are two outcomes here. If the person lives, the insurance company makes a pure profit of \$150. The probability that the person lives is 1-0.001=0.999. If the person dies, the company takes in \$150 and but must also pay out \$50,000 for a loss of \$49,850.

Outcome (insurance company's net profit or loss)	Person Lives \$150	Person Dies $\$150 - \$50,000 = -\$49,850$
Probability	0.999	0.001

The expected value for one policy is

$$EV = 150(0.999)+(-\$49,850)(0.001) = \$100$$

If the company sells 10,000 policies at a profit of $100 each, the total expected profit is $\$100(10,000) = \$1,000,000$

Try it Now 3

You take out a fire insurance policy on your home. The yearly premium is $250. In case of fire, the insurance company will pay you $200,000. The probability of a house fire in your area is 0.003%. What is the insurance company's expected value for selling one of these policy?

Answer

$244

3.4: Working with Events

The Complement Rule

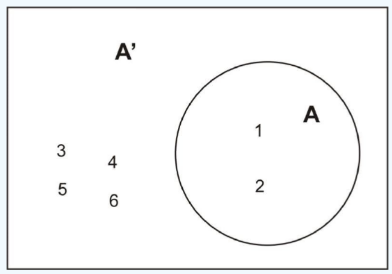
In the sets and logic chapters, we saw the complement of set A and the negation of statement A. We defined the complement of set A as another set containing the elements in the universal set that are not in set A, and we wrote the complement of A as A'. Similarly, we defined the negation of statement A as exactly the opposite of statement A and wrote the negation of A as $\sim A$.

In probability, we can think of the complement in a similar way. The **complement** of event A contains all outcomes in the sample space that are not in event A. We write the probability of the complement as $P(A')$ or $P(\text{not } A)$.

Example 1

Consider the experiment of rolling a regular six-sided die. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. If event A is "*rolling a number less than 3,* " then $A = \{1, 2\}$. The complement of event A', "*not rolling a number less than 3,*" contains all the outcomes in the sample space that are not in A. Thus, $A' = \{3, 4, 5, 6\}$.

A Venn diagram that illustrates the relationship between S, A, and A' is shown below:



Here, $P(A) = \frac{2}{6} = \frac{1}{3}$ and $P(A') = \frac{4}{6} = \frac{2}{3}$.

When we combining all outcomes in event A with all outcomes in its complement A', the result is all the outcomes in the sample space of the experiment. Therefore, the probabilities of an event and its complement must add to 1. This formula and its variations are known as the *Complement Rule of Probability.*

Complement Rule of Probability

$$\begin{aligned} P(A) + P(A') &= 1 \text{ or } P(A) + P(\text{not } A) = 1 \\ \text{Alternatively, } P(A) &= 1 - P(A') \text{ or } P(A) = 1 - P(\text{not } A) \\ P(A') &= 1 - P(A) \text{ or } P(\text{not } A) = 1 - P(A) \end{aligned}$$

As shown in the box, you may find that it is sometimes easier to calculate the probability of an event's complement than it is to calculate the probability of the event itself. Once this is done, the probability of the event, $P(A)$, is found using the relationship $1 - P(A')$.

Example 2

Suppose you know that the probability of getting the flu this winter is 0.43. What is the probability that you will not get the flu?

Solution

The probability of getting the flu is $P(F) = 0.43$.

The probability of not getting the flu is $P(F') = 1 - P(F) = 1 - 0.43 = 0.57$.

Example 3

The probability that a person does not live in an industrialized country of the world is $\frac{4}{5}$. Find the probability that a person lives in an industrialized country.

Solution

The probability that a person does not live in an industrialized country is $P(I') = \frac{4}{5}$.

The probability that a person lives in an industrialized country is $P(I) = 1 - P(I') = 1 - \frac{4}{5} = \frac{5}{5} - \frac{4}{5} = \frac{1}{5}$.

Example 4

A card is drawn from a standard deck of cards. What is the probability that the card is *not* a King?

Solution

The probability of getting a King is $P(K)=\frac{4}{52}=\frac{1}{13}$.

Therefore, the probability of not getting a King is $P(K')=1-\frac{1}{13}=\frac{13}{13}-\frac{1}{13}=\frac{12}{13}$.

Try it Now 1

A card is drawn from a standard deck of cards. What is the probability that the card is *not* a club?

Answer

$\frac{3}{4}$

Combining Events

Many probabilities in real life involve more than one event. If we draw a single card from a deck we might want to know the probability that it is either red **or** a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair **and** blue eyes. When we combine two outcomes to make a single event we connect the outcomes with the word "or" or the word "and." It is very important in probability to pay attention to the words "and" and "or" if they appear in a problem. The word "and" restricts the field of possible outcomes to only those outcomes that satisfy more than one event at the same time or in succession. The word "or" broadens the field of possible outcomes to those that satisfy one or more events.

Combining Events with "Or"

The event "A or B" refers to an event that includes outcomes of A, or B, or both. As we saw with sets and truth tables, the union $A\cup B$ and disjunction $A\vee B$ refer to where at least one of A or B occur. Let's begin the development of a rule for $P(A \text{ or } B)$ by exploring an example.

Example 5

A box contains 12 slips of paper, each numbered from 1 to 12. The sample space is $\{1,2,3,4,5,6,7,8,9,10,11,12\}$These twelve outcomes are all equally likely. What is the probability of selecting an even number or a number divisible by 3?

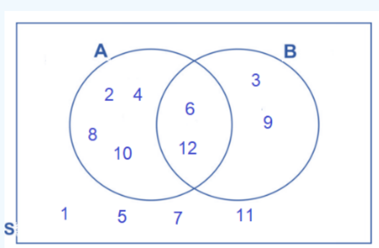
Solution

The event in question must meet one or more of the two criteria: *selecting an even number* or *a number divisible by 3*. This means we can select a 2, 4, 6, 8, 10, or 12 (an even number) or select a 3, 6, 9, or 12 (a number divisible by 3.) Here,

$$P(\text{even number})=\frac{6}{12} \text{ and } P(\text{divisible by } 3)=\frac{4}{12}.$$

It is tempting to add $P(\text{even number})+P(\text{number divisible by } 3)=\frac{6}{12}+\frac{4}{12}=\frac{10}{12}$. However, there is an error in this thinking.

To see this error, look at the sample space of this experiment represented by a Venn diagram. We are interested in finding $P(A \text{ or } B)$, which is the union of two sets in the diagram.

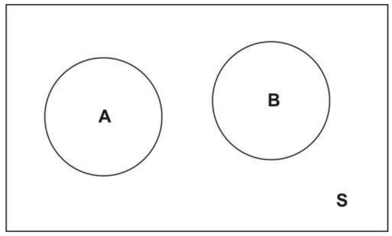


The outcomes 6 and 12 meet both of these criteria because they are even numbers **and** divisible by 3. Consequently, we have included these outcomes in both $P(\text{even number})$ and $P(\text{number divisible by } 3)$. To compensate for over-counting the outcomes 6 and 12, we must subtract the probability of these two over-counted outcomes, $P(\text{even number and divisible by } 3)=\frac{2}{12}$. So,

$P(\text{even number or number divisible by } 3)=\frac{6}{12}+\frac{4}{12}-\frac{2}{12}=\frac{8}{12}=\frac{2}{3}$.

Based on the previous example, it appears that $P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$. This thinking should not be new or a surprise. Joining two sets with "or" is a union of those two sets. And, in Chapter 1, we found that $n(A\cup B)=n(A)+n(B)-n(A\cap B)$.

But, before we state this general rule for finding the probability of A or B, we should consider a special case of a Venn diagram with event A and event B. The figure below is a Venn diagram showing the events are **mutually exclusive**. The circles do not overlap because there are no outcomes that are common to both events.



In this Venn diagram, event A could represent all the outcomes of drawing a red card from a deck of cards, and event B could represent all the outcomes of drawing a club from a deck of cards. These two events have no outcomes in common because there are no cards in a deck that are both red and a club. Therefore, the events have no outcomes in common.

Mutually Exclusive Events

Event A and event B are **mutually exclusive** if the events have no common outcomes.

That is, event A and event B are mutually exclusive if and only if $A \cap B = \{\ \}$ and $P(A \text{ and } B) = 0$.

In the case of mutually exclusive events, case $P(A \text{ and } B) = 0$. There is no over-counting outcomes because there are no outcomes that are in both A and B. So, when two mutually exclusive events are joined by "or," the formula we use for $P(A \text{ or } B)$ becomes much simpler:

$$\begin{aligned} P(A \text{ or } B) &= P(\mathrm{A}) + P(\mathrm{B}) - P(A \text{ and } \mathrm{B}) \\ &= P(A) + P(B) - 0 \\ &= P(A) + P(B) \end{aligned}$$

The formula for $P(A \text{ or } B)$ when A and B are mutually exclusive is just a special case of the general formula. Since the two events are mutually exclusive, there is no double counting and no need to subtract their overlap.

The box gives the general formula for $P(A \text{ or } B)$ and a more specialized formula when the events are mutually exclusive.

Addition Rule for "Or" Probabilities

If A and B are any events then

$$P(A \text{ or } B) = P(A) + P(B) - P(A, \text{ and } B) \quad \text{or } P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive events then

$$P(A \text{ or } B) = P(A) + P(B) \quad \text{or } P(A \cup B) = P(A) + P(B)$$

To apply the probability formulas for "or" scenarios successfully, we should be careful to distinguish whether the two events are mutually exclusive or not. We now look at several examples.

Example 6

A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that the card is a club or a face card (King, Queen, Jack.)

Solution

The events "club" and "face card" are not mutually exclusive because a card can be a club and also a face card. There are 13 cards that are clubs, 12 face cards (J, Q, K in each suit), and 3 cards that are both face cards and clubs.

$$\begin{aligned} P(\text{club or face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club and face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \approx 0.4231 \end{aligned}$$

The probability that the card is a club or a face card is approximately 0.4231 or 42.31%.

Example 7

A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that the card is a King or an Ace.

Solution

The event "King" and the event "Ace" have no common outcomes so these are mutually exclusive events.

$$\begin{aligned} P(\text{King or Ace}) &= P(\text{King}) + P(\text{Ace}) \\ &= \frac{4}{52} + \frac{4}{52} \\ &= \frac{8}{52} \\ &= \frac{2}{13} \end{aligned}$$

The probability of selecting a King or Ace is $\frac{2}{13} \approx 0.1538$, or about 15.39%.

Try it Now 2

A regular six-sided die is rolled. What's the probability that the die lands on an odd number or a number less than 5?

Answer

$\frac{5}{6}$

Try it Now 3

On a game spinner, the probability of spinning red is $\frac{1}{3}$, the probability of spinning white is $\frac{2}{5}$, and the probability of spinning blue is $\frac{4}{15}$. What is the probability of spinning once and getting white or blue?

Answer

$\frac{2}{3}$

Example 8

An experiment consists of tossing a coin and then rolling a die. Find the probability that the coin lands heads up or the number is five.

Solution

Let H represent heads up and T represent tails up. The sample space for this experiment is

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

- There are six ways the coin can land heads up: $\{H1, H2, H3, H4, H5, H6\}$
- There are two ways the die can land on five, $\{H5, T5\}$.
- There is one way for the coin to land heads up and the die to land on five, $\{H5\}$.

$$\begin{aligned} P(\text{heads or five}) &= P(\text{heads}) + P(\text{five}) - P(\text{both heads and five}) \\ &= \frac{6}{12} + \frac{2}{12} - \frac{1}{12} \\ &= \frac{7}{12} = \approx 0.5833 \end{aligned}$$

The probability that the coin lands heads up or the number is five is approximately 0.5833 or 58.33%.

Example 9

There is a 90% chance of snow, a 20% chance of strong winds, and a 15% chance of both snow and strong winds. What is the probability of snow or strong winds?

Solution

$$\begin{aligned} P(\text{snow or strong winds}) &= P(\text{snow}) + P(\text{strong winds}) - P(\text{snow and strong winds}) \\ &= 0.90 + 0.20 - 0.15 \\ &= 0.95 \end{aligned}$$

The probability of snow or strong winds is 0.95 or 95%.

Try it Now 4

In a subdivision, there is a 67% chance that a home has a garage and a 32% chance that a home has a patio. There is a 13% chance that a home has both a garage and a patio. What's the chance that a home has a garage or patio?

Answer

86%

Example 10

The table shows a group of cyclists and the routes they prefer. One cyclist is randomly chosen from this group.

Gender and Path Preference for a Group of Cyclists

Gender	Lake Path, L	Hilly Path, H	Wooded Path, W	Total
Male, M	45	38	27	110
Female, F	26	52	12	90
Total	71	90	39	200

a. What is the probability of selecting a cyclist who is a female or prefers the wooded path?
b. What is the probability of selecting a cyclist or prefers the lake path or the hilly path?
c. What is the probability of selecting a cyclist who does not prefer the lake path?

Solution

a. This question asks for $P(F \text{ or } W)$. These events are not mutually exclusive because there are cyclists who are both female and prefer the hilly path:

$$\begin{aligned} P(F \text{ or } W) &= P(F) + P(W) - P(F \text{ and } W) \\ &= \frac{90}{200} + \frac{39}{200} - \frac{12}{200} \\ &= \frac{117}{200} = 0.585 \end{aligned}$$

b. This question asks for $P(L \text{ or } H)$. These events are mutually exclusive because there are no cyclists who prefer both the Lake path and the Hilly path:

$$\begin{aligned} P(L \text{ or } H) &= P(L) + P(H) \\ &= \frac{71}{200} + \frac{90}{200} \\ &= \frac{161}{200} = 0.805 \end{aligned}$$

c. This question asks for $P(L')$:

$$\begin{aligned} P(L') &= 1 - P(L) \\ &= 1 - \frac{71}{200} \\ &= \frac{129}{200} = 0.645 \end{aligned}$$

Try it Now 5

A group of people were surveyed about the type of movies they prefer. Suppose a person is chosen at random from this group.

Gender and Movie Preference for a Group of People

Gender	Romantic, R	Action, A	Horror, H	Total
Male, M	8	25	6	39
Female, F	12	10	3	25
Total	20	35	9	64

a. What is the probability of selecting a person who is male or prefers horror movies?
b. What is the probability of selecting a person who prefers romantic or action movies?
c. What is the probability of selecting a person who does not prefer romantic movies?

Answer

a. $\frac{42}{64} = \frac{21}{32}$
b. $\frac{55}{64}$
c. $\frac{44}{64} = \frac{11}{16}$

Combining Events with "And"

Sometimes we need to calculate probabilities for compound events that are connected by the word "and." This requires that we look for outcomes that are common to both events. As we saw with sets and truth tables, the intersection $A \cap B$ and conjunction $A \wedge B$ refer to where both A and B occur. When working with probabilities, these events may occur at the same time or they could happen in a sequence, such as A followed by B.

How we calculate the theoretical probability of the event "A and B" depends on whether the two events are *independent* or *dependent*. So, we must discuss these concepts before developing a formula.

Two events A and B are **independent** if the probability that B occurs is the same whether or not A occurs. If the probability of B is affected by the occurrence of A, then we say that the events are **dependent**.

Coin tosses and die rolls are common examples of independent events – tossing heads does not change the probability of tossing heads the next time. The probability of tossing a heads on the second toss is still 1/2 no matter if there was a heads or tails on the first toss. Likewise, rolling a six on a first die does not change the probability that the next roll will be a six.

Another example of independent events is randomly selecting items from a container when the items are selected *with replacement*. By replacing the item after the first selection, we reset the probability back to what it was before we made the selection. Since the probabilities in the second selection are the same as the first selection, the events are independent.

If we select *without replacement*, however, we change the total number of possible outcomes, thereby changing the probabilities in subsequent selections. For example, if we select two people from a group for a committee, the first member selected cannot be selected again so the sample space (and therefore the probabilities) for the second selection has changed. So, if we draw without replacement, the events will be dependent.

What about drawing two cards from a deck? This can be a bit tricky. When you draw the first card and put it back before drawing a second card, the sample space for drawing the second card remains 52 cards just as it was for the first card. These events are independent. However, if you draw the first card and put it aside before drawing a second card, the sample space for drawing the second card changes to 51 because one card has been removed. This is an example of events that are dependent.

Independent and Dependent Events

Two events are **independent events** if the occurrence of one event has no effect on the probability of the occurrence of the other event. Examples of independent events include tossing coins, rolling dice, spinning spinners, and selecting items from a group when the items are replaced after each selection.

Two events are **dependent events** if the occurrence of one event has an effect on the probability of the occurrence of the other event. Examples of dependent events include selecting items from a group when the items are not replaced after each selection.

✔ Example 11

Determine whether each pair of events are independent or dependent.

a. Flipping a coin twice and getting tails both times.
b. Selecting a president and then a vice president at random from a pool of five equally qualified individuals.
c. The event that it will be cold in Washington, DC tomorrow and the event that it will be cold in Baltimore tomorrow.
d. Wearing your lucky socks and getting an A on your exam.

Solution

a. The probability of getting tails on the first flip is $\frac{1}{2}$. After flipping tails, the probability of getting tails on the second flip is still $\frac{1}{2}$. Since the probability of flipping tails on the second flip did not change because we flipped tails on the first flip, the events are independent.
b. Since two different people will be put in the role of president and vice president, we are drawing without replacement and the events are therefore dependent.
c. If it is cold in Baltimore it is more likely that it will be cold in Washington, so the events are dependent.
d. The socks you wear do not have a direct effect on how well you do on your exam, so these events are independent.

? Try it Now 6

Determine whether each pair of events are independent or dependent.

a. Rolling a die twice and getting '2' both times.
b. Getting two hearts when selecting one card from a well-shuffled deck, setting it aside, and then selecting another card.
c. Drawing two red marbles one-by-one from a bucket of marbles where the first marble is replaced before drawing the second marble.

Answer

a. independent
b. dependent
c. independent

To calculate "and" probabilities we always multiply, but we need to determine whether the events are independent or dependent. If they are independent, then we can multiply the individual probabilities of the events because one does not affect the other. If the events are dependent, we must remember that the first event has had an effect on the probability of the second event. A general formula for finding $P(A \text{ and } B)$ is given below:

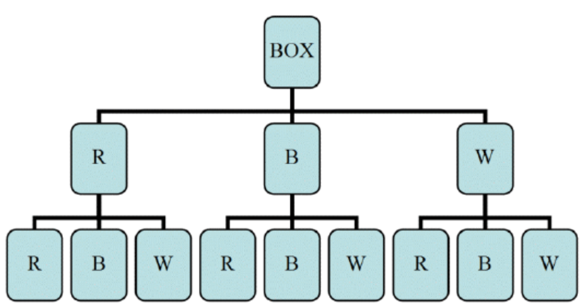
✎ Multiplication Rule for "And" Probabilities

For events A and B,

$$P(\text{A and B}) = P(A) \cdot P(B)^* \ .$$

* If events A and B are dependent, $P(B)$ depends on the fact that A happened.

So why does this multiplication rule work? Consider a box containing three balls -- one red, one blue, and one white. One ball is selected, its color is observed, and then the ball is placed back in the box. The balls are scrambled, and a ball is selected again. The tree diagram shows the sample space for the experiment.



From the tree diagram, there are 9 equally likely outcomes in the sample space is $S = \{RR, RB, RW, BR, BB, BW, WR, WB, WW\}$. By counting, we can see that the probability of selecting a red ball followed by a white ball $P(RW)$ is $\frac{1}{9}$.

We could have reached this same answer by thinking another way and without listing a sample space.

- We know there are 9 outcomes in the sample space because there are 3 options for the first selection of ball and 3 options for the second selection of ball. This means there are $3 \times 3 = 9$ ways to make the selection of two balls together.
- There is only one ball that is red during the first selection and one ball that is white during the second selection. This means there is $1 \times 1 = 1$ way to to make the selection of these two colors.
- Therefore, the probability of selecting red ball followed by a white ball is $\frac{1 \times 1}{3 \times 3}$ or $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. We have multiplied $P(\text{red ball on 1st selection}) \times P(\text{white ball on 2nd selection})$ to get $P(\text{red ball on 1st selection and white ball on 2nd selection})$.

Example 12

Suppose a fair coin is tossed.

a. If the coin is tossed twice, what is the probability that both tosses land heads up?
b. If the coin is tossed four times, what is the probability that all four tosses and heads up?

Solution

The tosses of the coins are independent events. We will use the formula for $P(A \text{ and } B)$ when A and B are independent.

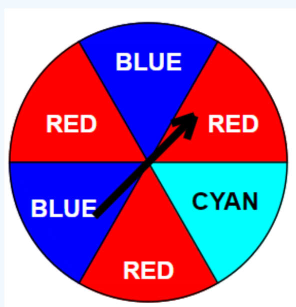
a. $$\begin{aligned} P(\text{two heads in a row}) &= P(\text{1st is heads and 2nd is heads}) \\ &= P(\text{1st is heads}) \cdot P(\text{2nd is heads}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

b. $$\begin{aligned} P(\text{four heads in a row}) &= P(\text{1st is heads and 2nd is heads and 3rd is heads and 4th is heads}) \\ &= P(\text{1st is heads}) \cdot P(\text{2nd is heads}) \cdot P(\text{3rd is heads}) \cdot P(\text{4th is heads}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^4 \\ &= \frac{1}{16} \end{aligned}$$

The probability that all four tosses land heads up is $\frac{1}{16}$.

Try it Now 7

The game spinner has 6 equal-sized sections. Jeremy will spin the spinner 3 times in a row.



a. Find the probability that Jeremy gets blue on all three spins.
b. Find the probability that Jeremy gets red on the first spin, blue on the second spin, and red on the third spin.

Answer

a. $\frac{1}{27}$
b. $\frac{1}{12}$

Example 13

A bag contains 5 red and 4 white marbles. Two marbles are selected one by one and colors are recorded. What is the probability that the first marble is red and the second marble is white if

a. the first marble is returned to the bag before the second marble is selected.
b. the first marble is not returned to the bag before the second marble is selected.

Solution

a. Since the first marble is returned to the bag before the second marble is selected, these are independent events.

$$\begin{aligned} P(\text{1st red and 2nd white}) &= P(\text{1st red}) \cdot P(\text{2nd white}) \\ &= \frac{5}{9} \cdot \frac{4}{9} = \frac{20}{81} \end{aligned}$$

The probability that the first marble is red and the second marble is white is $\frac{20}{81}$.

b. Since the first marble not returned to the bag before the second marble is selected, these are dependent events.

$$\begin{aligned} P(\text{1st red and 2nd white}) &= P(\text{1st red}) \cdot P(\text{2nd white}) \\ &= \frac{5}{9} \cdot \frac{4}{8} = \frac{20}{72} = \frac{5}{18} \end{aligned}$$

The probability that the first marble is red and the second marble is white is $\frac{5}{18}$.

Try it Now 8

A box contains 6 red blocks, 4 green blocks, and 3 black blocks. What is the probability of selecting

a. two red blocks in a row if the first red block is put back in the box?
b. two red blocks in a row if the first red block is not put back in the box?
c. a red block and then a green block if the red block is not put back in the box?

Answer

a. $\frac{36}{169}$
b. $\frac{5}{26}$
c. $\frac{2}{13}$

Example 14

Abby has an important meeting in the morning. She sets 3 battery-powered alarm clocks just to be safe. If each alarm clock has a 0.03 probability of malfunctioning, what is the probability that all three alarm clocks fail at the same time?

Solution

Since the clocks are battery-powered we can assume that one failing will have no effect on the operation of the other two clocks. The functioning of the clocks is independent.

$$\begin{aligned} P(\text{all three fail}) &= P(\text{first fails}) \cdot P(\text{second fails}) \cdot P(\text{third fails}) \\ &= (0.03)(0.03)(0.03) \\ &= (0.03)^3 \\ &= 2.7 \times 10^{-5} \end{aligned}$$

The probability that all three clocks will fail is approximately 0.000027 or 0.0027%. It is very unlikely that all three alarm clocks will fail.

Example 15

Approximately 80% of U.S. children are covered by some type of health insurance. Assume that whether one child has health insurance or not has no effect on whether the next child has health insurance. If 4 children are selected at random, find the probability that

a. all four of the children have insurance.

b. none of the four children have insurance

Solution

a. We will assume independence in this scenario.

$$\begin{aligned} P(\text{all four have insurance}) &= P(\text{1st has insurance}) \cdot P(\text{2nd has insurance}) \cdot P(\text{3rd has insurance}) \cdot P(\text{4th has insurance}) \\ &= (0.80)(0.80)(0.80)(0.80) \\ &= (0.80)^4 \\ &= 0.4096 \end{aligned}$$

The probability that all four children are covered by insurance is 0.4096 or 40.96%.

b. To find the probability that none of the four children have insurance, we will need to use the Complement Rule. If 80% of children are covered by insurance, then 100% - 80% = 20% are not covered by insurance.

$$\begin{aligned} P(\text{none of the four have insurance}) &= P(\text{1st no insurance}) \cdot P(\text{2nd no insurance}) \cdot P(\text{3rd no insurance}) \cdot P(\text{4th no insurance}) \\ &= (0.20)(0.20)(0.20)(0.20) \\ &= (0.20)^5) \\ &= 0.0016 \end{aligned}$$

The probability that none of the four children are covered by insurance is 0.0016 or 0.16%.

? Try it Now 9

According to some studies, it is believed that 15% of people are left-handed. Suppose 3 people are selected at random.

a. Find the probability that all three are left-handed.
b. Find the probability that none of the three are left-handed.
c. Find the probability that only the first one selected is left-handed (and the others are right-handed.)

Answer

a. 0.003375 = 0.3375%
b. 0.614125 = 61.4125%
c. 0.108375 = 10.8375%

'At Least Once' Rule for Independent Events

Many times we need to calculate the probability that an event will happen "at least once" in many trials. The calculation can get quite complicated if there are more than a couple of trials. Using the complement to calculate the probability can simplify the problem considerably. That is, we will find the probability of what we don't want to happen first, and then subtract that probability from 1. The following example will help you understand the formula.

✓ Example 16

The probability that a child forgets her homework on a given day is 0.15. What is the probability that she will forget her homework at least once in the next five days?

Solution

Assume that whether she forgets or not one day has no effect on whether she forgets or not the second day. The events are independent.

If $P(\text{forgets}) = 0.15$, then $P(\text{not forget}) = 0.85$.

$$\begin{aligned} P(\text{forgets at least once in 5 tries}) &= P(\text{forgets 1, 2, 3, 4 or 5 times in 5 tries}) \\ &= 1 - P(\text{forgets 0 times in 5 tries}) \\ &= 1 - P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \\ &= 1 - (0.85)(0.85)(0.85)(0.85)(0.85) \\ &= 1 - (0.85)^5 \\ &\approx 0.5563 \end{aligned}$$

The probability that the child will forget her homework at least one day in the next five days is 0.5563 or 55.63%

The idea in the previous example can be generalized as the **At Least Once Rule**.

At Least Once Rule

If an experiment is repeated n times where the n trials are independent and the probability of event A occurring one time is $P(A)$,

$$P(A\text{ occurs at least once}) = 1 - P(\text{not } A)^n \ .$$

Example 17

The probability of seeing a falcon near the lake during a day of bird watching is 0.21. What is the probability that a birdwatcher will see a falcon at least once in eight trips to the lake?

Solution

If $P(\text{sees falcon}) = 0.21$, then $P(\text{doesn't see a falcon}) = 1 - 0.21 = 0.79$.

$$\begin{aligned} P(\text{sees falcon at least once in 8 trips}) &= 1 - P(\text{doesn't see a falcon})^8 \\ &= 1 - (0.79)^8 \\ &\approx 0.8483 \end{aligned}$$

The probability of seeing a falcon at least once in eight trips to the lake is approximately 0.8483 or 84.83%.

Example 18

A multiple choice test consists of six questions. Each question has four choices for answers (A, B, C, or D), only one of which is correct. A student guesses on all six questions. What is the probability that he gets at least one answer correct?

Solution

Let A be the event that the answer to a question is correct. Since each question has four choices and only one correct choice, $P(\text{correct}) = \frac{1}{4}$. That means $P(\text{not correct}) = 1 - \frac{1}{4} = \frac{3}{4}$.

$$\begin{aligned} P(\text{at least one correct in six trials}) &= 1 - P(\text{not correct})^6 \\ &= 1 - \left(\frac{3}{4}\right)^6 \\ &\approx 0.8220 \end{aligned}$$

The probability that he gets at least one answer correct is 0.8220 or 82.20%.

Try it Now 10

Mary is in the archery club. The probability that she hits the target on an attempt is 65%. She will make 4 attempts to hit the target. What's the probability that Mary hits the target at least once on her attempts?

Answer

0.9850 or 98.50%

3.5: Conditional Probabilities

What do you think is the probability that a man is over six feet tall? If you know that both of his parents were tall, would you change your estimate of the probability? If you know that both of his parents were short, would that affect your estimate in a different way? Most likely. The chance a man is over six feet tall is probably higher if he has tall parents and lower if he has short parents. A *conditional probability* is a probability that is based on some prior knowledge.

While conditional probability may seem like a difficult concept, we use it all the time in our every day life.

- Weather forecasters use conditional probability to predict the likelihood of future weather conditions given current conditions. They may calculate the probability of rain *if* it is cloudy outside.
- Sports betting companies may use conditional probability to set the odds that particular teams win their game. These odds may rely on knowledge about the team such *knowing that* a key player is injured.
- A doctor may use conditional probability when discussing the efficacy of a vaccine with their patient. In other words, the chance of contracting a virus may be much less *when* the patient takes the vaccine compared to when they do not take the vaccine.
- An insurance company uses conditional probability when setting rates for car insurance. For example, the insurance company may believe the chance you have an accident is higher *if* you are younger than 27.

All these examples of conditional probability have one thing in common: we assume that something is known before calculating a probability.

In Section 3.4 we learned that when events are dependent, whether or not event A occurs affects the probability that event B occurs. When using the rule $P(A \text{ and } B) = P(A) \cdot P(B)$ with dependent events, we assume that event A has already occurred and affected the sample space of event B. In the sense of probability that means that the sample space of an experiment has been restricted before finding the probability of event B. We can express this idea as "the probability of B, given A."

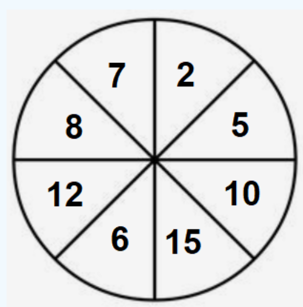
We will discuss this idea of restricting the sample space as we proceed through examples of conditional probability.

Conditional Probability

A **conditional probability** is the probability that an event will occur if some other condition has already occurred. This is denoted by $P(B|A)$, which is read “the probability of B given A.”

Example 1

You spin a spinner with the 8 equally likely outcomes shown below. A friend covers up the number where the spinner lands.



a. What's the probability it landed on an even number?
b. Your friend now tells you the spinner landed on a one-digit number. What's the probability it landed on an even number?
c. Your friend tells you the spinner landed on a two-digit number. What's the probability it landed on a number less than 12?

Solution

a. The sample space of spinning the spinner is $S = \{2, 5, 6, 7, 8, 10, 12, 15\}$ Of these outcomes, 5 of them are even numbers so $P(\text{even number}) = \frac{5}{8}$.

b. Now, you have some additional knowledge about there the spinner landed. The sample space is restricted to only one-digit numbers: $\{2, 5, 6, 7, 8\}$.Of these outcomes, 3 of them are even numbers. The probability that the spinner landed on an even number *given that* it landed on a one-digit number is $\frac{3}{5}$. We can write this as $P(\text{even} \mid \text{single digit}) = \frac{3}{5}$.
c. Here, you have some different additional knowledge which restricts the original sample space to those outcomes with two digits: $\{10, 12, 15\}$.Of these outcomes, 1 of them is less than 12. So, $P(\text{less than } 12 \mid \text{two digits}) = \frac{1}{3}$.

Example 2

One card is drawn from a well-shuffled deck of 52 cards. Find the following probabilities:

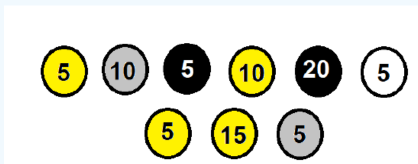
a. probability that the card is a heart given that it is red.
b. probability that the card is red given that it is a heart.
c. $P(\text{King} \mid \text{face card})$

Solution

a. We are told that the card is a red card. The sample space is restricted to only the 26 cards that are red. Of these 26 red cards, 13 are hearts. So, $P(\text{heart} \mid \text{red}) = \frac{13}{26} = \frac{1}{2}$.
b. We are told that the card is a heart. The sample space is restricted to only the 13 cards that are hearts. Since every heart is red, $(P(\text{red} \mid \text{heart}) = \frac{13}{13} = 1$.
c. In words, $P(\text{King} \mid \text{face card})$ means "*probability of selecting a King given that the card is a face card*." The sample space is restricted to the 12 face cards. Of these 12 face cards, 4 are Kings. $P(\text{King} \mid \text{face card}) = \frac{4}{12} = \frac{1}{3}$.

Try it Now 1

A box contains a collection of ping pong balls, all the same size but of different colors and numbers as shown below.



Find these probabilities.

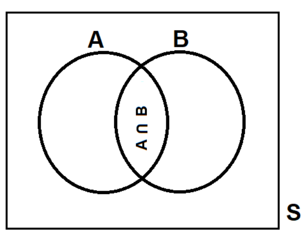
a. the probability of selecting a yellow ball given the ball shows a "5."
b. the probability of selecting a ball showing a number greater than 10 given the ball is black.
c. $P(\text{odd number} \mid \text{not gray})$

Answer

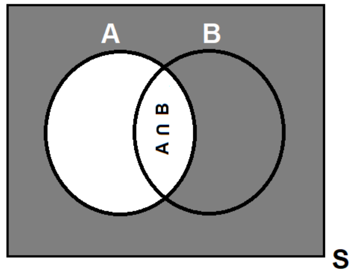
a. $\frac{2}{5}$
b. $\frac{1}{2}$
c. $\frac{5}{7}$

While we can find conditional probabilities by analyzing and restricting the sample space, it is useful to have a formula for conditional probability. Let's examine a Venn diagram to develop a general formula.

The Venn diagram shows event A and event B which overlap in the intersection of $A \cap B$.



Suppose we want to to find the probability of event B given event A, or $P(B|A)$. This means we need to restrict the sample space to only those outcomes in event A. A portion of the Venn diagram has been blackened to show that these outcomes are no longer part of the sample space, leaving only the outcomes in the given event A.



The only portion of event B that remains once the sample space has been restricted to event A is the portion of the Venn diagram $A \cap B$. Therefore, $P(B|A) = \dfrac{n(A \cap B)}{n(A)}$ or $\dfrac{n(A \text{ and } B)}{n(A)}$.

If we divide numerator and denominator of this conditional probability formula by the number of outcomes in the sample space $n(S)$, then we can compute the conditional probability in alternative way as

$$P(B|A) = \frac{\left(\frac{n(A \text{ and } B)}{n(S)}\right)}{\left(\frac{n(A)}{n(S)}\right)} = \frac{P(A \text{ and } B)}{P(A)}.$$

Conditional Probability Formula

For events A and B, $P(B|A) = \dfrac{n(A \text{ and } B)}{n(A)}$ or $P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)}$

Example 3

Two fair dice are rolled and the sum of the numbers is observed. What is the probability that the sum is at least 9 if it is known that a 5 was rolled?

Solution

We are given that the dice show a 5 so this is a conditional probability. We are asked to find $P(\text{sum is at least 9} \mid \text{5 was rolled})$. Here, event A is "5 was rolled." Event B is "sum is at least 9."

Find the number of outcomes in event A. The pairs of dice showing a 5 are

$$A = \{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5), (5,1), (5,2), (5,3), (5,4), (5,6)\}.$$

Therefore, $n(A) = 11$.

List the pairs of dice in event $(A \text{ and } B)$. The pairs of dice showing 5 was rolled and the sum is at least 9 is

$$A \text{ and } B = \{(4,5), (5,5), (6,5), (5,4), (5,6)\}.$$

Therefore, $n(A \text{ and } B) = 5$.

Applying the conditional probability formula,

$$P(B|A) = \frac{n(A \text{ and } B)}{n(A)} = \frac{5}{11}.$$

The probability that the sum is at least 9 if it is known that a 5 was rolled is $\dfrac{5}{11}$.

Try it Now 2

A coin is flipped twice and the results are recorded. Find each probability.

a. the probability that both coins land heads given at least one coin lands heads.
b. the probability that both coins land heads given the first coin lands heads.

Answer

a. $\frac{1}{3}$
b. $\frac{1}{2}$

Example 4

For a group of people, the probability of having blond hair is 25%. The probability of having blond hair and blue eyes is 10%. What is the probability that person has blue eyes given they have blond hair?

Solution

We need to find a conditional probability because it is given that a person has blond hair. We want to find $P(\text{ blue eyes } \mid \text{ blond hair })$. We don't have the sample space to count numbers of outcomes so we must use the second formula for conditional probability: $P(B \mid A)=\frac{P(A \text{ and } B)}{P(A)}$.

The given event A is "*blond hair*," so B is the event "*blue eyes*." The problem states that $P(A)=P(\text{blond hair })=0.25$ and $P(A \text{ and } B)=P(\text{blond hair and blue eyes })=0.10$, so

$$P(B \mid A)=\frac{P(A \text{ and } B)}{P(A)}=\frac{0.10}{0.25}=\frac{2}{5}=0.40.$$

The probability that a person has blond hair given they have blue eyes is 0.40, or 40%.

Try it Now 3

The label on a medicine bottle claims that there is a 14% chance of experiencing insomnia. There is a 5% chance of experiencing both a headache and insomnia. What's the probability that a person who takes this medicine has a headache given they have insomnia?

Answer

$\frac{5}{14} \approx 0.3571$ or 35.71%

Example 5

The table shows survey results from 250 people who recently purchased a car.

Satisfaction of Car Buyers

	Satisfied	**Not Satisfied**	**Total**
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

Use the results in the table to find

a. the probability that a person is satisfied with their car.
b. the probability that a person is satisfied given the person bought a used car.
c. the probability that a person is not satisfied given the person bought a new car.

d. the probability that a person bought a new car given that they are satisfied.

Solution

a. The total number of customers was 250 of which 175 are satisfied with their purchase. This is not a conditional probability because the sample space has not been restricted.

   So, $P(\text{satisfied})=\frac{175}{250}=\frac{7}{10}=0.7$

   The probability that a person is satisfied is 0.7 or 70%.

b. We are given that the customer bought a used car. This is a conditional probability. Let A represent the given condition "customer bought a used car." Let B be "customer is satisfied." We want to find $P(B \mid A)$.

   - The number of people who bought a used car is $n(A)=130$.
   - The number of people who bought a used car and were satisfied is $n(A \text{ and } B)=83$.

   So, $P(B \mid A)=\frac{n(A \text{ and } B)}{n(A)}=\frac{83}{130} \approx 0.6385$.

   The probability that a person is satisfied given the person bought a used car is approximately 0.6385 or 63.85%.

c. We are given that the customer bought a new car. This is a conditional probability. Let A represent the given condition "customer bought a new car." Let B be "customer is not satisfied." We want to find $P(B \mid A)$.

   - The number of people who bought a new car is $n(A)=120$.
   - The number of people who bought a new car and were not satisfied is $n(A \text{ and } B)=28$.

   So, $P(B \mid A)=\frac{n(A \text{ and } B)}{n(A)}=\frac{28}{120}=\frac{7}{30} \approx 0.2333$.

   The probability that a person is satisfied given the person bought a used car is approximately 0.2333 or 23.33%.

d. We are given that the customer is satisfied. This is a conditional probability. Let A represent the given condition "customer is satsified." Let B be "customer bought a new car." We want to find $P(B \mid A)$.

   - The number of people who are satisfied is $n(A)=175$.
   - The number of people who are satisfied and bought a new car is $n(A \text{ and } B)=92$.

   So, $P(B \mid A)=\frac{n(A \text{ and } B)}{n(A)}=\frac{92}{175} \approx 0.5257$.

   The probability that a person is satisfied given the person bought a used car is approximately 0.5257 or 52.57%.

You may also see conditional probability scenarios written in ways that do not use the word "given." For example, in the previous example we could describe the conditional event "a person is satisfied given they bought a used car" as "a person is satisfied *if* they bought a used car" or "a person *who* bought a used car is satisfied."

✔ Example 6

A survey of 350 students at a university revealed the following data about class standing and place of residence.

Living Arrangements of University Students

	Freshman	Sophomore	Junior	Senior	Total
Dormitory	89	34	46	15	184
Apartment	32	17	22	48	119
with Parents	13	31	3	0	47
Total	134	82	71	63	350

Use the results in the table to find each probability.

a. What is the probability that a student is a sophomore if the student lives in an apartment?
b. What is the probability that a student lives with parents if the student is a freshman?

Solution

a. We are given that the student lives in an apartment. This is a conditional probability. Let A represent the given condition "student lives in an apartment." Let B be "student is a sophomore." We want to find $P(B|A)$.

- The number of students who live in an apartment is $n(A) = 119$.
- The number of students who live in an apartment and are sophomores is $n(A \text{ and } B) = 17$.

So, $P(B|A) = \frac{n(A \text{ and } B)}{n(A)} = \frac{17}{119} \approx 0.1429$.

The probability that a student is a sophomore if they live in an apartment is approximately 0.1429 or 14.29%.

b. We are given that the student is a freshman. This is a conditional probability. Let A represent the given condition "student is a freshman." Let B be "student lives with parents." We want to find $P(B|A)$.

- The number of students who are freshmen is $n(A) = 134$.
- The number of people who are freshmen and live with parents is $n(A \text{ and } B) = 13$.

So, $P(B|A) = \frac{n(A \text{ and } B)}{n(A)} = \frac{13}{134} \approx 0.0970$.

The probability that a lives with parents if they are a freshman is approximately 0.0970 or 9.70%.

Try it Now 4

A group of people were surveyed about the type of movies they prefer. Suppose a person is chosen at random from this group.

Movie Preferences

Gender	Romantic, R	Action, A	Horror, H	Total
Male, M	8	25	6	39
Female, F	12	10	3	25
Total	20	35	9	64

Use the results in the table to find

a. the probability that a person prefers action movies given they are male.
b. the probability that a person is female given they prefer romantic movies.

Answer

a. $\frac{25}{39}$
b. $\frac{12}{20} = \frac{3}{5}$

3.6: Counting Methods

Counting is as easy as 1-2-3, right? You already know how to count, or you wouldn't be taking a college-level math class! Of course, but what we'll really be investigating in this section is *efficient ways* of counting outcomes. Part of that task is learning what to count and what makes something different from something that we have already counted. The ultimate goal of learning to use different counting methods is so that we can integrate these methods into probability. So far the probability experiments we have worked with have had rather small numbers of outcomes. We could easily count the number of outcomes in the sample space and in the events. If there are a larger number of outcomes in the sample space, we need to develop other ways to count the outcomes without listing them all.

The Fundamental Counting Principle

The simplest of the counting methods is the *Fundamental Counting Principle* (FCP). This method is generally used when there are choices that must be made in succession and there are several options for each choice.

Example 1

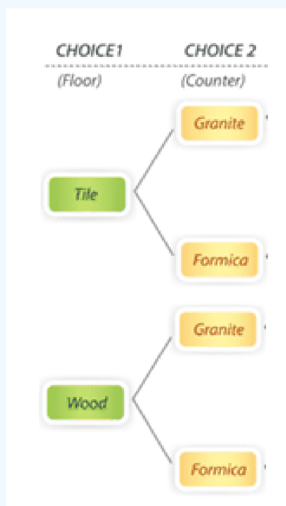
Gretchen is remodeling her kitchen. For her kitchen design package, she must choose a type of floor, a type of counter, and a type of sink from the options shown in the table below. Gretchen wants to know how many different kitchen designs she could make with the options she has available.

Gretchen's Kitchen Design Options

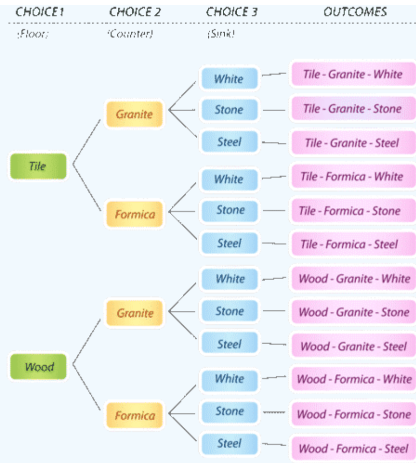
Choice 1: Floor	**Choice 2: Counter**	**Choice 3: Sink**
Tile	Granite	White
Wood	Formica	Steel
		Stone

Solution

A tree diagram is a useful tool to visually see all the possibilities. It can also help you organize the outcomes so that you don't miss any of them. Start the tree diagram by listing one of the options for the *Floor*, branching off to each of the two options for the *Counter.* Make sure each option of *Counter* repeats for each branch of the *Floor*. This results in $2 \times 2 = 4$ *Floor-Counter* patterns.



Next, make sure that each option for the *Sink* repeats for each of the 4 *Floor-Counter* patterns. The completed tree diagram of Gretchen's design choices is shown.



We see that there are 12 possible outcomes for the kitchen design, and they are all listed on the right side of the tree diagram above.

While tree diagrams provide a visual layout of choice options and outcomes, they can take time to create -- especially when there are many options for the choices or when there are lots of choices of make. A quicker way to calculate the number of final outcomes when provided different options for at each stage of choice is to multiply together the number of options at each stage. We can use multiplication to calculate the number of different design packages that Gretchen can consider: $2 \times 2 \times 3 = 12$ design packages. This counting technique is called the *Fundamental Counting Principle*.

Fundamental Counting Principle

If there are m possible outcomes for event A and n possible outcomes for event B, then there are a total of $m \times n$ possible outcomes for event A followed by event B.

This principle can be generalized for three, four, or even more events.

Example 2

Let's say that a person walks into a restaurant for a three course dinner. There are four different salads, three different entrees, and two different desserts to choose from. Assuming the person wants to eat a salad, an entree, and a dessert, how many different meals are possible?

Solution

There are three events: choose a salad, choose an entree, and choose a dessert. According to the Fundamental Counting Principle, multiply the number of outcomes possible at each event: $\underline{4} \times \underline{3} \times \underline{2} = 24$.

There are 24 different meals possible.

Try it Now 1

When purchasing a computer, the e-Box laptop computer offers customers several different options for screen, memory, and color as shown below. How many ways can a customer choose to personalize her selection of a new computer?

- **Screen:** *small, medium, or large*
- **Memory:** *standard, 1 GB, or 2 GB*
- **Color:** *pearl, gray, blue, or black*

Answer

36 ways

The Fundamental Counting Principle may seem like a very simple idea, but it is very powerful. Many complex counting problems can be solved using this strategy.

Example 3

Some license plates in Maryland consist of three letters followed by three digits. How many license plates of this type are possible if

1. There are 26 letters (A, B, C, ... Z) and 10 digits (0, 1, 2, 3, …, 9).

$$\underbrace{(26 \cdot 26 \cdot 26)}_{\text{letters}} \cdot \underbrace{(10 \cdot 10 \cdot 10)}_{\text{digits}} = 17,576,000 \text{ license plates}$$

2. Letters can be repeated but digits cannot?

$$\underbrace{(26 \cdot 26 \cdot 26)}_{\text{letters}} \cdot \underbrace{(\underline{10} \cdot \underline{9} \cdot \underline{8})}_{\text{digits}} = 12,654,720 \text{ license plates}$$

3. Both digits and letters can be repeated but the first digit cannot be zero.

$$\underbrace{(26 \cdot 26 \cdot 26)}_{\text{letters}} \cdot \underbrace{(\underline{9} \cdot \underline{10} \cdot \underline{10})}_{\text{digits}} = 15,818,400 \text{ license plates}$$

4. All letters and digits can be used but cannot be repeated.

$$\underbrace{(\underline{26} \cdot \underline{25} \cdot \underline{24})}_{\text{digits}} \cdot \underbrace{(\underline{10} \cdot \underline{9} \cdot \underline{8})}_{\text{letters}} = 11,232,000 \text{ license plates}$$

Try it Now 2

How many 3-digit area codes can be formed where the first and last digits are odd, and digits cannot be repeated?

Answer

180 area codes

Example 4

Four customers arrive at a grocery store checkout at the same time. In how many ways can the four people line up to pay for their items?

Solution:

We can solve this problem by thinking about making four successive choices. Any of the customers can be first so there are 4 options for the first choice. Then, there are 3 people left who can be second. Next, there are 2 customers left who can be third. Finally, there is only 1 person left to be last in the line.

Using the Fundamental Counting Principle there are $\underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 24$ ways for the four customers to line up.

The multiplication pattern above appears so often in counting that it has its own name, called a **factorial**, and its own symbol, which is '!'. We say "*four factorial*" and we write "4!".

Factorial

If n is a counting number, then n factorial ($n!$) is defined as

$$n! = n(n-1)(n-2)(n-3)\ldots(2)(1)$$

$$0! = 1$$

Example 5

Evaluate $5!$, $8!$, and $12!$.

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \\ 8! &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40,320 \\ 12! &= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 479,001,600 \end{aligned}$$

Factorials get very large, very fast. It is not pleasant to type so many factors when n is large. Most scientific calculators have a factorial command. You can find this command on the TI calculator as follows:

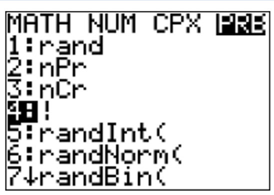
Technology Note: Factorial on the TI-83/84 Calculator

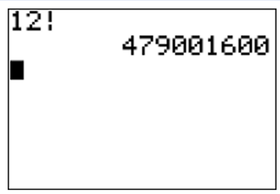
To find $n!$, type the value of n first.

Press **[MATH]** and use the arrow to scroll to the right to the **PRB** (probability) menu.

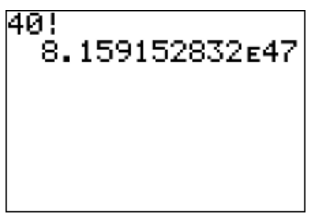
Select option **4: !** and then press **ENTER.**

The screenshots below show the menu and the proper uses of these commands for 12!.





A calculator will express large values of factorials in scientific notation. As shown below, the TI calculator finds $40!$ to be almost 8.16×10^{47}.



Examples 4 and 5 illustrate a type of common counting problem and method called a permutation. A *permutation* is an ordered arrangement of objects. The key to recognizing a permutation is that it is "ordered." For example, using only the three letters C, A, and T, there are six different three-letter permutations or sequences that we can make: ACT, ATC, CAT, CTA, TAC, and TCA.

If we add a fourth letter to the list, say S, then there are exactly 24 different four-letter permutations:

ACST CAST SACT TACS
ACTS CATS SATC TASC
ASCT CSAT SCAT TCAS
ASTC CSTA SCTA TCSA
ATCS CTAS STAC TSAC
ATSC CTSA STCA TSCA

Counting the number of ordered arrangements of the four letters C, A, T, S is the same problem as counting the number of ways four grocery customers can line up at the check out stand. Therefore, when counting how many ways a group of objects can be ordered, we use the factorial of the number of objects to be ordered.

Permutation

A **permutation** is an arrangement of a set of objects without repetition where a different order of the same set of objects counts as a different arrangement.

The number of permutations of n different objects, taken altogether, is $n!$.

Example 6

The school orchestra is planning to play eight pieces of music at their next concert. In how many different ways can the pieces of music be sequenced in the program?

Solution

This is a permutation because the orchestra is arranging all 8 songs in an order to make the program. We can use the Fundamental Counting Principle or a factorial to count the number of ordered arrangements:

$$\underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 40,320 \quad \text{or } 8! = 40,320$$

There are 40,320 different ways of sequencing the songs in the program.

Try it Now 3

There are 6 DVD's to be placed on a shelf. In how many ways can they be placed on the shelf from left to right?

Answer

$6! = 720\,\text{ways}$

But what if the orchestra in the previous example has time to only play 5 of the 8 musical pieces. How many ways could the orchestra design the program of music? This is still a permutation but a slightly different question. This type of problem will be explored next.

Permutations

We now consider permutations of a set of objects taken from a larger set.

Example 7

The school orchestra has learned 8 pieces of music to play at their next concert. However, due to time restraints, they can only chose 5 pieces to play. In how many different ways can the pieces of music be chosen and sequenced in the program?

Solution

We can think of this as making 5 choices in a row: There are 8 options for the first song choice, 7 options for the second song choice, 6 options for the third song choice, 5 options for the fourth song choice, and 4 options for the fifth song choice. We can use the Fundamental Counting Principle to calculate the number of ways to choose and arrange the five pieces of music:

$$\underline{8} \times \underline{7} \times \underline{6} \times \underline{5} \times \underline{4} = 6,720 \quad \text{ways.}$$

We say that the number of permutations of 8 songs taken 5 at a time is 6,720.

Notice that in this example of counting the number of ways to select and order only 5 of the 8 songs, we could not quite use $8!$ because not all 8 musical pieces are being used. Notice, however, that

$$8 \times 7 \times 6 \times 5 \times 4 = \frac{8!}{3!} = \frac{8!}{(8-5)!} \quad .$$

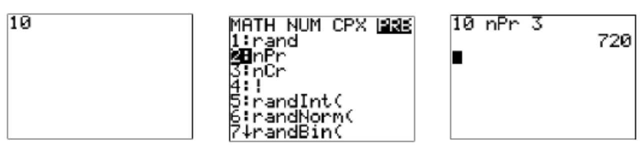
We can generalize the preceding observation to create a formula for counting the number of permutations of r objects from a group of n objects.

> **Permutation Formula**
>
> The number of permutations of n objects taken r at a time is given by the formula
>
> $$ {}_nP_r = \frac{n!}{(n-r)!}. $$

It should be emphasized that a permutation problem is nothing more than a special case of a Fundamental Counting Principle problem. Using either strategy to count the number of permutations will arrive at the same answer.

> **Example 8**
>
> There are 10 cars in a race. In how many ways can three cars be awarded 1st, 2nd, and 3rd place?
>
> **Solution**
>
> The order in which the cars finish is important to consider when counting the number of ways the cars can finish. We say this is a "permutation of 10 choose 3" and can write it symbolically as ${}_{10}P_3$.
>
> - There are 10 possible cars which can finish first.
> - Once a car has finished first, there are 9 cars left which can finish second.
> - After the second car has finished, any of the 8 remaining cars can finish third.
>
> Using the Fundamental Counting Principle, there are $\underline{10} \cdot \underline{9} \cdot \underline{8} = 720$ ways for cars to finish in the top three places.
>
> Alternately, using the permutation formula,
>
> $$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 10 \cdot 9 \cdot 8 $$
> $$ = 720 $$

It should be noted that most calculators have a permutation command. On a TI calculator, it is found in the same menu as the factorial. To evaluate ${}_{10}P_3$ on the TI calculator, type the value of n, go to the **MATH** menu and move right to the **PRB** sub-menu, select the ${}_nP_r$ command, type the value of r, and press **ENTER**. Here is the sequence of screens.



> **Example 9**
>
> There are 4 hooks in a row on a wall to hang some pictures. You have 7 pictures to display. Find the number of ways can you choose and arrange 4 pictures on the hooks from the group of 7 pictures.
>
> **Solution**
>
> The order in which the pictures are arranged is important. Choosing the same group of 4 pictures but placing them in a different order creates a different arrangement. This is a permutation of 7 choose 4, or ${}_7P_4$.
>
> Using the Fundamental Counting Principle, there are 4 choices to make with 7 options for the first choice, 6 options for the second choice, 5 options for the third choice, and 4 options for the fourth choice:
>
> $$ \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} = 840 \text{ ways} $$
>
> Using the permutation formula,
>
> $$ {}_7P_4 = \frac{n!}{(n-r)!} = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840 $$

There are 840 different ways the to choose and arrange 4 pictures on the hooks.

Example 10

The Volunteer Club has 18 members. An election is held to choose president, vice-president, and secretary. In how many ways can the three officers be chosen?

Solution

The order in which the officers are chosen matters. Choosing the same three people but assigning them to a different role would make a different way of choosing the officers. That is, A as president, B as vice-president, and C as secretary is different from B as president, C and vice-president, and A as secretary. This is a permutation of 18 choose 3, or ${}_{18}P_3$.

Using the Fundamental Counting Principle, there are 3 choices to make with 18 options for the first choice, 17 options for the second choice, and 16 options for the third choice:

$$\underset{\text{Pres.}}{\underline{18}} \cdot \underset{\text{V.P.}}{\underline{17}} \cdot \underset{\text{Sec.}}{\underline{16}} = 4896$$

Using the permutation formula,

$${}_{18}P_3 = \frac{n!}{(n-r)!} = \frac{18!}{(18-3)!} = \frac{18!}{15!} = \frac{18 \cdot 17 \cdot 16 \cdot \cancel{15!}}{\cancel{15!}} = 18 \cdot 17 \cdot 16 = 4,896$$

To make the simplification a bit shorter, several of the factors of $18!$ in the numerator cancel with $15!$ in the denominator.

There are 4,896 different ways the three officers can be chosen.

Example 12

How many 5 character passwords can be made using the letters A through Z if letters can be used only once?

Solution

As we all know, the order in which you type the letters in your password matters! Using the correct letters in the password but in the wrong order won't unlock your account. Since a different order of the same five letters makes a different password, this is a permutation of 18 choose 3, or ${}_{18}P_3$.

Using the Fundamental Counting Principle,

$$\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} \cdot \underline{22} = 7,893,600$$

Using the permutation formula,

$${}_{26}P_5 = \frac{n!}{(n-r)!} = \frac{26!}{(26-5)!} = \frac{26!}{21!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!}} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$$

There are 7,893,600 different passwords made of only 5 letters of the alphabet.

Try it Now 4

Twelve actresses are available to be cast in a play that has five female roles. In how many ways could the 5 roles be cast?

Answer

95,040 ways

Combinations

We have considered the situation where we chose r items from a group of n items where the order of selection is important in distinguishing one group from another. We now consider a similar situation in which the order of selection is not important.

A collection of items, in no particular order, is called a *combination*. Using our language of sets, a combination is a subset of a set of objects. For example, suppose that in a group of five students — Andy (A), Barry (B), Cheryl (C), Darren (D), and Ellen (E) — three students are to be selected to make a team. Each of the possible three-member teams is a combination. How many such combinations are there? We can answer this question by using our knowledge of permutations and the Fundamental Counting Principle.

If order *did* matter in the selection of the three students for the team, permutations would be counted as ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$ teams. But, since order doesn't matter in this case, this number of permutations counts more teams than there should be. We need to divide out the number of teams that have been repeated but are in a different order.

When order of team members makes no difference, the permutations ABC, ACB, BAC, BCA, CAB, and CBA are all really the same team, ABC. That is, there are six three-person permutations for each team of three people. This is consistent with the fact that three objects can be rearranged in $3! = 6$ different ways.

To compute the number of possible combinations of three students chosen from a group of five students, we can divide the number of permutations by the number of repeated teams. So, one technique for counting the number of combinations of 5 objects chosen 3 at a time is

$$ {}_5C_3 = \frac{{}_5P_3}{3!} = \frac{60}{6} = 10 \text{ teams,} $$

which could also be written as

$$ {}_5C_3 = \frac{{}_5P_3}{3!} = \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!\,3!} == \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(2 \times 1) \times (\cancel{3} \times \cancel{2} \times \cancel{1})} = 10 \quad \text{teams.} $$

We can generalize the preceding observations to write a formula for the number of combinations of r objects selected from a group of n objects.

Combination Formula

A **combination** is a selection of a set of objects without repetition in which the order of selection does not matter.

The number of combinations of n objects is taken r at a time is given by the formula

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!}\ . $$

Example 13

A college class has a reading list of eight books. A student must choose and read five of the books before the end of the course. In how many ways can the student choose five books to read?

Solution

The order of selecting the books is not important, only which books are read. That is, as long as the same 5 books are selected it is the same choice no matter which order they are selected. We say this is a "combination of 8 choose 5" and can write it symbolically as ${}_8C_5$.

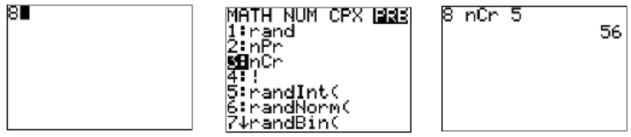
Using the formula for counting combinations and simplifying,

$$ C(8,5) = \frac{n!}{(n-r)!\,r!} = \frac{8!}{(8-5)!\,5!} = \frac{8!}{3! \times 5!} = \frac{8 \times 7 \times 6 \times 5!}{3! \times 5!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{3! \times \cancel{5!}} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = \frac{8 \times 7 \times 6}{6} $$

$$ = \frac{8 \times 7 \times \cancel{6}}{\cancel{6}} = 8 \times 7 = 56 $$

There are 56 ways to choose five of the books to read.

As with permutations, most calculators have a combination command. On the TI calculator, it is found in the same menu as factorial and permutation. To find ${}_8C_5$ on the TI calculator, type the value of n, go to the **MATH** menu and move right to the **PRB** sub-menu, select the ${}_nC_r$ command, type the value of r, and press **ENTER**. Here is the sequence of screens.



✔ Example 14

A child wants to pick three pieces of candy to take in her school lunch box. If there are 13 pieces of candy to choose from, how many ways can she pick the three pieces?

Solution

This is a combination because it does not matter in what order the candy is chosen. The same 3 pieces selected in any order gives the child the same candies. This is a combination of 13 choose 3, or ${}_{13}C_3$.

Using the combination formula,

$$ {}_{13}C_3 = \frac{n!}{(n-r)!\, r!} = \frac{13!}{(13-3)!\, 3!} = \frac{13!}{10! \times 3!} = \frac{13 \times 12 \times 11 \times 10!}{10! \times 3!} = \frac{13 \times 12 \times 11 \times \cancel{10!}}{\cancel{10!} \times 3!} = \frac{13 \times 12 \times 11}{3 \times 2 \times 1} $$
$$ = \frac{1716}{6} = 286 $$

There are 286 ways to choose the three pieces of candy to pack in her lunch.

✔ Example 15

The Volunteer Club has 18 members. A committee of members will be selected to plan the annual food drive. How many different 3-person committees can be selected?

Solution

Unlike selecting members for officers where each officer serves a different role, the order in which the members are chosen for a committee is not meaningful. Choosing the same group of three people but in a different order results in the same committee of three people. This is a combination of 18 choose 3, or ${}_{18}C_3$.

$$ {}_{18}C_3 = \frac{n!}{(n-r)!\, r!} = \frac{18!}{(18-3)!\, 3!} = \frac{18!}{15! \times 3!} = \frac{18 \times 17 \times 16 \times 15!}{15! \times 3!} = \frac{18 \times 17 \times 16 \times \cancel{15!}}{\cancel{15!} \times 3!} = \frac{18 \times 17 \times 16}{3 \times 2 \times 1} $$
$$ = \frac{4896}{6} = 816 $$

You could have also used the result from Example 10:

$$ {}_{18}C_3 = \frac{{}_{18}P_3}{3!} = \frac{4896}{6} = 816\ . $$

There are 816 ways to choose a committee of three members.

? Try it Now 5

You have 4 extra tickets to the Nationals game and 9 of your friends want to go. How many ways can you select a group of friends to join you at the game?

Answer

126 ways

Simplifying permutations and combinations by hand can be tedious for large quantities so most of the time we will want to use technology. The difficulty for most people is knowing whether a problem calls for a permutation, a combination, or only the Fundamental Counting Principle. The table gives a quick summary:

Distinguishing between Fundamental Counting Principle, Permutation, and Combination

Fundamental Counting Principle	Counts the number of ways for event A followed by event B when event A has m outcomes and event B has n outcomes: $$m \times n$$ • Can be used with a sequence of more than 2 events. • May be used with or without repetitions. • Particularly useful when there are specific placement requirements or conditions to meet (e.g. first digit must be odd.)
Permutation	Counts the number of ways to select r items from a group of n items when the order of items is important $${}_nP_r = \frac{n!}{(n-r)!}$$ • ABC is different than BAC. • Items cannot be repeated. • May also be solved using the Fundamental Counting Principle.
Combination	Counts the number of ways to select r items from a group of n items when the order of items is not important $${}_nC_r = \frac{n!}{(n-r)!\,r!}$$ • ABC is the same as BAC. • Items cannot be repeated. • May also be solved by dividing the number of permutations by $r!$

In the examples that follow, we concentrate on identifying which method to use and calculate with technology.

Example 16

A jury pool consists of 20 people. How many different 12-person juries can be selected from the jury pool?

Solution

Choosing the same 12 people in a different order for the jury would not result in a different outcome. Order does not make a difference so this is a combination: ${}_{20}C_{12} = 125,970$ juries.

Example 17

A swimming event has 16 contestants. The swimmers with the five fastest speeds will be listed, in the order of their speed, on the leader board. How many ways are there for the names of five swimmers to be listed?

Solution

Listing the same 5 names in a different order on the leader board would be a different result to the race. Order is important so this is a permutation: ${}_{16}P_5 = 524,160$ ways.

Example 18

You are completing a four-question survey on your experience at Taco Bell. You can answer *Below Average*, *Average*, or *Above Average* for each question. In how many different ways could you answer this survey?

Solution

Recall that when items can be selected more than once you cannot use the permutation or combination formulas. It is possible to respond "Average" to more than one of the four questions. There are four choices to be made, and there are three options for each choice. Use the Fundamental Counting Principle: $\underline{3} \times \underline{3} \times \underline{3} \times \underline{3} = 81$ ways.

Example 19

When playing poker, players are dealt 5 cards from a regular deck of cards. How many different hands of poker could a player be dealt?

Solution

Receiving the same five cards but in a different order would not mean that you had a different set of cards. Order does not make a difference so this is a combination: ${}_{52}C_5 = 2,598,960$ hands.

Try it Now 6

Decide whether the scenario requires permutations, combinations, or the Fundamental Counting Principle. Then, use technology to compute the answer.

a. Four of 15 people attending a meeting will be selected to receive door prizes. One receives a book, one receives a gift card, one receives a notepad, and another receives a box of candy. In how many ways can the door prizes be awarded?
b. A serial number is formed using two letters of the alphabet, followed by two digits, followed by another letter of the alphabet. If letters and digits can be repeated, how many different serial numbers can be formed?
c. There are 12 standby passengers who hope to get on a flight to Hawaii, but only 6 seats are available on the plane. How many different ways can the 6 people be selected?

Answer

a. permutation: 32,760 ways
b. Fundamental Counting Principle: 1,757,600 serial numbers
c. combination: 924 ways

Using More than One Method

Sometimes a counting problem may require more than one counting method. For example, you may need to compute combinations and use the Fundamental Counting Principle together. Let's look at a relatively simple but common example.

Example 20

A sandwich shop offers a special: choose exactly one kind of bread, one kind of protein, and three toppings from the menu below and get a special price. How many ways can a customer select menu items for the special?

Sandwich Menu

Bread	Protein	Vegetable Toppings
White	Ham	Lettuce
Wheat	Turkey	Pickle
	Roast Beef	Onion
	Cheese	Tomato
		Hot Peppers

Solution

This problem looks very similar to examples where we used the Fundamental Counting Principle. However, in those problems, we chose only one option for each stage of choice. Here, we are choosing one type of bread, one type of protein, but three types of vegetable toppings.

The Fundamental Counting Principle can be applied by multiplying the number of ways a bread can be selected (2), the number of ways a protein can be selected (4), and the number of ways 3 vegetables can be selected from 5 vegetables $({}_5C_3 = 10)$:

$$\underline{2} \times \underline{4} \times \underline{{}_5C_3} = 2 \times 4 \times 10 = 80 \quad .$$

So, there are 80 different ways of selecting menu items for he special.

Example 21

Brenda will choose 5 movies to rent over the weekend, and she has decided to rent 3 science fiction movies and 2 comedies. She can choose from 6 science fiction movies and 4 comedies. How many different ways can Brenda choose the group of 5 movies?

Solution

There are two different stages to selecting the group of movies.

First, compute how many ways Brenda can select 3 science fiction movies from a group of 6 science fiction movies using ${}_6C_3 = \frac{6!}{(6-3)!\ 3!} = 20$.

Then, compute how many ways Brenda can choose 2 comedies from the group of 4 comedies using ${}_4C_2 = \frac{4!}{(4-2)!\ 2!} = 6$.

Each group of science fiction movies can be paired with each group of the comedies so we use the *Fundamental Counting Principle* to find how many different groups of 3 science fiction movies and 2 comedies can be selected together:

$${}_6C_3 \times {}_4C_2 = 20 \times 6 = 120 \text{ .}$$

There are 120 ways for Brenda to choose her group of 5 movies.

Try it Now 7

An art gallery has a total of 11 paintings by a certain artist. Of these paintings, 5 are oil paintings and 6 are watercolor paintings. The art gallery will display a special exhibition of this artist's work but is restricted to showing only 7 paintings. Calculate the number of ways in which the 7 paintings can be selected for the exhibition if it includes 3 oil paintings and 4 watercolor paintings.

Answer

150 ways

In the next section we will apply these counting methods when we return to calculating theoretical probability.

3.7: Probability with Counting Methods

Recall our definition of theoretical probability: If outcomes of an experiment are equally likely, then

$$P(E)=\frac{\text{number of outcomes in event } E}{\text{number of outcomes in the sample space } S}\ .$$

Earlier in this chapter, we determined the number of outcomes in event E and in the sample space S by listing them to count them. However, some experiments are so complex that it would take too long to list all the outcomes and then count them. When that is the case, we depend on counting methods.

Probabilities Involving Counting Methods

As we begin to find probabilities with counting methods, we must continue to focus on whether order is important in distinguishing one outcome from another in the event.

Example 1

There is a box that contains 13 marbles: 8 blue and 5 red. We will select two marbles from the box. What's the probability that both marbles are blue?

Solution

The order in which we select the two blue marbles is not important to the answer. We just want both to be blue, so this problem can be considered a combination probability problem. We can use combinations to find the number of outcomes in the event "select 2 blue marbles." Then, we use combinations to find the number of outcomes in the sample space of "selecting 2 marbles of any color."

- The number of ways to select 2 blue marbles from the 8 blue marbles in the box is

$${}_8C_2=\frac{8!}{(8-2)!\ 2!}=\frac{8!}{6!\ \times\ 2!}=\frac{8\times 7\times 6!}{6!\ \times\ 2!}=\frac{8\times 7\times \cancel{6!}}{\cancel{6!}\ \times\ 2!}=\frac{8\times 7}{2\times 1}=\frac{56}{2}=28\quad .$$

- The number of ways to select 2 marbles of any color from the 13 marbles in the box is

$${}_{13}C_2=\frac{13!}{(13-2)!\ 2!}=\frac{13!}{11!\ \times\ 2!}=\frac{13\times 12\times 11!}{11!\ \times\ 2!}=\frac{13\times 12\times \cancel{11!}}{\cancel{11!}\ \times\ 2!}=\frac{13\times 12}{2\times 1}=\frac{156}{2}=78\quad .$$

- So, $P(\text{selecting 2 blue marbles})=\frac{{}_8C_2}{{}_{13}C_2}=\frac{28}{78}=\frac{14}{39}\approx 0.3590$.

The probability of selecting 2 blue marbles from the box is 0.3590 or 35.90%.

Example 2

There are 8 finalists in the 100-meter sprint at the Olympic games. Suppose 4 of the runners are from the United States and that all the runners have an equal chance of winning. What's the probability that runners from the United States finish in 1st, 2nd, and 3rd place?

Solution

Order is important in counting the number of ways the runners could finish 1st, 2nd, and 3rd place, so this problem can be considered a permutation probability problem.

We can use permutations to find the number of orders in which 3 of the 4 U.S. runners could finish the race. Then, we use permutations to find the number of orders in which 3 of the 8 runners could finish the race.

- The number of orders in which any 3 of the 4 U.S. runners can finish the race is

$${}_4P_3=\frac{4!}{(4-3)!}=\frac{4!}{1!}=\frac{4\times 3\times 2\times 1}{1}=\frac{24}{1}=24\quad .$$

- The number of orders in which any 3 of the 8 runners can finish the race is

$${}_8P_3=\frac{8!}{(8-3)!}=\frac{8!}{5!}=\frac{8\times 7\times 6\times 5!}{5!}=\frac{8\times 7\times 6\times \cancel{5!}}{\cancel{5!}}=8\times 7\times 6=336\quad .$$

- So, $P(3 \text{ runners from U.S. win 1st, 2nd, and 3rd place }) = \frac{{}_4P_3}{{}_8P_3} = \frac{24}{336} = \frac{1}{14} \approx 0.0714$.

The probability that U.S. runners finish in 1st, 2nd, and 3rd place is 0.0714 or 7.14%.

From this point on, we will use technology to count permutations and combinations rather than writing out all steps.

Example 3

There are 20 students in a class, of which 12 are females. The names of the students are put into a hat and five names are drawn. What is the probability that all of the students chosen are females?

Solution

We need to find $P(\text{ select all females }) = \frac{\text{number of ways to select 5 females}}{\text{number of ways to select 5 students}}$.

This is a combination problem because the order of selecting the students is not important in counting the outcomes as different. We just want them all to be females.

The number of ways to select 5 females is ${}_{12}C_5 = 792$. The number of ways to select 5 students from the class is ${}_{20}C_5 = 15,504$.

$$P(\text{ select all females }) = \frac{\text{number of ways to select 5 females}}{\text{number of ways to select 5 students}} = \frac{{}_{12}C_5}{{}_{20}C_5} = \frac{792}{15,504} \approx 0.0511$$

The probability that all the students chosen are females is 0.0511 or 5.11%.

Example 4

A research laboratory requires a four-digit passcode to enter. A passcode may contain any four digits 0, 1, 2, 3, ... 8, 9 but no digit can be repeated. What is the probability that a scientist is assigned a passcode with the digits 6, 7, 8, and 9 in any order?

Solution

The order of the digits in a passcode is meaningful. That is, a passcode of 6-7-8-9 is different than 7-6-9-8 even though the same four digits are used. Therefore, we need to count permutations in this problem.

The number of ordered passcodes that can be formed with the digits 6, 7, 8, and 9 is ${}_4P_4 = 24$, or $4! = 24$. The number of ordered passcodes that can be formed using any 4 digits is ${}_{10}P_4 = 5,040$.

$$P(\text{passcode contains the digits 6, 7, 8, and 9 }) = \frac{\text{number of passcodes containing the digits 6, 7, 8, and 9}}{\text{number of passcodes containing any 4 digits}}$$

$$= \frac{{}_4P_4}{{}_{10}P_4} = \frac{24}{5,040} = \frac{1}{210} \approx 0.0048$$

The probability that a scientist is assigned a passcode with the digits 6, 7, 8, and 9 is 0.0048 or 0.48%.

Try it Now 1

There are 10 canned drinks in a cooler: 6 are colas and 4 are fruit drinks. Danielle will reach into the cooler and pull out 5 drinks without looking to see what she gets. Find the probability that all 5 drinks are colas.

Answer

$\frac{1}{42}$

Try it Now 2

A motorcycle license plate consists of 5 digits that are randomly selected. No digit is repeated. What is the probability of getting a license plate with all odd numbers?

Answer

$\frac{1}{252}$

Example 5

There are 12 males and 8 females in a jury pool. A group of 8 of them will be selected to sit on a jury. What's the probability of selecting 5 males and 3 females for the jury?

Solution

$P(\text{ select 5 males and 3 females }) = \frac{\text{ways to select 5 male jurors and 3 female jurors}}{\text{ways to select 8 jurors}}$

This problem involves combinations because the order in which jurors are selected is not important to counting a jury as different.

First, we count the number of outcomes in the event "select 5 male jurors and 3 female jurors." The number of ways to select 5 male jurors is ${}_{12}C_5 = 792$. The number of ways to select 3 females jurors is ${}_{8}C_3 = 56$. Using the Fundamental Counting Principle as we did at the end of Section 3.6, the number of ways of selecting 5 males and 3 females is ${}_{12}C_5 \times {}_{8}C_3 = 792 \times 56 = 44,352$.

Next, we count the number of outcomes in the sample space of the experiment "select any 8 jurors." The number of ways to select 8 jurors from the group of $12 + 8 = 20$ people is ${}_{20}C_8 = 125,970$ ways.

Finally,

$$P(\text{ 5 males and 3 females }) = \frac{\text{ways to select 5 male jurors and 3 female jurors}}{\text{ways to select 8 jurors}} = \frac{{}_{12}C_5 \times {}_{8}C_3}{{}_{20}C_8} = \frac{44,352}{125,970}$$
$$\approx 0.3521.$$

The probability the jury will contain 5 males and 3 females is 0.3521 or 35.21%.

Try it Now 3

There are 10 canned drinks in a cooler: 6 are colas and 4 are fruit drinks. Danielle will reach into the cooler and pull out 5 drinks without looking to see what she gets. Find the probability that 3 drinks are colas and 2 drinks are fruit drinks.

Answer

$\frac{10}{21}$

Example 6

A technician is launching fireworks near the end of a show. Of the remaining 15 fireworks, 8 are blue, 4 are red, and 3 are white. If she launches 5 of them in a random order, what is the probability that she launches no blue fireworks?

Solution

This problem involves combinations because the order in which fireworks are selected is not important to counting the outcomes as different.

If no blue fireworks are launched, then only red and white fireworks are to be selected. There are $4 + 3 = 7$ fireworks that are red or white. The number of ways 5 fireworks can be selected from this group of 7 red or white fireworks is ${}_{7}C_5 = 21$.

The total number of possible outcomes for the experiment of selecting 5 fireworks is ${}_{15}C_5 = 3,003$.

$$P(\text{select no blue fireworks}) = \frac{\text{ways to select 5 red or white fireworks}}{\text{ways to select 5 fireworks}} = \frac{{}_{7}C_5}{{}_{15}C_5} = \frac{21}{3,003} = \frac{1}{143} \approx 0.0070.$$

The probability that no blue fireworks are launched is 0.0070 or 0.70%.

Example 7

Compute the probability that a 5-card poker hand is dealt to you that contains exactly two hearts.

Solution

This problem involves combinations because the order in which cards are dealt is not important to counting poker hands as different. In solving this problem, it is important to understand that the event must be made up of 5 cards: 2 of the cards must be hearts and the remaining 3 cards can be anything else except hearts.

First, count the number of outcomes in the event "2 hearts and 3 non-hearts." The number of ways to select 2 hearts is ${}_{13}C_2 = 78$. The number of ways to select 3 non-hearts is ${}_{39}C_3 = 9,139$. Using the Fundamental Counting Principle , the number of ways to get 2 hearts and 3 non-hearts is ${}_{13}C_2 \times {}_{39}C_3 = 78 \times 9,139 = 712,842$.

Next, we count the number of outcomes in the sample space of a 5-card poker hand. The number of different poker hands is ${}_{52}C_5 = 2,598,960$.

Finally,

$$P(\text{ exactly 2 hearts}) = \frac{\text{ways to select 2 hearts and 3 non-hearts}}{\text{ways to select 5 cards}} = \frac{{}_{13}C_2 \times {}_{39}C_3}{{}_{52}C_5} = \frac{78 \times 9,139}{2,598,960} = \frac{712,842}{2,598,960}$$

$$\approx 0.2743.$$

The probability of being dealt exactly 2 hearts in a five-card poker hand is 0.2743 or 27.43%.

Try it Now 4

A bag contains 6 real diamonds and 5 fake diamonds. If 4 diamonds are picked from the bag at random, what is the probability that

a. none of them are fake.
b. exactly 2 of them are real.

Answer

a. $\frac{1}{22}$
b. $\frac{5}{11}$

Example 8

A basket contains 6 good apples and 4 bad apples. A distracted shopper reaches into the basket and picks 3 apples without looking. What is the probability he gets at least one bad apple?

Solution

This is a combination since the order in which the apples are picked is not important. Getting "at least one bad apple" when selecting 3 apples means there could be 1 bad apple, 2 bad apples, or 3 bad apples. This is an "at least one" problem so we can use the Complement Rule:

$$P(\text{select at least 1 bad apple}) = 1 - P(\text{select no bad apples}) \; .$$

Selecting no bad apples means selecting 3 good apples. There are 6 good apples from which to choose. The number of ways of selecting 3 good apples is ${}_6C_3 = 20$. The number of outcomes in the experiment of selecting any of the 3 apples from the basket is ${}_{10}C_3 = 120$.

Therefore,

$$P(\text{ select 3 good apples }) = \frac{\text{ways to select 3 good apples}}{\text{ways to select 3 apples}} = \frac{{}_6C_3}{{}_{10}C_3} = \frac{20}{120} = \frac{1}{6}$$

Finally,

$$\begin{aligned}P(\text{at least 1 bad apple}) &= 1 - P(\text{select no bad apples}) \\ &= 1 - P(\text{select 3 good apples}) \\ &= 1 - \frac{1}{6} \\ &= \frac{5}{6} \approx 0.8333\end{aligned}$$

The probability that the customer gets at least one bad apple is 0.8333 or 83.33%.

Try it Now 5

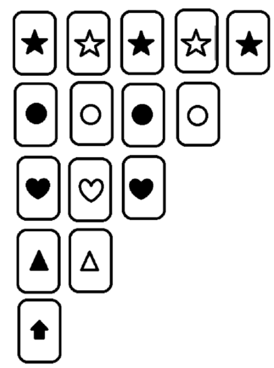
A jar contains 5 black buttons and 4 brown buttons. If 3 buttons are picked at random, what is the probability that at least one of them is brown?

Answer

$\frac{37}{42}$

3.8: Review Exercises

1. According to the results of a survey, 200 people said they prefer cats, 160 said they prefer dogs, while 40 said they don't like either pet.
   a. Find the empirical probability that a person prefers dogs. Give the probability as a fraction in simplest form, as a decimal, and as a percent.
   b. Based on this probability, how many people should prefer dogs if a survey of 500 people is taken?
2. A six-sided die (with sides numbered 1 through 6) is tossed. Find the probability of each event as a simplified fraction.
   a. getting an even number.
   b. getting a number less than 5.
   c. getting a 7.
3. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of each event as a simplified fraction.
   a. a red ball is drawn
   b. a white or yellow ball is drawn
   c. a yellow ball is not drawn
4. This set of 15 cards is used in a game. The cards are well-shuffled, and then you pick one card. Calculate each of the following.



   a. P(selecting a circle)
   b. P(selecting a black shape)
   c. P(selecting a black heart)
   d. P(selecting a triangle or a star)
   e. P(selecting a heart or a black shape)
   f. P(selecting a card that is not a star)
   g. P(selecting a circle given the shape is white)
   h. P(selecting a white shape given it is a circle)
   i. odds in favor of selecting a triangle
   j. odds against selecting a black shape
5. The sample space for rolling two dice is shown as ordered pairs. Use the sample space to find the probability of each event.

   a. P(roll the same value on both dice)
   b. P(roll a 5 on at least one of the two dice)
   c. P(sum of the dice is 8)
6. When a button is pressed, a computer program outputs a random odd number greater than 1 but less than 9. Consider the experiment "press the button twice."
   a. Write the sample space for this experiment. Use an ordered pair for each outcome.
   b. List the outcomes in the event $A =$ "*the sum of the two numbers is 10 or more.*" Then find $P(A)$.
   c. List the outcomes in the event $B =$ "*both numbers are the same.*" Then find $P(B)$.
7. A box contains one green, one red, one yellow, and one blue marble. You will select one marble and record its color. Without replacing the marble, you will select another marble and record its color.
   a. Write the sample space for this experiment. Use an ordered pair for each outcome.
   b. What is the probability that you get a red or yellow marble?
   c. What is the probability that you won't get a yellow marble?
8. An urn contains 10 red balls, 15 white balls, and 20 black balls. A single ball is selected at random. Find these odds written as ratios in lowest terms.
   a. odds in favor of selecting a red ball

b. odds against drawing a white ball.

9. The odds in favor of Julio winning the game are 9 : 14 What is the probability that Julio wins?
10. The probability your English professor is in their office is $\frac{3}{5}$.
    a. What is the probability that your English professor is not in their office?
    b. What are the odds in favor of your English professor being in their office?
11. There is a 35% probability that the bus is late. What are the odds in favor of the bus being late?
12. For a certain tutor the probabilities for various numbers of student no-shows per day are shown in the following table. What is the expected value for the number of no-shows per day?

Probabilities of No Shows for Tutoring

	Number of No Shows		0	1	2	3	4
	Probability		0.40	0.25	0.20	0.10	0.05

13. An insurance company insures a house worth $250,000 for an annual premium of $500. If the probability of the house being destroyed is 0.0015 and assuming either total loss or no loss, what is the insurance company's expected annual profit for the policy?
14. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. What is your expected gain or loss for playing this game. Is this a fair game?
15. A student believes he has an 80% chance of passing his English class, 30% chance of passing his math class, and a 25% chance of passing both classes.
    a. What is the probability he will not pass his English class?
    b. What is the probability he passes English or math?
    c. What is the probability he passes neither class?
    d. What is the probability of passing math given he passed English?
16. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.
17. On any given night, the probability that Nick has a cookie for dessert is 15%. The probability that Nick has ice cream for dessert is 50%. The probability that Nick has a cookie and ice cream is 10%. What is the probability that Nick has a cookie or ice cream for dessert?
18. Suppose $P(B) = 0.68$, $P(A \text{ or } B) = 0.90$, and $P(A \text{ and } B) = 0.45$. Find $P(A)$.
19. The table below shows the number of credit cards owned by a group of individuals. One person will be selected at random. Find the probability of each event below.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

    a. the person is female
    b. the person has exactly 0 or 1 credit card
    c. the person is male and has exactly one credit card
    d. the person is male or has no credit cards.
    e. the person is female given that they have 2 or more credit cards
    f. the person has 0 credit cards if they are male

20. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card. Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that
    a. Bert gets a Jack and Ernie rolls a five.
    b. Bert gets a heart and Ernie rolls a number less than six.
    c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
    d. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.

21. A multiple-choice test consists of 4 questions. Each question has answer choices *a* through *e* so the chance of guessing the correct answer is $\frac{1}{5}=0.2$. Suppose a student guesses on all 4 questions.
    a. What is the probability the student guesses all the answers correctly?
    b. What is the probability the student guesses none of the answers correctly?
    c. What is the probability the student guesses at least one of the answers correctly?
22. A jar contains 6 red balls, 2 white balls, and 5 yellow balls. Two balls are randomly selected *with replacement*. Find the probability of each event.
    a. both balls are red.
    b. neither of the balls are red.
    c. first ball is red and second ball is white.
23. A jar contains 6 red balls, 2 white balls, and 5 yellow balls. Two balls are randomly selected *without replacement*. Find the probability of each event.
    a. both balls are red.
    b. neither of the balls are red.
    c. first ball is red and second ball is white.
24. A math class consists of 25 students, 14 female and 11 male. Three different students are selected at random to participate in a probability experiment. Compute the probability that
    a. a male is selected, then two females.
    b. a female is selected, then two males.
    c. three males are selected.
25. Suppose that 21% of people own dogs. You pick three people at random. Assuming independence between the people, find each probability:
    a. all 3 people own a dog.
    b. none of the 3 people own a dog.
    c. at least one of the 3 people own a dog.
26. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. even-numbered given that the marble is red.
    b. red given that the marble is even-numbered.
27. The probability that Jack oversleeps is 40%.The probability Jack oversleeps and is late for work is 10%. What is the probability that Jack is late for work given he oversleeps?
28. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?
29. A computer password must be 6 characters long. How many passwords are possible if
    a. only the 26 lowercase letters of the alphabet are allowed and letters may be repeated?
    b. only the 26 lowercase letters of the alphabet are allowed and letters may not be repeated?
    c. both lowercase and uppercase letters are allowed and letters may be repeated?
30. A doctor needs to schedule 4 patients for a vaccine at different times. In how many ways can she she schedule the 4 patients?
31. In how many ways can first, second, and third prizes be awarded in a contest with 50 contestants?
32. In how many ways can you select 4 different pizza toppings from 12 available toppings on the menu?
33. At a baby shower 15 guests are in attendance and 5 of them are randomly selected to receive identical door prizes. How many ways can the prizes be awarded?
34. In a lottery game, a player picks six different numbers from 1 to 50. How many different choices of numbers does the player have for his lottery ticket?
35. A congressional committee consists of 10 members: 6 Democrats and 4 Republicans. A sub-committee of 5 members must be formed containing 3 Democrats and 2 Republicans. How many ways can the sub-committee be chosen?
36. A gardener has 12 identical-looking tulip bulbs, of which 7 will produce yellow tulips and 5 will be red. She randomly chooses 4 of the bulbs and plants them.
    a. What's the probability that all 4 tulips will be yellow when they bloom?

b. What's the probability that 2 of the tulips will be yellow and 2 of the tulips will be red when they bloom?

37. Find the probability of being dealt each of the following 5-card poker hands:
    a. all hearts.
    b. exactly three red cards.

CHAPTER OVERVIEW

4: Statistics

4.1: Introduction to Statistics and Sampling

In all aspects of our professional and personal lives we make (or try to make) data driven decisions. **Statistics** is the science of gathering, analyzing, and making predictions from the data we gather. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, statistics is one of the most important things that you can study.

Here are some statistical claims that we have heard many times. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar. If not, we bet that you have heard other claims like them. Notice how diverse the examples are. They come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950's and 60's to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

Populations and Samples

Before we begin gathering and analyzing data we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be

- All first-year community college students in the state of Maryland.
- All first-year students at public colleges and universities in the state of Maryland.
- All first-year students at all colleges and universities in the state of Maryland.
- All first-year students at all colleges and universities in the entire United States.
- and so on...

Population

The **population** of a study is the group the collected data is intended to describe.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Maryland might take slightly more diverse courses than those at PGCC, and some of these courses may require less popular textbooks that cost more. Or, on the other hand, the University of Maryland Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case, the data we gather from PGCC will probably different in nature than that from the University of Maryland. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 1

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

Solution

While the population may have been intended to be all people, the real population of the survey is readers of the website because data is gathered only from the website.

If we were able to gather data from *every member* of the population under study and find the average amount of money spent on textbooks by first-year students at PGCC during the 2022-2023 academic year, the resulting number would be called a **parameter**.

Parameter

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

However, we seldom see parameters. This is because surveying an entire population is usually very time-consuming and expensive unless the population is very small or the data has already been collected for us. In those cases where data is collected from the entire population, the process is called a **census**.

Census

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: 1) the official government Census that attempts to count the population of the U.S. every ten years, and 2) voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study.

Sample

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail momentarily. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample of the entire group we want to study (say, 100 first-year students PGCC) and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Statistic

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example 2

A researcher wants to know how citizens of Tacoma feels about a voter initiative. To study this, she goes to the Tacoma Mall and randomly selects 500 shoppers to ask them their opinion. The researcher finds that 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

Solution

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tacoma citizens, the population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tacoma citizens.

The 60% value was based on the sample, so it is a statistic.

Try it Now 1

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Answer

The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch with the bait.

Sampling Methods

Now that you know that you have to take samples in order to gather data, the next question is how to best gather a sample. There are many ways to take samples. Not all of them will result in a representative sample. Also, just because a sample is large does nt mean it is a good sample. As an example, you can take a sample involving one million people to find out if they feel there should be more gun control. But if you only ask members of the National Rifle Association (NRA) you may get biased results. The same is true if you only ask members of the Coalition to Stop Gun Violence. You need to make sure that you ask a cross-section of individuals. Let's look at the types of samples that can be taken. Do realize that no sample is perfect and any sample may not result in a representation of the population.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is *simple random samplin*g.

Simple random sample

A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 3

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1,000 slips out of the hat, we would have a simple random sample.

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear. The best procedure to select a random sample is to use a random number generator program that you can find in *Excel* or on a TI graphing calculator.

Another sampling technique that helps to assure that various groups of population has equal provability to be represented in a sample is known as *stratified sampling*. Stratified sampling is done by breaking the population into groups of individuals sharing some common trait (like gender, ethnicity, political affiliation, grade level, etc.*)* Then, some individuals are random selected from each of these groups to form the stratified sample.

Stratified sampling

In **stratified sampling**, a population is divided into a number of subgroups (called strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 4

In a particular state, previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. In a sample of 1,000 people, they would then expect to get about 390 Democrats, 370 Republicans and 240 independents.

To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Another sampling method is *cluster sampling.* Again, the population is divided into groups, but this time one or more whole groups are randomly selected to be in the sample.

Cluster sampling

In **cluster sampling**, the population is divided into subgroups (called clusters), and a set of subgroups are selected to be in the sample.

Cluster sampling is often done by selecting a neighborhood, block, or street at random from within a town or city. It is also used at large public gatherings or rallies. This is how the National Park Police determines the number of people who show up for events on the National Mall. They may take a picture of a small, representative area of the crowd, count the individuals in just that area, and then use that count to estimate the total crowd in attendance.

Cluster sampling is very useful in geographic studies such as about the opinions of people in a state or measuring the diameter at breast height of trees in a national forest. In both situations, a cluster sample reduces the traveling distances that occur in a simple random sample. For example, suppose that the Gallup Poll needs to perform a public opinion poll of all registered voters in Colorado. In order to select a good sample using simple random sampling, the pollsters would have to have the names of all the registered voters in Colorado and then randomly select a subset of these names. This may be very difficult to do. So, they will use a cluster sample instead. Start by dividing the state of Colorado up into categories or groups geographically. Randomly select some of these groups. Now ask all registered voters in each of the chosen groups. This makes the job of the pollsters much easier because they will not have to travel over every inch of the state to get their sample. But, is still a random sample.

Example 5

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Another sampling method is *systematic sampling*. This method is often used when there is organizational order in the population. After choosing a starting individual at random, individuals for the sample are selected using a pattern. If you have ever chosen teams or groups by counting off by threes or fours, you were engaged in systematic sampling.

Systematic sampling

In **systematic sampling**, every n^{th} member of the population is selected to be in the sample.

Example 6

To select a sample using systematic sampling, a pollster calls every 100^{th} name in the phone book.

Systematic sampling is not as random as a simple random sample. In the previous example, if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample. But, systematic sampling can yield acceptable samples.

Perhaps the worst types of sampling methods are *convenience samples* and *voluntary response samples*.

Convenience sampling and voluntary response sampling

Convenience sampling is done by collecting data from selecting whoever is convenient to reach.

Voluntary response sampling allows individuals to choose themselves for the sample.

Example 7

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample because the pollster gathers data from those who are easy to each.

Example 8

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Convenience samples should be avoided because they do not use randomization. Individuals that are convenient to reach generally are alike in some way and may not be representative of the entire population. Likewise, voluntary response samples do not use randomization either. Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Example 9

Determine if the sample type is simple random sample, stratified sample, systematic sample, cluster sample, or convenience sample.

1. A researcher wants to determine the different species of trees that are in the Coconino National Forest. She divides the forest using a grid system. She then randomly picks 20 different sections and records the species of every tree in each of the chosen sections.

   Solution: This is a *cluster sample* since she randomly selected some of the groups and all individuals in the chosen groups were surveyed.

2. A pollster stands in front of an organic foods grocery store and asks people leaving the store how concerned they are about pesticides in their food.

   Solution: This is a *convenience sample* since the person is just standing out in front of one store. Most likely the people leaving an organic food grocery store are concerned about pesticides in their food, so the sample would be biased.

3. The Pew Research Center wants to determine the education level of mothers. They randomly ask mothers to say if they had some high school, graduated high school, some college, graduated from college, or have an advanced degree.

   Solution: This is a *simple random sample*, since the individuals were picked randomly.

4. Penn State wants to determine the salaries of their graduates in the majors of agricultural sciences, business, engineering, and education. They randomly ask 50 graduates of agricultural sciences, 100 graduates of business, 200 graduates of engineering, and 75 graduates of education what their salaries are.

   Solution: This is a *stratified sample* since all groups were used and then random samples were taken inside each group.

5. In order for the Ford Motor Company to ensure quality of their cars, they test every 130th car coming off the assembly line of their Ohio Assembly Plant in Avon Lake, OH.

   Solution: This is a systematic sample since the sample was determined using a numerical pattern: every 130th car was chosen.

Try it Now 2

In each case, indicate which sampling method was used.

a. Homework was collected from every 4th person who entered the classroom.
b. A sample was selected to contain 25 men and 35 women.
c. Viewers of a new show are invited to vote for their favorite character on the show's website.
d. A website randomly selects 50 of their customers for a satisfaction survey.
e. To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

Answer

a. Systematic b. Stratified c. Voluntary response d. Simple random e. Cluster

4.2: Frequency Distributions and Statistical Graphs

Once we have collected data, then we need to start analyzing the data. One way to display and summarize data is to use statistical graphing techniques. The type of graph we use depends on the type of data collected. Qualitative data use graphs like bar graphs and pie graphs. Quantitative data use graphs such as histograms and frequency polygons.

In order to create graphs, we must first organize and create a summary of the individual data values in the form of a *frequency distribution*. A **frequency distribution** is a listing all of the data values (or groups of data values) and how often those data values occur.

> **Frequency and Frequency Distributions**
>
> **Frequency** is the number of times a data value or groups of data values (called *classes*) occur in a data set.
>
> A **frequency distribution** is a listing of each data value or class of data values along with their frequencies.
>
> **Relative frequency** is the frequency divided by n, the size of the sample. This gives the proportion of the entire data set represented by each value or class. Relative frequencies are expressed as fractions, decimals, or percentages.
>
> A **relative frequency distribution** is a listing of each data value or class of data values along with their relative frequencies.

The method of creating a frequency distribution depends on whether we are working with *qualitative data* or *quantitative data*. We will now look at how to create each type of frequency distribution according to the type of data and the graphs that go with them.

Organizing Qualitative Data

Qualitative data are pieces of information that allow us to classify the items under investigation into various categories. We usually begin working with qualitative data by giving the frequency distribution as a *frequency table*.

> **Frequency Table**
>
> A **frequency table** is a table with two columns. One column lists the categories, and another column gives the frequencies with which the items in the categories occur (how many data fit into each category).

> **Example 1**
>
> An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some colors of cars are more likely to ve involved in accidents. To research this, the insurance company examines police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Graphing Qualitative Data in Bar Graphs and Pie Charts

Once we have organized and summarized qualitative data into a frequency table, we are ready to graph the data. There are various ways to visualize qualitative data. In this section we will consider two common graphs: *bar graphs* and *pie graphs*.

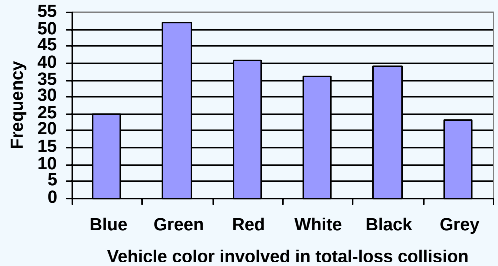
Bar graph

A **bar graph** is displays a bar for each category. The length of each bar indicates the frequency of that category.

To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction has a scale and measures the frequency of each category. The horizontal axis has no scale in this instance but lists the categories. The construction of a bar chart is most easily described by use of an example.

Example 2

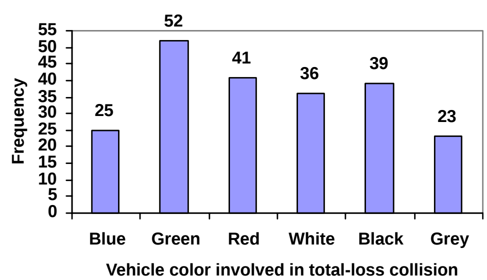
Using the car color data from Example 1, note the highest frequency was 52, so the vertical axis needs to go from 0 to 52. We might as well use 0 to 55 so that we can put a hash mark every 5 units:



You should notice a few things about the correct construction of this bar graph.

- The height of each bar is determined by the frequency of the corresponding color.
- Both axes are labeled clearly.
- The bars do not touch and they are the same width.

The horizontal grid lines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs on graph paper so the grid lines will already be in place or use technology to create the graph. Instead of grid lines, we might also list the frequencies at the top of each bar, like this:



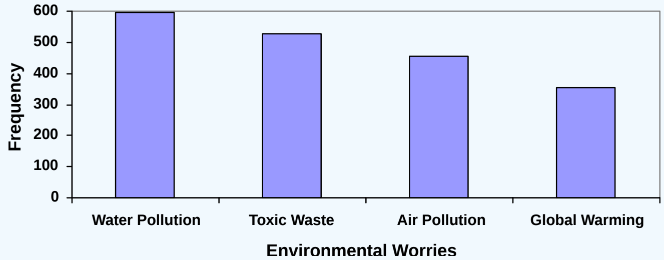
Example 3

In a survey, adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried "a great deal" about some selected concerns are summarized below.

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

Display the data using a bar graph.

Solution



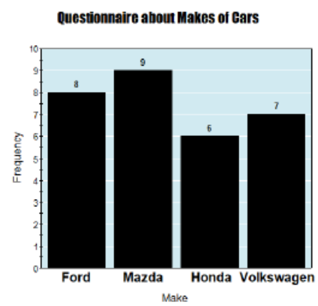
? Try it Now 1

A questionnaire on the makes of people's vehicles showed the following responses from 30 participants. Construct a frequency table and a bar graph to represent the data. (F = Ford, H = Honda, V = Volkswagen, M = Mazda)

F M M M V M F M F V H H F V F H H F M M V H M V V F V H M F

Answer

Make	Frequency
Ford	8
Mazda	9
Honda	6
Volkswagen	7



✔ Example 4

A class was asked for their favorite soft drink with the following results:

Coke	Pepsi	Mt. Dew	Coke	Pepsi	Dr. Pepper	Sprite	Coke	Mt. Dew
Pepsi	Pepsi	Dr. Pepper	Coke	Sprite	Mt. Dew	Pepsi	Dr. Pepper	Coke
Pepsi	Mt. Dew	Coke	Pepsi	Pepsi	Dr. Pepper	Sprite	Pepsi	Coke
Dr. Pepper	Mt. Dew	Sprite	Coke	Coke	Pepsi			

a. Create a frequency distribution table for the data.
b. Create a relative frequency distribution table for the data.
c. Draw a bar graph of the frequency distribution.
d. Draw a bar graph of the relative frequency distribution.

Solution

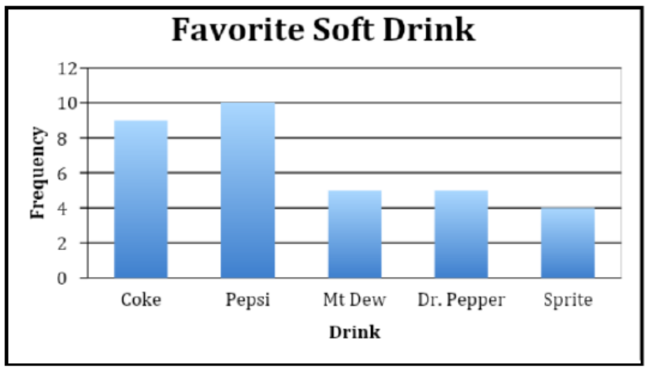
a. To make a frequency distribution table, list each drink type and and then count how often each drink occurs in the data. Notice that Coke happens 9 times in the data set, Pepsi happens 10 times, and so on.

Drink	Coke	Pepsi	Mt Dew	Dr. Pepper	Sprite
Frequency	9	10	5	5	4

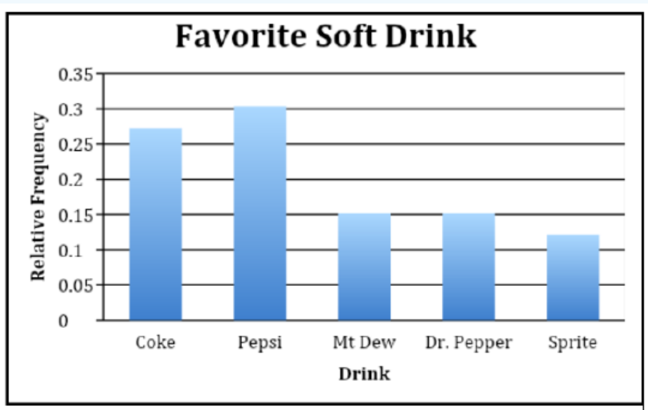
b. To make a relative frequency distribution table, use the previous results and divide each frequency by 33, which is the total number of data responses.

Drink	Coke	Pepsi	Mt Dew	Dr. Pepper	Sprite
Frequency	9	10	5	5	4
Relative Frequency	$\frac{9}{33} \approx 0.273$ or 27.3%	$\frac{10}{33} \approx 0.303$ or 30.3%	$\frac{5}{33} \approx 0.152$ or 15.2%	$\frac{5}{33} \approx 0.152$ or 15.2%	$\frac{4}{33} \approx 0.121$ or 12.1%

c. Along the horizontal axis you place the drinks. Space these apart equally, and allow space to draw bars above the axis. The vertical axis shows the frequencies. Make sure you create a scale along that axis in which all of the frequencies will fit. Notice that the highest frequency is 10, so you want to make sure the vertical axis goes to at least 10, and you may want to count by two for every tick mark. Here is what the graph looks like using *Excel*.



d. A bar graph for the relative frequency distribution is similar to the bar graph for the frequency distribution except that the relative frequencies are used along the vertical axis instead. Notice that the graph does not actually change except the numbers on the vertical scale.



Let's use the last example to introduce another way of visualizing data – a *pie chart* also known as *circle graph*.

Pie Chart

A **pie chart** is a graph where the "pie" represents the entire sample and the "slices" represent the categories or classes. The size of the slice of the pie corresponds to the relative frequency for that category.

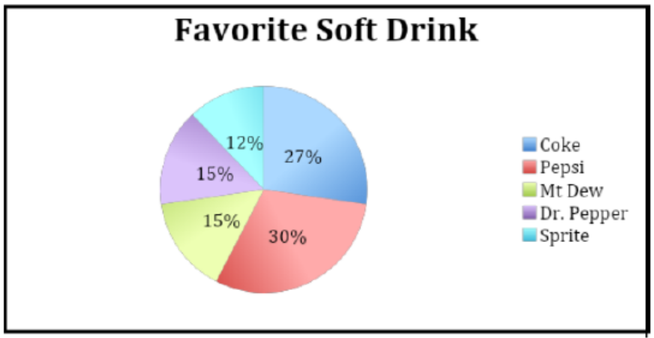
To find the angle that each “slice” takes up, multiply the relative frequency of that slice by 360°.

Note: Theoretically, the percentages of all slices of a pie chart must add to 100%. In practice, the percentages may add to be slightly more or less than 100% if percentages are rounded.

To draw a pie chart, multiply the relative frequencies of each drink by 360°. Then, use a protractor to mark off the corresponding angle in a circle. Usually it is easier to use *Excel* or some other spreadsheet program to draw the graph.

Drink	Coke	Pepsi	Mt Dew	Dr. Pepper	Sprite
Frequency	9	10	5	5	4
Relative Frequency	$\frac{9}{33} \approx 0.273$ or 27.3%	$\frac{10}{33} \approx 0.303$ or 30.3%	$\frac{5}{33} \approx 0.152$ or 15.2%	$\frac{5}{33} \approx 0.152$ or 15.2%	$\frac{4}{33} \approx 0.121$ or 12.1%
Angle Measures	$\frac{9}{33} \times 360^o \approx 98.2^o$	$\frac{10}{33} \times 360^o \approx 109.1^o$	$\frac{5}{33} \times 360^o \approx 54.5^o$	$\frac{5}{33} \times 360^o \approx 54.5^o$	$\frac{4}{33} \times 360^o \approx 43.6^o$

The pie graph from *Excel* is shown below.



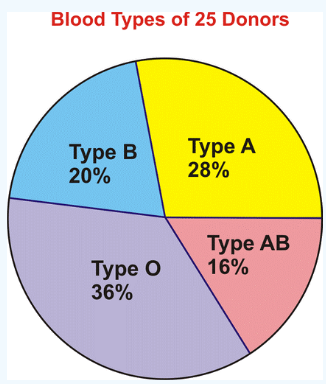
Try it Now 2

The Red Cross Blood Donor Clinic had a very successful morning collecting blood donations. Within 3 hours, many people had made donations. The table shows the frequency distribution of the blood types of the donations. Construct a pie chart to display the relative frequency distribution.

Frequency Table for Blood Types

Blood Type	A	B	O	AB
Number of Donors	7	5	9	4

Answer



Organizing Quantitative Data

Quantitative is data that is the result of counting or measuring some aspect of items under investigation. For this reason, this type of data is also known as numerical data. Quantitative data can also be summarized in a table to show its frequency distribution.

Example 5

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by counting how many times each particular data value occurs.

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

In the previous example, the table listed every different data value that occurred and how often each value occurred. We call this type of frequency distribution presentation **ungrouped**. Sometimes it is helpful to group the data into *classes* to observe information about the distribution of data that otherwise wouldn't be noticeable. This is particularly true if there are many different values or each value only occurs once. You can think about classes as "bins" that we create to sort the data. When we group the data into classes, we call this type of frequency distribution presentation **grouped**.

When data are grouped, the following guidelines about the classes should be followed

1. Classes should have the same width.
2. Classes should not overlap.
3. Each piece of data should belong to only one class.

Let's use the data from the previous example to create a grouped frequency distribution.

Example 6

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

Create a grouped frequency table in two ways:

A. with classes of width 5 beginning at a score of 0, and
B. with classes of width 6 beginning at a score of 0.

Solution

A. The first class contains the scores 0, 1, 2, 3, and 4 -- if any occur. Likewise, the second class will contain scores 5, 6, 7, 8, and 9 -- if any occur. This pattern continues until classes are no longer needed.

The first two columns of the table shows the classes and the frequency of the data in each class.

Frequency Table for Quiz Scores

Class	Frequency	Class Mark (Midpoint)
0-4	2	$\frac{0+4}{2}=2$
5-9	1	$\frac{5+9}{2}=7$
10-14	1	$\frac{10+14}{2}=12$
15-19	20	$\frac{15+19}{2}=17$

Class	Frequency	Class Mark (Midpoint)
20-24	6	$\frac{20+24}{2}=22$

In the first column, the numbers 0, 4, 10, 15, and 20 are called the **lower class limits** and the numbers 4, 9, 14, 19, and 24 are the **upper class limits.** You can see these limits increase by 5. The class width can be determined as the difference between any two consecutive lower or upper class limits. The **class mark** is the midpoint of the class and is determined by averaging the lower and upper limits of the class. The class marks are shown in the third column of the table.

The **modal class** of a frequency distribution is the class with the highest frequency. Here the modal class is 15-19 with a frequency of 20 students. This grouping of the data allows us to more clearly see the grade distribution. Always be sure that the sum of the frequencies is the number of data values.

B. Here is another grouping for the same data but with a class width of 6.

Frequency Table for Quiz Scores

Class	Frequency	Class Mark (Midpoint)
0-5	3	$\frac{0+5}{2}=2.5$
6-11	0	$\frac{6+11}{2}=8.5$
12-17	9	$\frac{12+17}{2}=14.5$
18-23	18	$\frac{18+23}{2}=20.5$

When the data are grouped using this structure, the modal class is 18–23.

? Try it Now 3

The data below indicates number of children in a sample of 16 families:

2 1 2 1 2 5 5 3 2 3 5 2 5 2 2 1

a. Create a non-grouped frequency table for the data.
b. Create a grouped frequency table with first class 0-2. Identify the class width, the class mark for each class, and the modal class.

Answer

a.

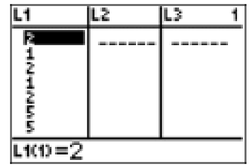
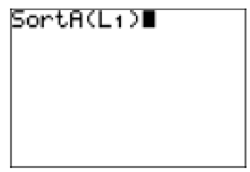
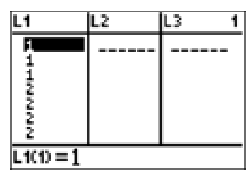
Number of children	Frequency
1	3
2	7
3	2
4	0
5	4

b.

Number of children	Frequency	Class Mark
0-2	10	1
3-5	6	4

Class width: 3 Modal Class: 0-2

There is a "sort feature" on the TI calculator that sorts data in ascending or descending order for you. This makes organizing data and counting frequencies much easier. The steps for entering data and sorting it is shown here for the data presented in *Try it 3*.

	Technology Note: Entering and Sorting Data on the TI-83/84 Calculator	
1.	Press **[STAT]** to open the Statistics menu. Choose **1:Edit** on the **EDIT** menu. Press **[ENTER]**.	
2.	You now have a screen that allows you to enter the data in a column. Enter the number of children data in L1. Be sure to press **[ENTER]** after each value.	
3.	**[QUIT]** the Edit screen, press **[STAT]**. From the **EDIT** menu, choose **SortA(** to sort the data in ascending order (smallest to largest.) **SortD(** will sort the data in descending order (largest to smallest.)	
4.	After **SortA(**, give the list name of the data. If the data were entered in L1, press **[2ND] [1]** **)** for list L1. Press **[ENTER]** and the result **DONE** should appear.	 
5.	Press **[STAT]** and choose **1:Edit** on the **EDIT** menu to return to list L1 for the ordered data. The data will now be ordered.	

Let's consider the reverse situation when we have a frequency table with grouped data and determine information about the original data. This scenario is important because you will often see grouped data due to data storage capacities.

✔ Example 7

Answer the questions using the frequency table.

Class	Frequency
9 – 15	4
16 – 22	7
23 – 29	1
30 – 36	0
37 – 43	3
44 – 50	5

a. What is the total number of data values in this data distribution?

Adding the frequencies of each class, we have $4+7+1+0+3+5=20$.

b. What class width is used to group the data?

Subtract any two consecutive lower class limits or any two consecutive upper class limits. For example, $16-9=7$.

c. What is the class mark of the second class ?

The class mark is the midpoint of the class. Average the lower and upper class limit: $\frac{16+22}{2}=19$.

d. What is the modal class?

The class with the highest frequency is 16-22.

e. If an additional class were added to the end of the table, what would be the upper and lower class limits?

Add the class width 7 to the last lower and upper class limits to get 51-57.

Graphing Quantitative Data in Histograms and Frequency Polygons

A *histogram* is a statistical graph commonly used to visualize frequency distributions of quantitative data. A histogram is like a bar graph, but where the horizontal axis is a number line.

Histogram

A **histogram** is a graph with observed values or classes of values along the horizontal axis and frequencies along the vertical axis. A bar with a height equal to the frequency (or relative frequency) is built above each observed value or class.

In a histogram, classes may be identified by their class marks (midpoints of the classes) or by their class limits. The horizontal scale may or may not begin at 0, and but the vertical scale should always start at zero. The bars generally touch in a histogram - unless the frequency is 0 for a particular data value or class of values.

Let's illustrate how a histogram is constructed with the following example.

Example 8

Each member of a class is asked how many plastic beverage bottles they use and discard in a week. Suppose the following (hypothetical) data are collected.

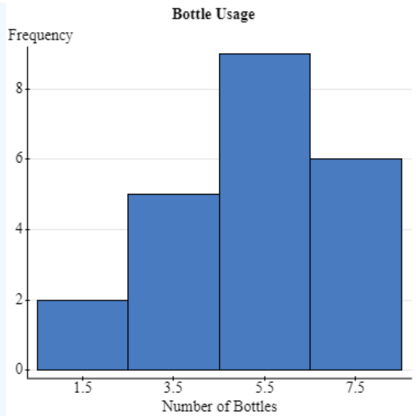
Hypothetical Class Data: Number of Water Bottles Used Per Week														
6	4	7	7	8	5	3	6	5	7	6	7	6	6	7
7	5	2	6	1	3	5	4	7	4	5	4	6	5	3

First, we organize the data by grouping it and presenting it in a frequency table. The classes have width 2 and begin at 1.

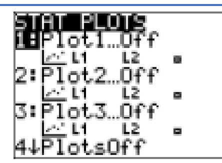
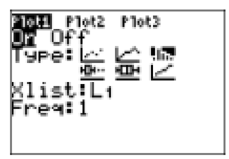
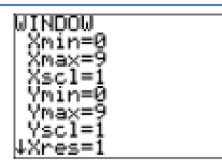
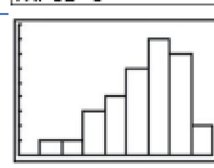
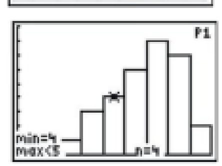
Frequency Table for Number of Bottles Used

Number of Bottles Used	Frequencies	Class Marks (Midpoints)
1-2	2	1.5
3-4	7	3.5
5-6	14	5.5
7-8	9	7.5

Next, we draw a bar for each class so that its height represents the frequency of students using those numbers of bottles. We label the midpoints of each bar with the class marks along the horizontal axis.



Graphing data can get tedious and complicated, especially if there are lots of data to organize. *Excel* and other software can easily make graphs. So can a TI graphing calculator. The steps to creating a histogram for these data is given below.

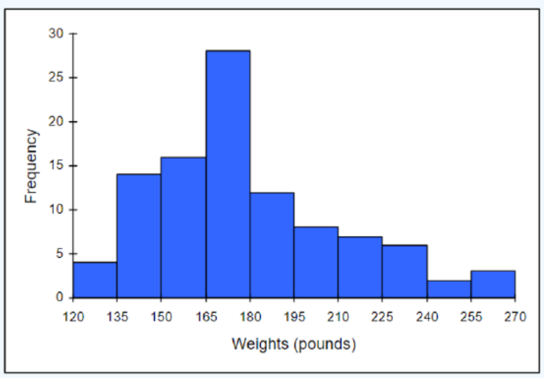
	Technology Note: Making a Histogram on the TI-83/84 Calculator	
1.	To draw a histogram, you must first enter the data into a list as we have done in the past. Press **[STAT]** and then choose **1:Edit** to input the data.	
2.	Press **[2ND] [Y=]** to enter the **STAT PLOTS** menu.	
3.	You can plot up to three statistical plots at one time. Choose **Plot 1**. Move the cursor to the word **On** and press **[ENTER]** to turn the plot on. Change the **Type** of plot to a histogram, and choose L1 for **Xlist.** Enter '1' for **Freq** by pressing **[2ND][A-LOCK]** to turn off alpha lock, which is normally on in this menu. If you have already made a frequency table, you enter the values of the variables in L1 and the frequencies in L2 as we did in Chapter 1 rather than typing in all the raw data.	
4.	We need to set a window. Press **[WINDOW]** and enter an appropriate window to display the plot. In this case, **Xscl** is what determines the bin width. Also notice that the **Xmax** value needs to extend right to 9 to show the last bin, even though the data values stop at 8. Enter all the values shown.	
5.	Press **[GRAPH]** to display the histogram. If you press **[TRACE]** and then use the left or right arrows to trace along the graph, notice how the calculator uses the notation to properly represent the frequency of values in each bin. **Note:** Instead of setting the WINDOW manually, you can let the calculator set its own window by pressing **[ZOOM]** and then choosing **9:ZoomStat**.	 

✓ Example 9

Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Interval	Frequency
120 – 134	4
135 – 149	14
150 – 164	16
165 – 179	28
180 – 194	12
195 – 209	8
210 – 224	7
225 – 239	6
240 – 254	2
255 – 269	3

A histogram of this data would look like

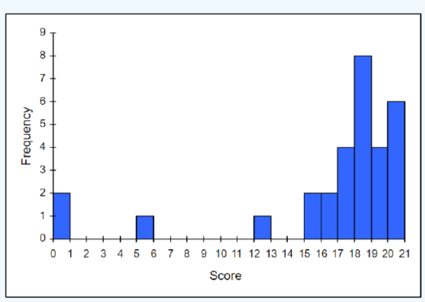


You can see the modal class is 165-179. You can also conclude there is a higher frequency of males in the lower part of the distribution of weights because the bars are taller there.

? Try it Now 4

Create a histogram for the data given in Example 5 using the frequency table of ungrouped data.

Answer



Another way to visualize frequency distribution data is to construct a *frequency polygon*.

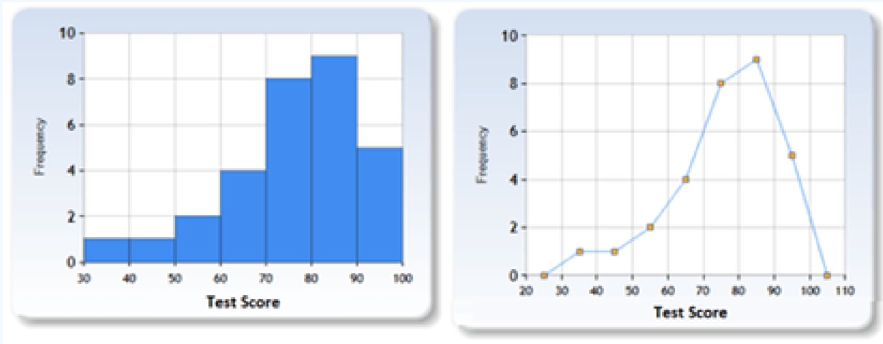
Frequency Polygon

An alternative representation of a histogram is a **frequency polygon**. A frequency polygon starts like a histogram, but instead of drawing a bar, a point is placed at the midpoint of each interval at a height equal to the frequency. Typically, the points are connected with straight lines to emphasize the shape of the data distribution.

The following example illustrates the relationship between a histogram and a frequency polygon for the same data.

Example 10

Ms. Winter made a histogram and frequency polygon of the science test scores from 5^{th} period.



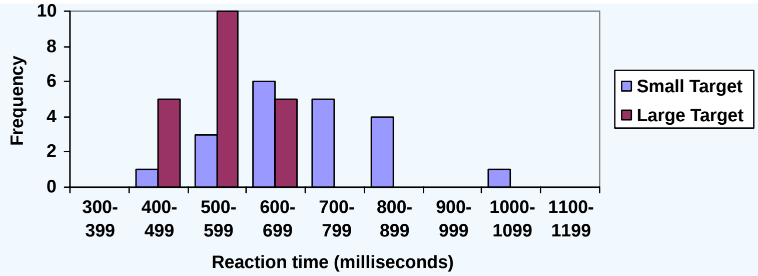
From either the histogram or the frequency polygon, we can see the class width is 10 points. We can also see that the modal class is 80-89. Finally, you can conclude that there is a larger frequency of students who scored high on the test than low on the test because the bars of the histogram and peak on the frequency polygon are taller on the right side of the horizontal axis.

Example 11

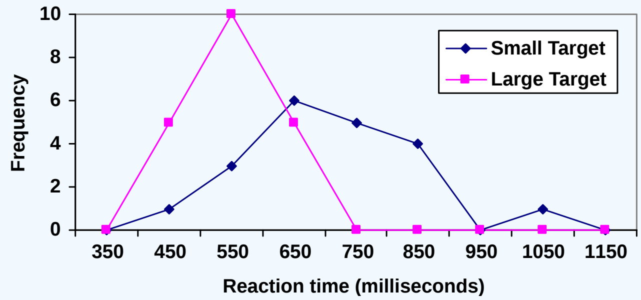
The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	Frequency small target	Frequency large target
300 – 399	0	0
400 – 499	1	5
500 – 599	3	10
600 – 699	6	5
700 – 799	5	0
800 – 899	4	0
900 – 999	0	0
1000 – 1099	1	0
1100 – 1199	0	0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.



A pair of frequency polygons in the same graph for the same two sets of data makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



In the next section, we will begin to analyze and describe data distributions numerically rather than graphically.

4.3: Measures of Central Tendency

In addition to graphical and verbal descriptions of data, we can use numbers to summarize quantitative data distributions. We want to know what a typical, average, or representative value for a set of data is (the center of the data), and how spread out the values are in the data set. In this section we explore measures of central tendency, and in the next section we will explore measures of spread.

Mean

We need to be careful with using the word "*average*" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. The mean is what most people think of when they use the word "average," but we should try to use statistical terms to be precise.

Mean

The **mean** of a set of data is found as the sum of the data values divided by the number of values.

Symbolically, the formula for the sample mean is

$$\bar{x}=\frac{\sum x_i}{n}=\frac{x_1+x_2+x_3+x_4+\ldots+x_n}{n}$$

where each x_i is the i^{th} data value and n is the sample size. $\sum x_i$ is a short way to write a bunch of x's added together.

Statisticians use the symbol $\bar{x}$ to represent the mean while x is the symbol for a single measurement. We say $\bar{x}$ as "*x bar*."

Example 1

Marci's exam scores for her last math class were 79, 86, 82, and 94. What is the mean of Marci's exam scores?

$$\frac{79+86+82+94}{4}=85.25$$

Typically, we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 2

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below. Find the mean number of TD passes.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Solution

Adding these values, we get 634 total TDs. Dividing by the number of data values, 31, we get $\frac{634}{31}\approx 20.4516$. It would be appropriate to round this to 20.5 TDs.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Try it Now 1

The price of a jar of peanut butter at 5 stores was \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the mean price.

Answer

\$3.68.

We can also find the mean when data has been organized into a frequency table.

✓ Example 3

A sample of 100 families in a particular neighborhood are asked their annual household income, to the nearest 5 thousand dollars. The results are summarized in a frequency table below. What is the mean annual household income for this neighborhood?

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

Calculating the mean by hand could get tricky if we try to actually add 100 values. We want to add all 100 values and divide by 100 such as

$$\bar{x} = \frac{15+\cdots+15+20+\cdots+20+25+\cdots+25+\cdots+50+50}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $15 \cdot 6 = 90$. Using this shortcut, we get

$$\bar{x} = \frac{15 \cdot 6+20 \cdot 8+25 \cdot 11+30 \cdot 17+35 \cdot 19+40 \cdot 20+45 \cdot 12+50 \cdot 7}{100} = \frac{3390}{100} = 33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

✓ Example 4

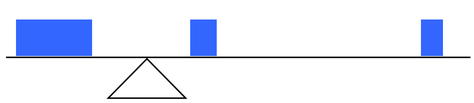
Continuing from the previous example, suppose a new family with a household income of 5 million dollars moves into the neighborhood. (This is 5,000 thousand dollars). Including this in the sample, the mean is now

Solution

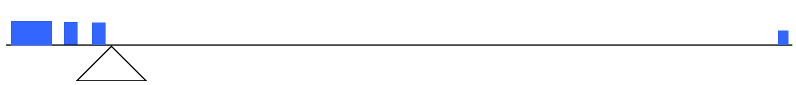
$$\bar{x} = \frac{15 \cdot 6+20 \cdot 8+25 \cdot 11+30 \cdot 17+35 \cdot 19+40 \cdot 20+45 \cdot 12+50 \cdot 7+5000 \cdot 1}{101} = \frac{8390}{101} \approx 83.069$$

While 83.1 thousand dollars ($83,100) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



In the graph of the household income data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance.



For this reason, when working with data sets that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center -- the *median*.

Median

Most of us are familiar with the median of a roadway. It is usually an area of grass or concrete that separates two halves of the roadway. The median is defined similarly in statistics.

Median

The **median** is the value found in the middle of an *ordered* data set.

There is no symbol or formula for median. To find the median, order (or rank) the data values from smallest to largest and then count from both ends inward towards the center one data value at a time until reaching the middle:

- If there are an odd number of data values, then there is one middle data value and that is the median.
- If there are an even number of data values, then there are two middle data values. The median is the mean of those two data values.

Example 5

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7 6

Solution

We must start by listing the data in order: 2 4 5 5 6 6 7 7 8 8 10

It is helpful to mark or cross off the numbers in the original data set as you list them to make sure you don't miss any. Also, be sure to count the number of data values in the ordered list to make sure it matches the number of data values in the original list.

In this example there are $n = 11$ quiz scores. When the distribution contains an odd number of data values, there will be a single value in the middle and that value is the median. For small data sets, we can "walk" one value at a time from the ends of the ordered list towards the center to find the median

$$\underbrace{2\ 4\ 5\ 5\ 6}_{\text{lower half}} \quad \underbrace{6}_{\text{median}} \quad \underbrace{7\ 7\ 8\ 8\ 10}_{\text{upper half}}$$

The median test score is 6 points.

Example 6

Suppose another quiz score needs to be included in the set of quiz scores in the previous example. Someone in the class got a perfect score of 20 points on this very difficult quiz.

Solution

The ordered list of data is now: 2 4 5 5 6 6 7 7 8 8 10 20

There are now $n = 12$ quiz scores in our sample. When the distribution contains an even number of data values, there will be a pair of values in the middle rather than a single value. We find the mean of those middle two values.

$$\underbrace{2\ 4\ 5\ 5\ 6}_{\text{lower half}} \quad \underbrace{6\ 7}_{\text{middle pair}} \quad \underbrace{7\ 8\ 8\ 10\ 20}_{\text{upper half}}$$

The median test score is $\dfrac{6+7}{2} = 6.5$ points.

It is important to notice that despite adding an outlier quiz score to the data set, the median is largely unaffected. The median quiz score for the new distribution is 6.5 points when it was 6 points before.

Example 7

The students in a math class were asked to report the number of children that live in their house (including brothers and sisters temporarily away at college). The data are recorded below. Find the mean and median for these data.

1 3 4 3 1 2 2 2 1 2 2 3 4 5 1 2 3 2 1 2 3 6

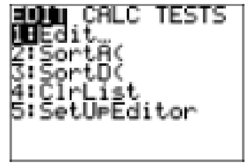
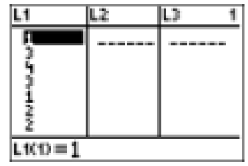
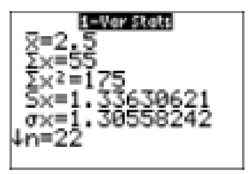
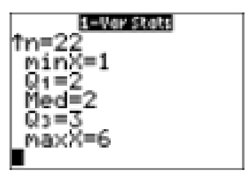
Solution

To find the mean, first determine that the sum of all data values is $\sum x = 55$. Then, divide the sum by the number of data values in the list, $n = 22$. The mean is $\overline{x} = \frac{55}{22} = 2.5$ children.

To find the median, begin by ordering the data: 1 1 1 1 1 2 2 2 2 2 2 2 2 3 3 3 3 3 4 4 5 6

Because there is an even number of data values ($n = 22$), the median will be between the two middle values located at position #11 and position #12. The median is $\frac{2+2}{2} = 2$ children.

For larger data sets, technology will be useful. The procedure for finding the mean and median using a TI calculator is shown.

	Technology Note: Finding the Mean and Median on the TI-83/84 Calculator	
1.	Press **[STAT]** to open the Statistics menu. Choose **1:Edit** on the **EDIT** menu. Press **[ENTER]**.	
2.	You now have a screen that allows you to enter the data in a column. Enter the number of children data in L1. Be sure to press **[ENTER]** after each value.	
3.	Again, press **[STAT]** and use the right arrow to access the **CALC** menu.	
4.	Choose **1:1-Var Stats** on the CALC menu. You will have one of the two screens shown: If you have a screen like the one on the top, press **[2ND] [1]** for the list L1 and press **[ENTER]**. If you have a STAT WIZARD screen that looks like the one on the bottom, enter the name of the list by pressing **[2ND] [1]**. Leave **FreqList** blank. Highlight **Calculate** and press **[ENTER]**.	
5.	The calculator will display several values. We will learn about all these statistics later. The first screen shows the mean $\bar{x}$, the sum of the data values Σx, and the number of data values n. If you use the down arrow, you will also see the median (Med).	

The Mean vs. the Median

Both the mean and the median are important and widely used measures of center for quantitative data. Is one better to use than the other? It depends -- as the following example shows.

Example 8

Suppose you got an 85 and a 93 on your first two quizzes, but then you had a really bad day and got a 14 on your next quiz!

- The mean of your three grades is $\frac{85+93+14}{3}=\frac{192}{3}=\mathbf{64}$
- The median of your three grades is 85 because 85 is in the middle once the data are ordered: 14 **85** 93.

Which result do you think gives a more accurate picture of your performance?

Solution

The median does not change if the lowest grade is an 84 or if the lowest grade is a 14. However, when you add the three numbers to find the mean, the sum will be much smaller if the lowest grade is a 14. You would probably argue that the median is more representative of your performance because you had done well except for one bad day.

The mean and the median are so different in this example because there is one grade that is extremely different from the rest of the data. In statistics, we call such an extreme value an **outlier**. The mean is affected by the presence of an outlier; however, the median usually is not affected as much or even at all. A statistic that is not significantly affected by outliers is called **resistant.**

The median is a resistant measure of center, and the mean is a **non-resistant** measure of center. In a sense, the median can resist the pull of a far away value, but the mean is drawn to such a value. It cannot resist the influence of outlier values. When we have a data set that contains an outlier, it is often better to use the median to describe the center rather than the mean.

Example 9

In 2005, the CEO of Yahoo, Terry Semel, was paid almost $231,000,000. This is certainly not typical of what the average worker at Yahoo could expect to make. Instead of using the mean salary to describe how Yahoo pays its employees, it would be more appropriate to use the median salary of all the employees. The CEO's salary will have a big impact on the mean and inflate it to the point where it might no longer be representative.

You will often see medians used to describe the typical salary or to represent the value of houses in a neighborhood as the presence of a very few extremely well-paid employees or expensive homes could make the mean appear misleadingly large.

Try it Now 2

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F.)

Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

Find the mean and median high temperature.

Answer

The mean is 62.7°F and the median is 63°F.

The next two examples use the concepts of the mean and then median to solve problems.

Example 10

To qualify for a tournament, Mara must have at least a 180 average bowling score in her most recent 5 games. So far, Mara has bowled 4 games with scores of 170, 184, 160, and 195. What score must Mara get in her 5^{th} game to have a 180 average?

Solution

We are told that $n = 5$ and $\bar{x} = 180$. We also know four of the individual data values but not the fifth one: $x_1 = 170$, $x_2 = 184$, $x_3 = 160$, $x_4 = 195$, and $x_5 = ?$.

$$\begin{aligned} \bar{x} &= \frac{\sum x}{n} \\ \sum x &= n \cdot \bar{x} \\ 170 + 184 + 160 + 195 + x_5 &= 5 \cdot 180 \\ 709 + x_5 &= 900 \\ x_5 &= 191 \end{aligned}$$

Mara needs to score 191 to bring her bowling average to 180.

Example 11

Design a set of data to represent nine 10-point quiz scores where the mean is 7 and the median is 8.

Solution

The sum of the nine quiz scores must be $9 \times 7 = 63$ points. There are an odd number of data values and the median is 8. The middle value must be 8.

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad \mathbf{8} \quad x_6 \quad x_7 \quad x_8 \quad x_9$$

The sum of the missing data values must be $63 - 8 = 55$, the x-values to the left of 8 must be 8 or less, and the x-values to the right of 8 must be 8 or more (but 10 or less.) There are many possibilities. Here is one solution: 2, 5, 6, 7, 8, 8, 8, 9, 10.

Mode

In addition to the mean and the median, there is another common measurement of the "typical" value of a data set -- the *mode*. The mode is fairly useless with quantitative data like weights or heights where there are a large number of possible values. The mode is more commonly used with qualitative data, for which median and mean cannot be computed.

Mode

The **mode** is the value in the data set that occurs most often. It is possible for a data set to have more than one mode if several categories or values have the same frequency. It is also possible for there to be no mode if every category occurs only once.

Example 12

In the vehicle color survey, we collected the data

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

For this data, "green" is the mode color because it is the data value that occurred most frequently.

? Try it Now 3

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F.)

Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

Find the mode high temperature.

Answer

57^{o}F

? Try it Now 4

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating
b. The median rating
c. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Answer

a. 2.5
b. 2
c. 2

Midrange

The last measure of central tendency we will consider in this text is the *midrange*. The midrange is halfway between the two extreme values in a data set.

Midrange

Midrange is the mean of the lowest value and the highest values in the sample data.

✓ Example 13

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F.)

Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

Find the midrange high temperature.

Solution

Once sorted, the smallest data value is 57ºF and the largest data value is 71ºF. The midrange is the mean of these two extreme values:

$$\frac{57+71}{2}=64\,^{\mathrm{o}}\mathrm{F}.$$

4.4: Measures of Spread and Position

Consider these three sets of student quiz scores on a 10-point quiz:

- **Class A:** 5, 5, 5, 5, 5, 5, 5, 5, 5, 5
- **Class B:** 0, 0, 0, 0, 0, 10, 10, 10, 10, 10
- **Class C:** 4, 4, 4, 5, 5, 5, 5, 6, 6, 6

All these data sets have mean $\overline{x} = 5$ and median of 5, yet the three sets of scores are clearly quite different.

In Class A, everyone had the same score. In Class B, half the class got no points and the other half got a perfect score of 10 points. Scores in Class C were not as consistent as those in Class A but also not as widely varied as those in Class B.

This scenario shows that, in addition to the mean and median which measure the "typical" value of a data set, we also need a way to measure how "spread out" or varied each data set is. There are several ways to measure the variation and locate positions in a data distribution. In this section we explore *range*, *standard deviation*, *percentiles*, *quartiles*, and the *interquartile range* (IQR). We also examine a graphical representation of spread using a *box plot*.

Range

The first and simplest way to measure spread is the **range**. Calculation of the range uses only two values from the data set - the largest value and the smallest value. The range is the distance between these two values.

Range

The **range** is the difference between the maximum value and the minimum value of the data set.

Example 1

Refer to the three sets of student quiz scores from the introduction to this section.

- For Class A, the range is 0 since both the maximum and minimum are the same: $5 - 5 = 0$.
- For Class B, the range is 10 since $10 - 0 = 10$.
- For Class C, the range is 2 since $6 - 4 = 2$.

In this example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth set of quiz scores:

Class D: 0, 5, 5, 5, 5, 5, 5, 5, 5, 10.

Quiz scores from this class also have a mean and median of 5. The range is 10 like Class B, yet this data set is quite different than Class B. To more accurately measure the difference in spreads between these two sets of data, we'll have to turn to more sophisticated measures of variation.

Example 2

Find the range for each data set.

Set A: 10, 20, 30, 40, 50 **Set B:** 10, 35, 36, 37, 50

Solution

For both sets of data, the range is $50 - 10 = 40$. However, most of the data in Set B is closer together, except for the extremes. There seems to be less variability in the data in Set B than in the data in Set A. The range focuses only only the two extreme values yet ignores all the data between the extremes. So, we need a better way to quantify the spread.

Standard Deviation

We saw that the range focuses on the difference between the maximum and minimum values. What if we focused on the differences between each of the data values and and the center? The center we will use is the mean. The difference between a data value x and the mean of the distribution $\overline{x}$ is called a *deviation*.

> **Deviation**
>
> The difference between a data value x and the mean of the data distribution is called the **deviation from the mean.**
>
> $$\text{deviation from the mean } = x - \overline{x}$$

To see how deviations work, let's return to the temperature data set from the previous section.

Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

We computed the mean earlier and it was $\overline{x} = 62.7^o F$. We will create a table showing each of the 11 data values in the first column and the deviation from the mean for each data value in the second column.

x	$x - \overline{x}$
71	$71 - 62.7 = 8.3$
59	$59 - 62.7 = -3.7$
69	$69 - 62.7 = 6.3$
68	$68 - 62.7 = 5.3$
63	$63 - 62.7 = 0.3$
57	$57 - 62.7 = -5.7$
57	$57 - 62.7 = -5.7$
57	$57 - 62.7 = -5.7$
57	$57 - 62.7 = -5.7$
65	$65 - 62.7 = 2.3$
67	$67 - 62.7 = 4.3$
sum	0.3

Notice that some of the deviations and positive and some of them are negative. The sum of the deviations is around zero. If there had been no rounding of the mean, then the sum of the deviations would have been exactly 0.

So what does that tell us? Does this imply that on average the data values are a distance of zero units from the mean? No. It just means that some of the data values are above the mean and some are below the mean. The negative deviations are for data values that are below the mean and the positive deviations are for data values that are above the mean. The positive and negative deviations from the mean cancel each other out.

We need to eliminate the signs of the deviations so we can measure the distance from the mean. How do we get rid of a negative sign? Squaring a number is a widely accepted way to make all of the numbers positive. We continue building the table by adding a third column that contains the squares of the deviations from the mean.

x	$x-\overline{x}$	$(x-\overline{x})^2$
71	$71-62.7=8.3$	$(8.3)^2=68.89$
59	$59-62.7=-3.7$	$(-3.7)^2=13.69$
69	$69-62.7=6.3$	$(6.3)^2=39.69$
68	$68-62.7=5.3$	$(5.3)^2=28.09$
63	$63-62.7=0.3$	$(0.3)^2=0.09$
57	$57-62.7=-5.7$	$(-5.7)^2=32.49$
57	$57-62.7=-5.7$	$(-5.7)^2=32.49$
57	$57-62.7=-5.7$	$(-5.7)^2=32.49$
57	$57-62.7=-5.7$	$(-5.7)^2=32.49$
65	$65-62.7=2.3$	$(2.3)^2=5.29$
67	$67-62.7=4.3$	$(4.3)^2=18.49$
sum	0.3	304.19

Now that we have the sum of the squared deviations, we should find the mean of these values. However, since this is a sample, the normal way to find the mean (summing and dividing by n) does not estimate the true population spread correctly. It would underestimate the true value. So, to calculate a better estimate, we will divide by a slightly smaller number, $n-1$. This strange average is known as the *sample variance*. The **sample variance** is the sum of the squared deviations from the mean divided by $n-1$. The symbol for sample variance is s^2 and the formula for the sample variance is

$$s^2=\frac{\sum(x-\overline{x})^2}{n-1}.$$

For this data set, the sample variance is

$$s^2=\frac{304.19}{11-1}=\frac{304.19}{10}=30.419.$$

The variance measures the average squared distance from the mean. Since we want to know the average distance from the mean, we will need to take the square root at this point and the result will be the *sample standard deviation*. The **sample standard deviation** is the square root of the variance and measures the average distance the data values are from the mean. The symbol for sample standard deviation is s and the formula for the sample standard deviation is

$$s=\sqrt{s^2}=\sqrt{\frac{\sum(x-\overline{x})^2}{n-1}}.$$

Thus, for this data set, the sample standard deviation is

$$s=\sqrt{30.419}\approx 5.52^{\circ}F.$$

Note: The units are the same as the original data.

Sample Standard Deviation

The standard deviation is a measure of spread based on how far each data value deviates from the mean.

$$s=\sqrt{\frac{\sum(x-\overline{x})^2}{n-1}}$$

To compute the sample standard deviation by hand,

1. Find the deviation of each data value from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by one fewer than the number of data values, $n-1$. This value is the variance.
5. Take the square root of the result.

Example 3

A random sample of unemployment rates for 10 counties in the EU for March 2013 is given below.

Unemployment Rates for EU Countries

11.0%	7.2%	13.1%	26.7%	5.7%	9.9%	11.5%	8.1%	4.7%	14.5%

(Eurostat, n.d.)

Find the range, variance, and standard deviation.

Solution

The maximum is 26.7% and the minimum is 4.7% so the range is $26.7\% - 4.7\% = 22.0\%$.

To find the variance and the standard deviation, it is easier to use a table than the formula. The table helps us find all the calculations needed in the formula.

The mean is $\overline{x} = 11.24\%$.

x	$x-\overline{x}$	$(x-\overline{x})^2$
11.0	$11.0-11.24=-0.24$	$(-0.24)^2=0.0576$
7.2	$7.2-11.24=-4.04$	$(-4.04)^2=16.3216$
13.1	$13.1-11.24=1.86$	$(1.86)^2=3.4596$
26.7	$26.7-11.24=15.46$	$(15.46)^2=239.0116$
5.7	$5.7-11.24=-5.54$	$(-5.54)^2=30.6916$
9.9	$9.9-11.24=-1.34$	$(-1.34)^2=1.7956$
11.5	$11.5-11.24=0.26$	$(0.26)^2=0.0676$
8.1	$8.1-11.24=-3.14$	$(-3.14)^2=9.8596$
4.7	$4.7-11.24=-6.54$	$(-6.54)^2=42.7716$
14.5	$14.5-11.24=3.26$	$(3.26)^2=10.6276$
sum	0	354.664

Apply the formula for the sample variance:

$$s^2 = \frac{354.664}{10-1} = \frac{354.664}{9} \approx 39.40711111$$

Take the square root of the sample variance to find the sample standard deviation:

$$s = \sqrt{39.4071111} \approx 6.28\%$$

So, the typical unemployment rate for countries in the EU is approximately 11.24% with an average spread of about 6.28%. Since the sample standard deviation is fairly high compared to the mean, then there is a great deal of variability in unemployment rates for countries in the EU. This implies that some countries in the EU have rates that are much lower than the mean and some that have rates much higher than the mean.

There are a few important characteristics of the standard deviation that we should note. Standard deviation is always zero or positive. Standard deviation will be zero if all the data values are equal and the standard deviation will get larger as the data spreads out. Standard deviation, like the mean, can be highly influenced by outliers.

Try it Now 1

According to the U.S. Department of Agriculture, ten to twenty earthworms per cubic foot is a sign of healthy soil. Mr. Green checked the soil in his garden by digging 7 one-cubic-foot holes and counting the earthworms. Here is the number of earthworms he found in each hole. What are the mean and standard deviation?

Number of Worms

4	24	15	10	8	12	18

Answer

$\overline{x} = 13$ worms; $s \approx 6.66$ worms

The computation of the sample variance and standard deviation is not complicated, but it is tedious and time consuming. Later in this section there are instructions for using a TI calculator to find various measures of variation and position.

Percentiles

There are other calculations that we can do to look at spread and position of data within a data set. One of those is called a *percentile*. The percentile is a value in the data set which has a certain percent of the data less than or equal to its value.

Percentile

The k^{th} percentile is a value of the data set where $k\%$ of the data set is less than or equal to that data value.

For example, if a data value is at the 80$^{\text{th}}$ percentile, then 80% of the data values fall at or below this value (and 20% of the data values fall above this value.)

We see percentiles in many places in our lives. If you take any standardized test, your score is usually given as a percentile. If you take your child to the doctor, their height and weight are given as percentiles so they can be compared to other children their age. If your child is tested for gifted or behavior problems, the score is given as a percentile. If your child has a score on a gifted test that is at the 92$^{\text{nd}}$ percentile, then that means 92% of all of the children who took the same gifted test scored the same or lower than your child. Of course, that also means that 8% scored higher than your child. This may mean that your child is gifted.

A percentile is a measure that helps you determine where a data value is located relative to the other data values. For example, a test grade reported as a percentile does not tell you whether you did well or poorly. It does not tell you whether you passed or failed. It only tells you how well you did relative to the rest of the students who took the same test. For this reason, we often refer to a percentile as a measure of position.

Example 4

Suppose you took your biology final exam and received your score as a percentile.

a. What does a score at the 90$^{\text{th}}$ percentile mean?

   90% of the scores were at or below your score. You did the same as or better than 90% of the test takers. Only 10% scored higher than you.

b. What does a score at the 30$^{\text{th}}$ percentile mean?

   30% of the scores were at or below your score. You did the same or better than 30% of the test takers, and 70% scored higher than you.

c. If the test was out of 200 points and you scored at the 80$^{\text{th}}$ percentile, what was your score on the test?

   You do not know! All you know is that you scored the same or better than 80% of the students who took the test. If all the scores were really low, you could have still failed the test. On the other hand, if many of the scores were high you could have gotten a very good score on the test.

d. If your score was at the 95$^{\text{th}}$ percentile, does that mean you passed the test?

   No, it just means you did the same or better than 95% of the other students who took the test. You could have failed the test, but still did the same as or better than 95% of the rest of the people. It just means there were many others who also failed.

Five Number Summary

Three very common percentiles are the *first*, *second*, and *third quartiles*. **Quartiles** are locations in the data set that split the data distribution into quarters, or sections that each contain 25% of the data values.

Quartiles

Quartiles are values that divide the data in quarters.

- The **first quartile** (Q_1) is the value so that 25% of the data values are at or below this value. This is also known as the 25$^{\text{th}}$ percentile.
- The **second quartile** (Q_2) is the value so that 50% of the data values are at or below this value. This is also known as the 50$^{\text{th}}$ percentile, but more commonly called the median.
- The **third quartile** (Q_3) is the value so that 75% of the data values are at or below this value. This is also known as the 75$^{\text{th}}$ percentile.

To find the quartiles,

1. Order the data from smallest to largest.
2. Find the median. This is the second quartile, Q_2.
3. Find the median of the lower half of the data values (all values to the left of the median's location.) This is the first quartile, Q_1.
4. Find the median of the upper half of the data values (all values to the right of the median's location.) This is the third quartile, Q_3.

Like the standard deviation, the quartiles are used to measure how spread out the data are, but unlike the standard deviation the quartiles are not a single-number summary of spread. The three quartiles, together with the maximum and minimum values, create a measure of spread called the *five-number summary*.

Five Number Summary and IQR

The **five number summary** takes the form: Minimum, Q_1, Median, Q_3, Maximum.

> These five values divide the data into quarters: 25% of the data is between the minimum and Q_1, 25% is between Q_1 and the median, 25% is between the median and Q_3, and 25% is between Q_3 and the maximum value.
>
> Moreover, 50% of the data lies between Q_1 and Q_3. The distance between Q_1 and Q_3 is called the interquartile range.

The **interquartile range (IQR)** measures the spread in the middle 50% of the data. Subtract Q_1 from Q_3 to find its value.

$$IQR = Q_3 - Q_1$$

Examples should help make this clearer.

Example 5

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F) shown below. Find the five-number summary and IQR.

Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

(Weather Underground, n.d.)

Solution

To find the five-number summary, you must first order the data from smallest to largest:

57 57 57 57 59 63 65 67 68 69 71

Then find the median. There are $n=11$ data values so the median will be a single value in the middle. Here, 63°F is located at the middle of the data set. To find Q_1 and Q_3, look at the numbers in each half on each side of the median. Since 63 is the median, it is not included in either half.

$$\underbrace{57\ 57\ 57\ 57\ 59}_{\text{numbers below median}}\quad \underbrace{63}_{\text{median}}\quad \underbrace{65\ 67\ 68\ 69\ 71}_{\text{numbers above median}}$$

- There are 5 numbers below the median: {57 57 **57** 57 59}. The median of these numbers is 57. So, $Q_1 = 57^\circ F$.
- There are 5 numbers above the median: {65 67 **68** 69 71}. The median of these numbers is 68. So, $Q_3 = 68^\circ F$.
- The minimum is 57°F and the maximum is 71°F.

Thus, the five-number summary is Min = 57°F, Q_1 = 57°F, Med = 63°F, Q_3 = 68°F, Max = 71°F. The $IQR = Q_3 - Q_1 = 68 - 57 = 11^\circ F$.

Example 6

The scores for a women's golf team in tournament play are listed below. Find the five-number summary and the IQR.

89 90 87 95 86 81 111 108 83 88 91 79

Solution

First, order the $n=12$ data values from smallest to largest. The median will be the mean of the two middle values since there are an even number of data values.

$$\underbrace{79\ 81\ 83\ 86\ 87\ 88}_{\text{numbers below median}}\quad \underbrace{\quad}_{\text{median}}\quad \underbrace{89\ 90\ 91\ 95\ 108\ 111}_{\text{numbers above median}}$$

- The median is $\frac{88+89}{2} = 88.5$.
- There are 6 numbers below the median: {79 81 **83 86** 87 88}. The median of these six numbers is $\frac{83+86}{2} = 84.5$.
- There are 6 numbers above the median: {89 90 **91 95** 108 111}. The median of these six numbers is $\frac{91+95}{2} = 93$.
- The minimum is 79 and the maximum is 111.

Thus, the five-number summary is Min = 79, Q_1 = 84.5, Med = 88.5, Q_3 = 93, Max = 111. The $IQR = Q_3 - Q_1 = 93 - 84.5 = 8.5$.

Try it Now 2

Firman's Fitness Factory is a new gym that offers reasonably priced family packages. The following shows the number of family packages sold during the opening month. Find the five-number summary and the IQR. (The data have been ordered for your convenience.)

20 21 22 22 23 24 24 24 26 27

27 27 27 28 28 28 28 29 30 30

31 32 32 32 32 32 32 34 34 35

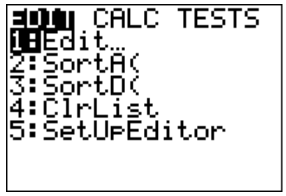
Answer

Minimum = 20, Q_1 = 24, Median = 28, Q_3 = 32, Maximum = 35, IQR = 8.

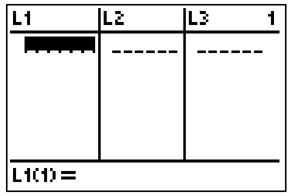
Finding Descriptive Statistics Using the TI Calculator

We have already used the TI calculator to find the mean and the median in the previous section. Now, we expand the previous explanation to measures of spread and position. The procedures for finding the descriptive statistics for the Flagstaff, AZ temperature data used in examples throughout this section are shown below.

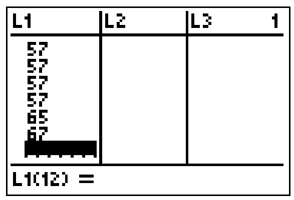
First, enter the data into the calculator. To do this, press **STAT**. The **STAT** button is in the third row of buttons, next to the arrow keys. Once you press **STAT**, you will see the following screen:



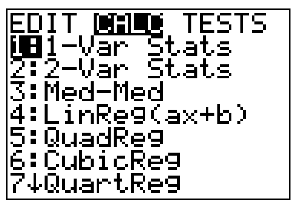
Choose **1:Edit…** and you will see the following screen. If there is already data in List 1 (L1), then you should move the cursor up to L1 by using the arrow keys. Then, press **CLEAR** and **ENTER**. This should clear all data from List 1 (L1).



Now type all of the data into List 1 (L1). Be sure to press **ENTER** after each value. You can only see the last six data values entered on the screen, but all the data has been entered.

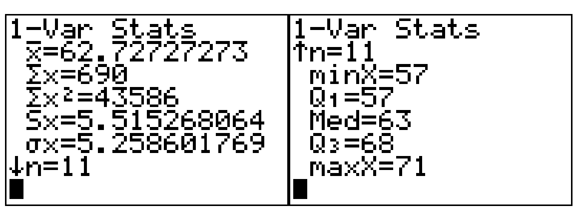


Next, press **STAT** again and move over to **CALC** using the right arrow button. You will see the following screen:



Choose **1:1-Var Stats**. This will put 1-Var Stats on your home screen. Type the name of the list containing the data L1 (2nd 1), and the calculator will show the following:

At this point press **ENTER**, and you will see the results. You will need to use the down arrow button to see all of the results.



Therefore, the mean is $\overline{x} \approx 62.7^{\circ} F$, the standard deviation is $s \approx 5.515^{\circ} F$, and the five-number summary is Min = 57°F, Q_1 = 57°F, Med = Q_2 = 63°F, Q_3 = 68°F, Max = 71°F.

You can find the range by subtracting the max and min. You can find IQR by subtracting Q_3 and Q_1, and you can find the variance by squaring the standard deviation. You cannot find the mode from the calculator.

The calculator also gives the population standard deviation $\sigma \approx 5.259^{\circ} F$. Notice it is different than the value for s, since they are calculated differently. The value the calculator gives you for the population standard deviation is not the actual true value. The calculator gives you both values because it does not know if you typed in data from a sample or a population. You can ignore the population standard deviation σ in almost all cases.

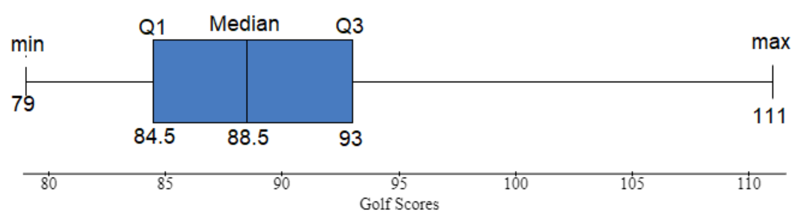
Box-and-Whiskers Plots

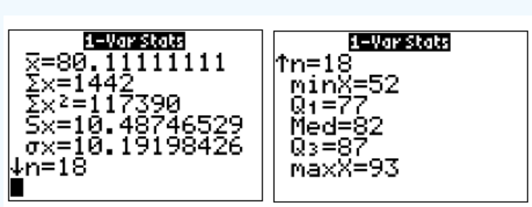
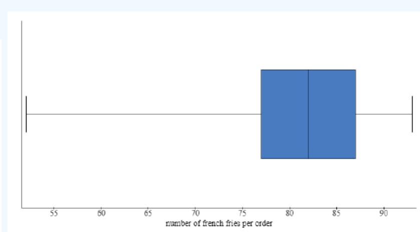
There are times when we want to look at the five-number summary as a graphical representation. This is known as a *box-and-whiskers plot*, or merely just a *box plot*.

> **Box plot**
>
> A **box plot** is a graphical representation of the five-number summary.

A box plot is created by first setting a scale (number line) as a guideline for the box plot. Then, draw a rectangle that spans from Q_1 to Q_3 above the number line. Mark the median with a vertical line through the rectangle. Next, draw symbols (dots, small vertical lines, etc.) for the minimum and maximum points to the sides of the rectangle. Finally, draw horizontal lines from the sides of the rectangle out to the symbols. These horizontal lines are known as "whiskers."

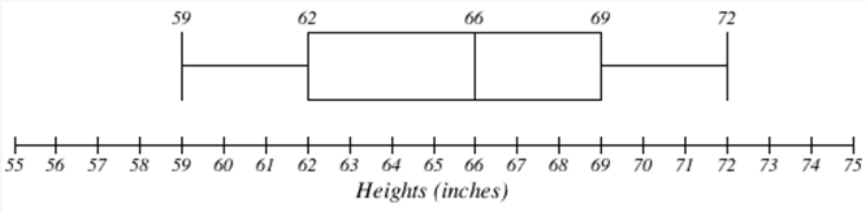
Using the results of the golf scores tournament from Example 6, the box plot has been constructed below:



> **Try it Now 3**
>
> A chain restaurant advertises that a typical number of French fries in a large order is 82. Roberta is a bit curious about this claim, so she bought a large order of fries each day for 18 days and counted the number of fries in the orders. Her data are shown below.
>
> 80 72 77 80 90 85 93 52 84 87 80 86 92 88 67 86 66 77
>
> Use a TI calculator to find the mean, median, Q_1, Q_3, IQR, standard deviation, minimum, and maximum for the data in her sample. Then, sketch a box plot.
>
> **Answer**
>
> 
>
>
> 
>
> The mean is 80.1, the median is 82, Q_1 is 77, Q_3 is 87, IQR is 10, standard deviation is about 10.5, minimum is 52, and maximum is 93.

Example 7

The box plot represents the heights of a group of several females.



a. What is the median height for the females in this group?

The median is located at the vertical line inside the box of the box plot. Here, the median is 66 inches.

b. What is the interquartile range of the heights for the females in this group?

$IQR = Q_3 - Q_1 = 69 - 62 = 7$ inches.

c. What percent of the females in this group are 62 inches or shorter?

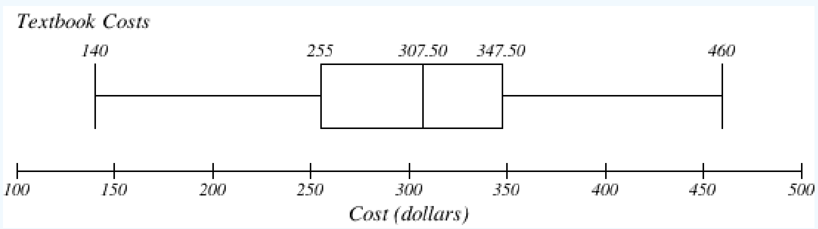
62 inches corresponds to Q_1. This value is also the 25th percentile, so 25% of the females are 62 inches or shorter.

d. How tall are the tallest 25% of females in this group?

The tallest 25% of females would be taller than the value of Q_3 = 69 inches.

Try It Now 4

The box plot shows the total cost of textbooks for the fall semester for a sample of PGCC students.



a. What is the most any student spent on textbooks for the semester?
b. What percent of students spent between $255 and $347.50 on textbooks for the semester?
c. What percent of students spent $347.50 or less on textbooks for the semester?

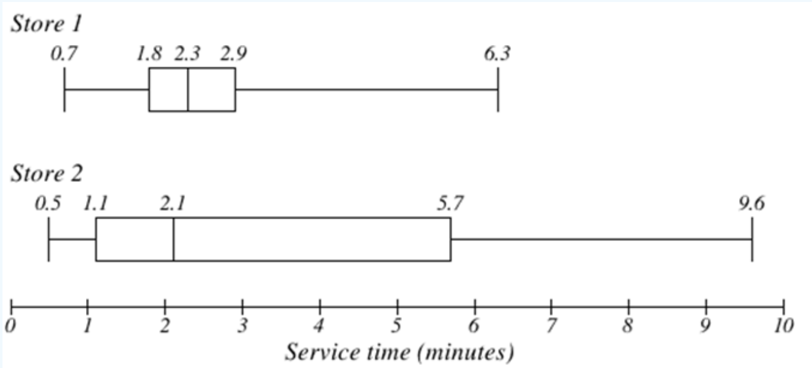
Answer

a. $460
b. 50%
c. 75%

Box plots are particularly useful for comparing data from two samples.

Example 8

The box plot of service times for two fast-food restaurants is shown below.



While Store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), Store 2 is less consistent in the amount of time needed to provide service. That is, Store 2 has a wider spread in the data.

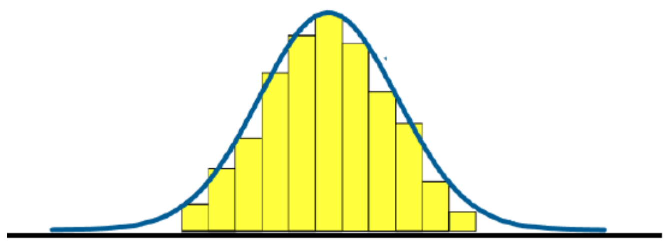
At Store 1, 75% of customers were served within 2.9 minutes, while at Store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck and probability – 25% of customers at Store 2 had to wait between 5.7 and 9.6 minutes, while all of the customers at Store 1 had been served within 6.3 minutes.

4.5: The Normal Distribution

There are many different shapes of distributions of quantitative data. In Section 4.2, we examined how a data set was distributed by displaying it in a histogram and frequency polygon. Sometimes the data may have been distributed symmetrically with the highest frequency in the center of the graph while in other instances there appeared to be a higher frequency of data on the left side or right side of the graph.

Look at the histogram below where there is a high frequency of data in its middle, and then the frequency goes down quickly and at the same rate as you move away from the middle towards both ends of the histogram. If we "smoothed out" the bars, what's left looks like a bell. This smoothed curve is sometimes called a *bell curve*.



A wide variety of quantities in the world are distributed in this special way. Given a big enough sample size, a histogram of the sample data would show that the frequency distribution of shoe sizes, IQ scores, life spans of batteries, amounts of coffee distributed by an automatic coffee dispenser, and waiting times for customer service calls, to name a few, all resemble this special distribution.

Moreover, these examples show a typical pattern that seems to be a part of many real-life phenomena. In statistics, because this pattern is so pervasive, it seems fit to call this pattern *normal*, or more formally, the **normal distribution**. The normal distribution is an extremely important concept because it occurs so often in data we collect from the natural world, as well as in many of the more theoretical ideas that are the foundation of statistics. Because so many real data sets closely approximate a normal distribution, we can use the idealized normal curve as a model to learn a great deal about such data. In practical data collection, the distribution will never be exactly normal. A true normal distribution only results from an infinite collection of data.

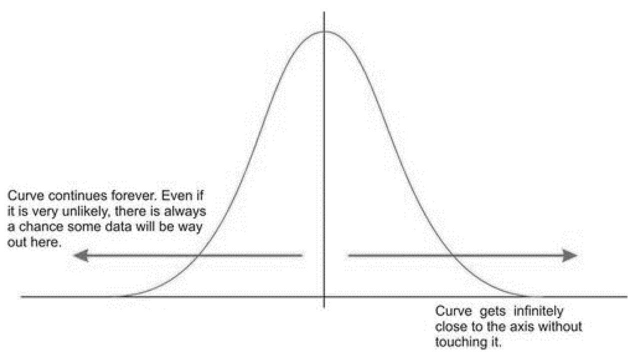
This section explores the basic properties of the normal distribution.

Characteristics of the Normal Distribution

There are several important characteristics that are true about any data that follows a normal distribution.

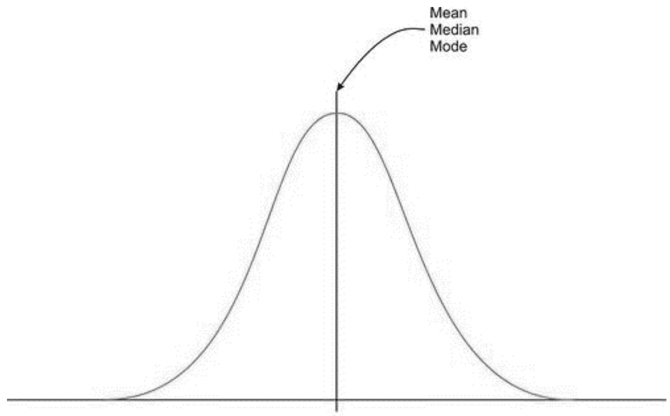
Shape

When the data from each of the situations described in the introduction are graphed, the distributions would be mound-shaped and symmetric. A *normal model* is an idealized, perfectly symmetric, unimodal distribution. In its perfect form, a normal curve extends forever leftwards to $-\infty$ and forever rightwards to $+\infty$. The normal curve is *asymptotic* to the x-axis. That is, the curve gets shorter and shorter as you move left or right along the x-axis, but the curve never touches the axis. This indicates that the frequency of observing a data value decreases to almost, but not quite, 0 as you move away from the center of the curve.



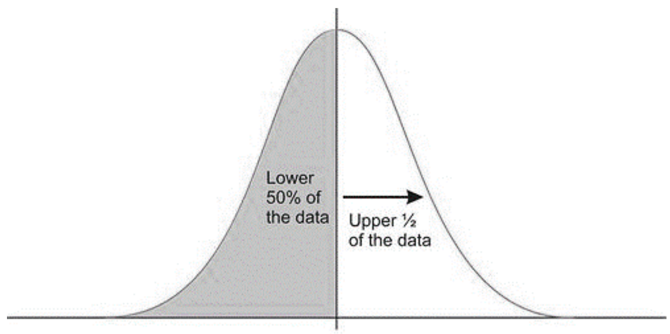
Center

Due to the exact symmetry of a normal curve, the center lies directly below the highest point of the distribution, and all the statistical measures of center we have already studied (the mean, median, and mode) are equal. The symbol we use to represent the mean of a normal curve is the Greek letter μ.



Normal Distribution Center

It is also important to realize that this location divides the data into two equal parts and corresponds to the 50th percentile.

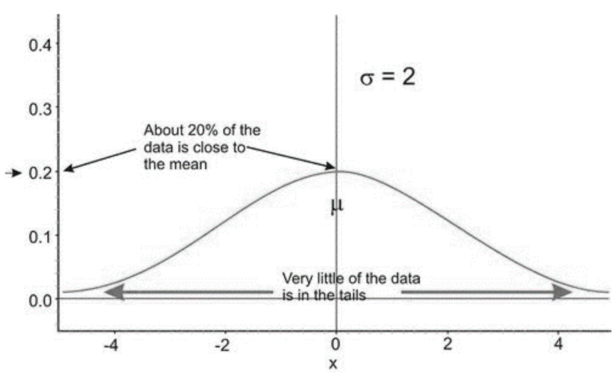


Normal Distribution divided into equal halves by the center

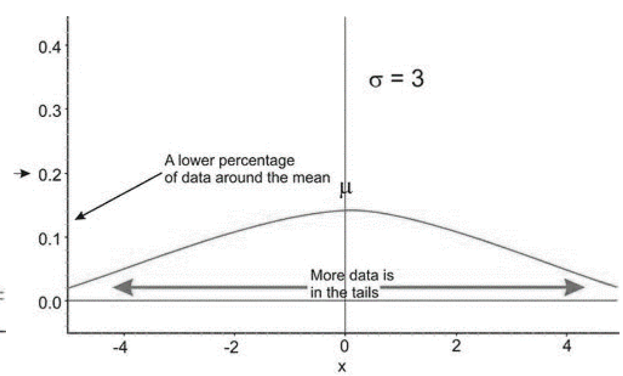
Spread

Because of the infinite spread of a normal model, the range would not be a useful statistical measure of spread. The most common way to measure the spread of a normal distribution is with standard deviation as the typical distance away from the mean. We use the Greek letter σ to represent standard deviation.

Below are two normal distributions. Both have the same mean $\mu = 0$, but they have different standard deviations. Look at how they differ.



normal distribution, standard deviation = 2



normal distribution, standard deviation = 3

The distribution pictured on the left has a smaller standard deviation, and so more of the data are heavily concentrated around the mean than in the distribution on the right. Also, in the first distribution, there are fewer data values at the extremes than in the second distribution. Because the second distribution has a larger standard deviation, the data are spread farther from the mean value, with more of the data appearing in the tails.

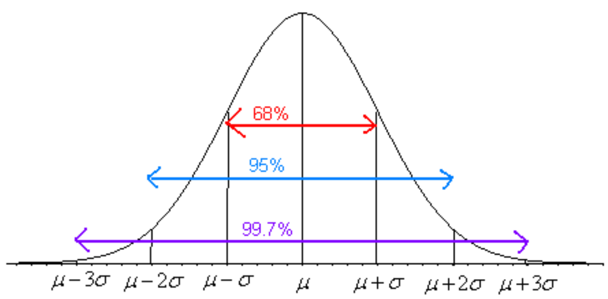
The Empirical Rule

Another property of the normal distribution has to do with the relative frequency of data that falls within a given distance from the mean. And since empirical probability is nothing more than relative frequency, we can also consider this property to tell us about the probability that a certain value falls within a given distance from the mean.

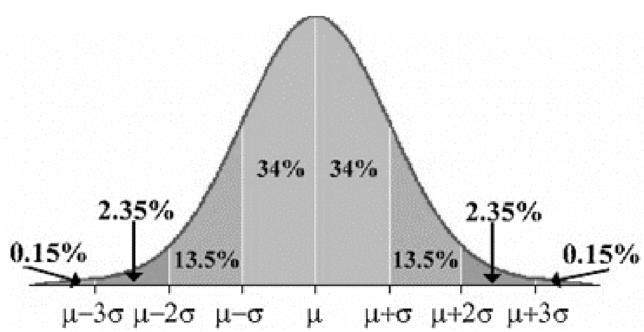
The **Empirical Rule** is a property that holds for all normal curves and can be used as a guide to approximate relative frequencies and probabilities for certain intervals of the normal curve. The *Empirical Rule* is also known as the 68-95-99.7 Rule for reasons that will be obvious.

The Empirical Rule for Normal Distributions

Approximately **68%** of the outcomes are within one standard deviation of the mean.	$P(\mu - \sigma < X < \mu + \sigma)$
Approximately **95%** of the outcomes are within two standard deviations of the mean.	$P(\mu - 2\sigma < X < \mu + 2\sigma)$
Approximately **99.7%** of the outcomes are within three standard deviations of the mean.	$P(\mu - 3\sigma < X < \mu + 3\sigma)$



Be careful: There is still a portion of the normal curve left over in each end beyond $\mu - 3\sigma$ and $\mu + 3\sigma$. The total relative frequency for the entire distribution is 100%. If you calculate $100\% - 99.7\%$ you will see that for both ends together there is 0.3% of the data remaining. Because of symmetry, you can divide this equally between both ends and find that there is 0.15% in each tail beyond $\mu \pm 3\sigma$. Here is a more detailed picture of the percentages in each of the eight intervals created when counting 3 standard deviations on each side of the mean:



The *Empirical Rule* is effective to use only when we have endpoints of the interval that are exactly 1, 2, or 3 standard deviations from the mean. If we want to find percentages or probabilities involving values of X that are not integer multiples of the standard deviation, we will need to use technology as shown further in this section.

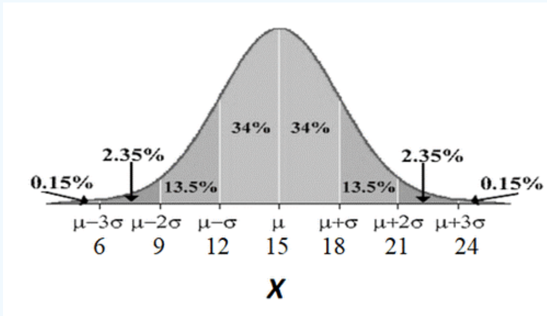
Let's consider some scenarios where we can apply the *Empirical Rule*.

Example 1

The mean and standard deviation for the age when a child starts walking are 15 months and 3 months, respectively. Assume X, the age at which children first walk, is normally distributed.

a. What percent of children start walking between 12 and 18 months of age?

The *Empirical Rule* can be used because the problem states that X follows a normal distribution. Start with a picture that relates the percentages and the values of the mean and standard deviation. In the problem we are given that $\mu = 15$ and $\sigma = 3$.



Looking at the portion of the curve from $X = 12$ to $X = 18$, we can combine the percentages shown in the middle two sections of the normal curve to find the total percentage: $P(12 < X < 18) = 34\% + 34\% = 68\%$.

We can say that about 68% of all children walk for the first time when they are between the ages of 12 and 18 months.

b. What percentage of children walk for the first time before age 18 months?

Using the same picture from part a, we can add the percentages to the left of $X = 18$. One way to do this is to add the area of each section separately: $P(X < 18) = 0.15\% + 2.35\% + 13.5\% + 34\% + 34\% = 84\%$.

You could also add the percentage left of the mean (50%) to the percentage from the mean μ to $\mu + \sigma$ (34%) to get 84%.

About 84% of children walk for the first time before 18 months of age.

c. Approximate the percentage of children who start walking after 21 months of age.

Using the same picture from part a, we can add the percentages to the right of $X = 21: 2.35\% + 0.15\% = 2.5\%$. About 2.5% of all children first walk after they are 21 months old.

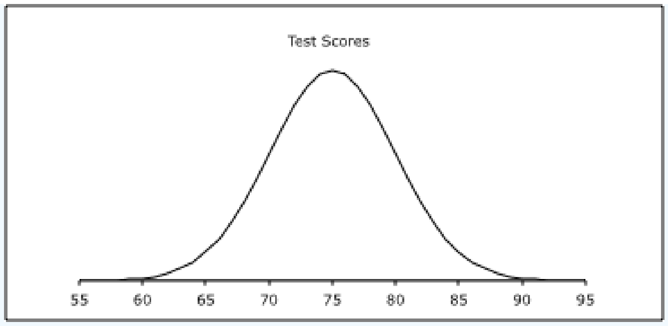
Example 2

Your class took a test and the mean score was 75 points and the standard deviation was 5 points. The test scores follow an approximately normal distribution. Sketch a normal curve and use it to answer these questions.

a. What percentage of the students had scores between 65 and 85?
b. What percentage of the students had scores between 65 and 75?
c. What percentage of the students had scores between 70 and 80?
d. What percentage of the students had scores above 85?

Solution

To solve each of these questions, it would be helpful to draw the normal curve that represents this situation. The mean is 75, so place 75 below the peak. The standard deviation is 5, so to label the axis, add 5 to the mean 3 times and subtract 5 from the mean 3 times. The graph looks like the following:



a. From the graph, $X = 65$ and $X = 85$ are both two standard deviations from the mean. According to the *Empirical Rule*, this percentage must be 95%..
b. The scores from $X = 65$ to $X = 75$ make up half of the percentage from $X = 65$ to $X = 85$. Because of symmetry, that means that $P(65 < X < 75$ is ½ of the answer to part a: $\frac{1}{2} \times 95\% = 47.5\%$. You could have also added $13.5\% + 34\%$.
c. From the graph, $X = 70$ and $X = 80$ are both one standard deviation from the mean. According to the *Empirical Rule*, this percentage must be 68%.
d. One way to find $P(X > 85)$ is to add $2.35\% + 0.015\% = 2.5\%$. There are other strategies too.

? Try it Now 1

The distribution of wait times at a drive-through restaurant is approximately normal with mean 185 seconds and standard deviation 15 seconds. Use a sketch and the *Empirical Rule* to find the percentage of wait times that are

a. between 155 and 215 seconds.
b. more than 170 seconds.
c. less than 155 seconds.
d. between 170 and 215 seconds.

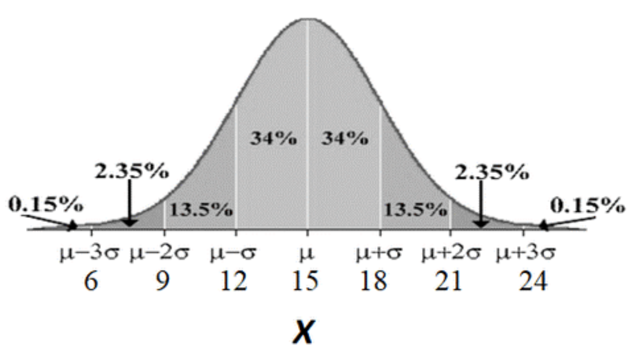
Answer

a. 95%
b. 84%
c. 2.5%
d. 81.5%

z-scores

Now we will begin to consider situations when the *Empirical Rule* cannot be used -- that is, when a value is not exactly one, two, or three standard deviations from the mean.

Let's return to the scenario in Example 1. Suppose we want to know the percentage of children who start walking after 10 months, or $P(X > 10)$. Recall the sketch of the normal curve:



The value of $X = 10$ does not fall exactly on any of the numbers we marked on the axis of the normal distribution. It is more than one but less than two standard deviations away from the mean of 15 months. We can compute what is called a *z-score* to measure how many standard deviations the value $X = 10$ is from the mean.

z-score

A ***z*-score** measures how many standard deviations a data value is from the mean of the distribution. To calculate the *z*-score for a value, find its deviation from the mean and divide by the standard deviation.

The formula for a z-score is $z = \dfrac{x-\mu}{\sigma}$ where

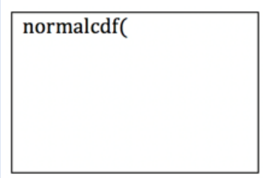
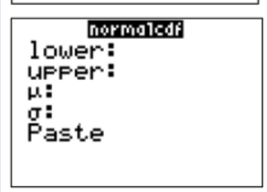
- x is the data value (raw score),
- z is the standardized value (z-score or z-value)
- μ is the mean
- σ is the standard deviation.

For a child that walks at $X = 10$ months, the z-score is $z = \dfrac{x-\mu}{\sigma} = \dfrac{10-15}{3} = \dfrac{-5}{3} \approx -1.67$. This means that this child is about 1.67 standard deviations below the mean age when children first begin to walk.

Using Technology to Find Percentages of a Normal Distribution

Once upon a time, finding percentages using the normal curve involved first computing *z*-scores and then consulting tables of normal areas in textbooks. Fortunately, as technology has developed to the point where we are now, we can use calculators and computers to find percentages when a variable follows a normal model. We will assume the use of technology in this course, but sketching a picture of the situation is still very useful.

The table below shows the general steps for finding percentages with the TI calculator. Example 3 below will demonstrate use of the steps.

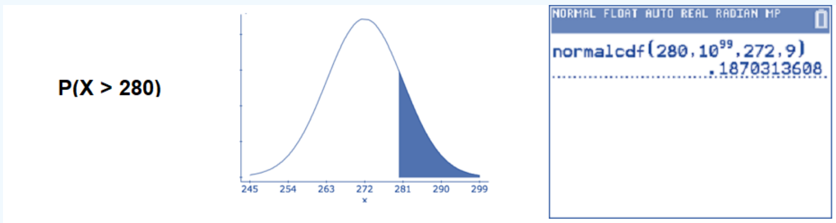
Technology Note: Computing Normal Percentages on the TI-83/84 Calculator		
1.	Push **[2ND] [VARS]** to open the **DISTR** (Distribution) menu. Choose **2:normalcdf(**.	
2.	You will have one of the two screens shown here: If you have a screen like the one on the top with a blinking cursor, you need to type the lower endpoint of the interval, the upper endpoint of the interval, the mean, and the standard deviation. Then, close the parentheses. If you have a STAT WIZARD screen that looks like the one on the bottom, enter the same information in the table. NOTE: If the lower endpoint is -∞, enter -10^{99} by pushing **[(-)] 10 [^] 99**. If the upper endpoint is ∞, enter 10^{99} by pushing **10 [^] 99**. These are close enough approximations for the infinities when we use them in statistics.	
3.	Press **[ENTER]**, or highlight **Paste** and push **[ENTER]**.	

Example 3

Lengths of human pregnancies are normally distributed with a mean $\mu = 272$ days and standard deviation $\sigma = 9$ days.

a. Find the percentage of pregnancies that last more than 280 days.

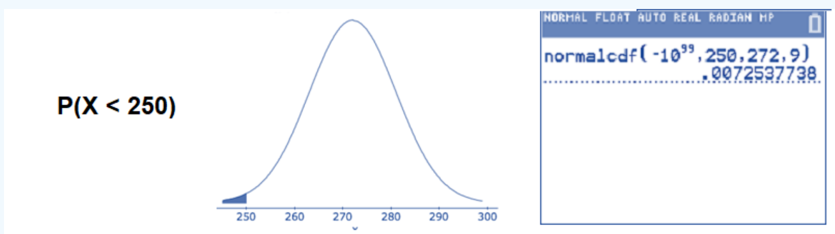
Translate the question into a mathematical statement and draw a sketch of a normal curve with $\mu = 272$ and $\sigma = 9$. Then, use the **normalcdf(** command. The lower endpoint of the interval is 280 and the upper endpoint of the interval is $+\infty$.



$P(X > 280) \approx 0.1870$: We can say that 18.70% of all human pregnancies last more than 280 days.

b. Find the percentage of pregnancies that last less than 250 days.

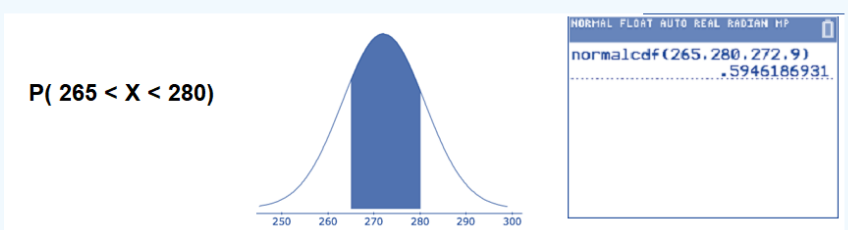
Translate the question into a mathematical statement and draw a sketch of a normal curve with $\mu = 272$ and $\sigma = 9$. Then, use the **normalcdf(** command. The lower endpoint of the interval is $-\infty$ and the upper endpoint of the interval is 250.



$P(X < 250) \approx 0.0073$: We can say that 0.73% of all human pregnancies last less than 250 days. This appears to be quite unusual.

c. Find the percentage of pregnancies that last between 265 and 280 days.

Translate the question into a mathematical statement and draw a sketch of a normal curve with $\mu = 272$ and $\sigma = 9$. Then, use the **normalcdf(** command. The lower endpoint of the interval is 265 and the upper endpoint of the interval is 280.



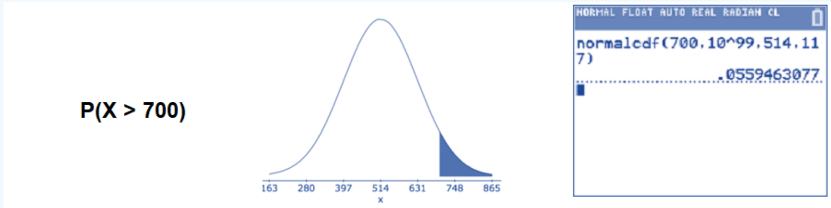
$P(265 < X < 280) \approx 0.5946$: About 59.46% of all human pregnancies last between 265 and 280 days.

Example 4

The mean score on a reading test for 4^{th} graders is 514 points with standard deviation 117 points. Assume that scores on this test are normally distributed.

a. Find the percentage of students who score at least 700 on this test.

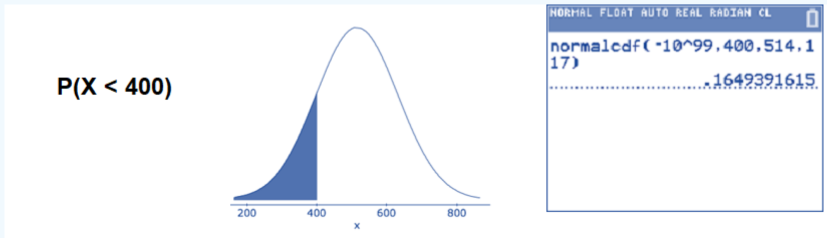
The question asks for the percentage to the right side of 700. Recall that '*at least*' means 700 or more. Translate the problem into a mathematical statement, draw a picture, and use the **Normalcdf(** command on the TI calculator.



$P(X > 700) \approx 0.0559$: About 5.59% of 4^{th} graders scored at least 700 on the reading test.

b. What percentage of 4^{th} graders score less than 400 on the reading test?

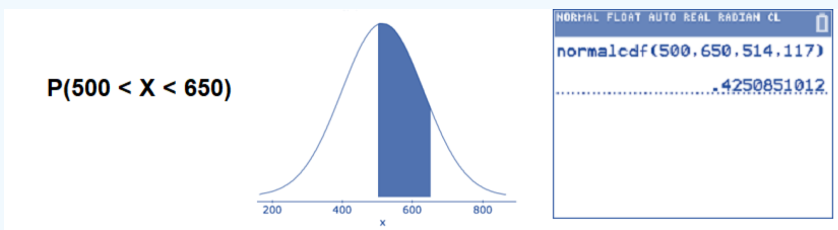
This asks for the percentage to the left side of 400. Use the **normalcdf(** command.



$P(X < 400) \approx 0.1649$: Approximately 16.49% of 4^{th} graders scored less than 400 on this reading test.

c. Find the percentage of 4^{th} graders who score between 500 and 650.

The question asks for the percentage between 500 and 600. Use the **normalcdf(** command.



$P(500 < X < 650) \approx 0.4251$: The percentage of 4^{th} graders with scores between 500 and 650 is approximately 42.51%.

? Try it Now 2

The average amount of caffeine consumed daily by an adult is normally distributed with mean of 250 mg and standard deviation of 48 mg.

a. Use technology to find the percentage of adults who consume these amounts of caffeine daily:
   i. more than 310 mg.
   ii. less than 250 mg.
   iii. between 202 mg and 346 mg.
   iv. more than 400 mg.

b. In a random sample of 500 adults, how many should consume at least 310 mg?

Answer

a. i. 10.56% ii. 50% iii. 81.86% iv. 0.09%
b. 53 adults

Using Technology to Find a Value from a Normal Distribution

In Example 3 we found the relative frequency of pregnancies that lie within a particular interval. All the questions in Example 3 asked for a percentage when given intervals of values for X such as $P(X > 280)$, $P(X < 250)$, or $P(165 < X < 280)$. When we were given a value of the variable and were asked to find the percentage, we used the **normalcdf(** command on the TI calculator.

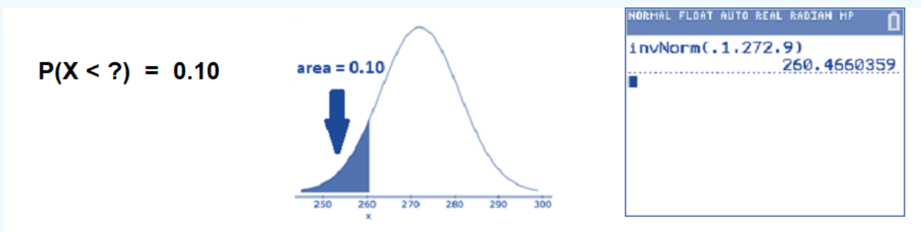
Now, we will consider the reverse operation. In other words, we will be concerned with finding a value for X that cuts off a given relative frequency on its left side or on its right side. Again, the TI calculator is convenient and accurate. The command on the TI calculator for this process is **invNorm(**. You may have seen it already in the **Distribution** menu.

Example 5

Revisiting the scenario from Example 3, lengths of human pregnancies are normally distributed with a mean $\mu = 272$ days and standard deviation $\sigma = 9$ days. Find the lengths of the shortest 10% of pregnancies.

Solution

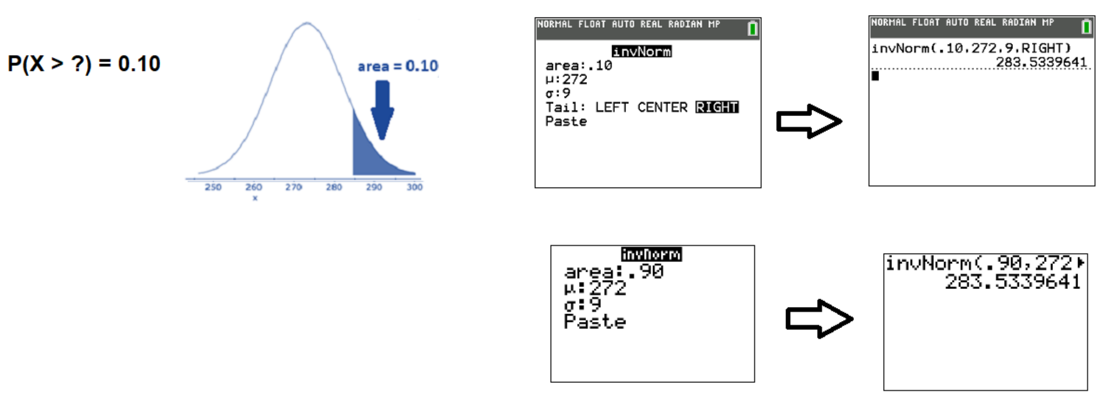
The question asks for an X-value in the normal distribution. You want to locate the X-value on the axis that has 10% of pregnancies *to the left of it.* Translate the problem into a mathematical statement and make a sketch showing that the area to left of some unknown X is 0.10. Then, use the **invnorm(** command with left-sided area 0.10.



Thus, 10% of all human pregnancies last less than about 260 days. We could also say that 90% of all human pregnancies last longer than 260 days. The value $X = 260$ separates the shortest 10% of human pregnancies from the longest 90%. As you may recall from an earlier section, we call 260 days the 10$^{\text{th}}$ percentile.

If we had wanted to find the longest 10% of pregnancies instead, we would have used the same command. But, we need to be careful about choosing the correct side of the normal curve.

For the longest 10% of pregnancies, we want to find the value for X so that $P(X > ? = 0.10$. Depending on the calculator version, you may be able to specify that you want the "right tail." If you don't have the option to choose the right tail, then you must find the X-value that cuts off the left 90% from the right 10% in a similar way to what we did in Example 5. Calculator screens for both versions of the calculator are shown.



Thus, 10% of all human pregnancies last longer than about 284 days. We could also say that 90% of all human pregnancies are shorter than 284 days. The value $X = 284$ separates the shortest 90% of human pregnancies from the longest 10%. We call 284 days the 90$^{\text{th}}$ percentile.

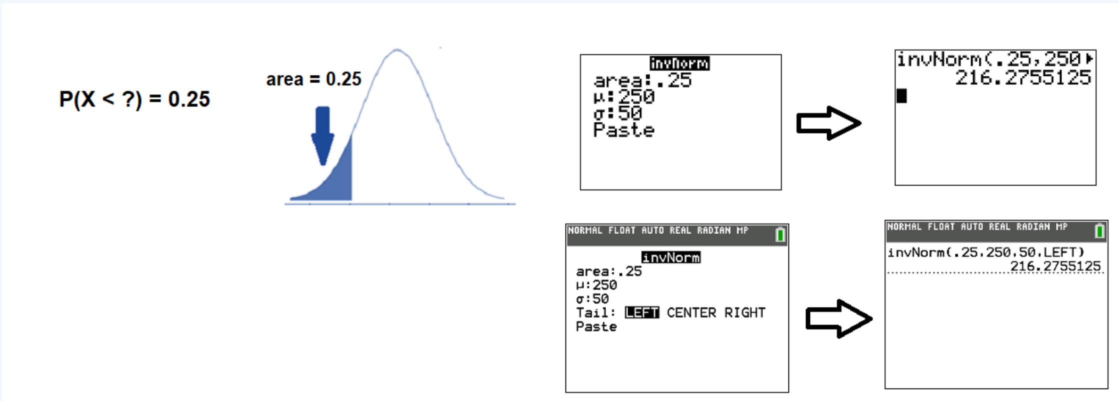
Example 6

Sarah keeps statistics for the baseball team. Based on her records, the distances that fly balls are hit to the outfield are approximately normally distributed with a mean 250 feet and standard deviation 50 feet.

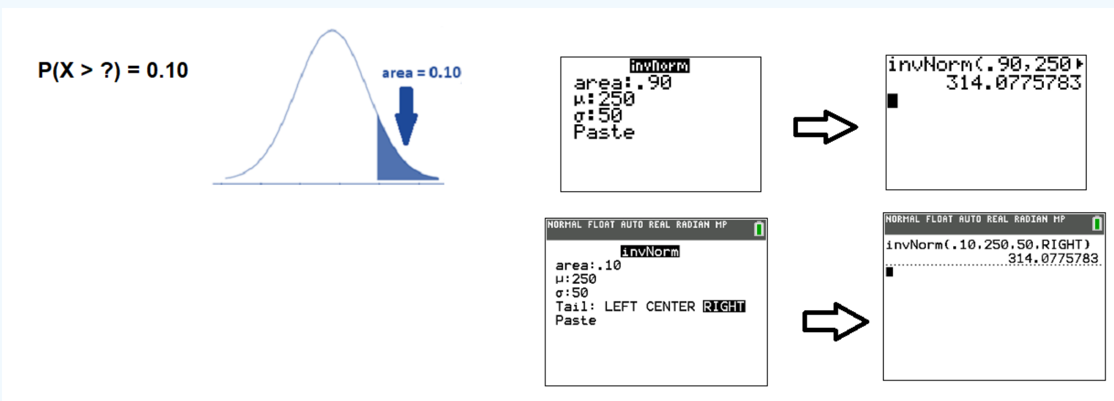
a. How far are the shortest 25% of all fly balls hit?
b. How far will a player need to hit a fly ball for it to be in the top 10% of all fly balls that are hit?

Solution

a. We want to find the X-value so that 25% of fly ball distances are lower than X. This means the area to the left of X is 0.25. As the calculator screen shows, the shortest 25% of all fly balls travel less than 216 feet.



b. We want to find the X-value so that 10% of fly ball distances are greater that X. This means the area to the right of X is 0.10. As the calculator screen shows, a player needs to hit the fly ball a distance of 314 feet or more.



Try it Now 3

A citrus farmer who grows tangerines finds that the diameters of tangerines harvested on his farm follow a normal distribution with a mean diameter 5.85 cm and standard deviation of 0.25 cm. Use a TI calculator to complete each statement.

a. The smallest 15% of tangerines have diameters that measure _____ cm or smaller.
b. The largest 25% of tangerines have diameters that measure _____ cm or bigger.

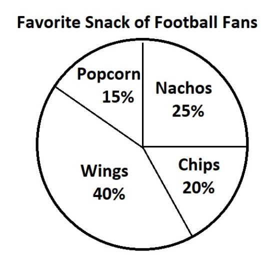
Answer

a. 5.59
b. 6.02

- **2.4: The Normal Distribution** by Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, & Kim Sonier is licensed CC BY-SA 4.0. Original source: https://www.coconino.edu/open-source-textbooks#college-mathematics-for-everyday-life-by-inigo-jameson-kozak-lanzetta-and-sonier.

4.6: Review Exercises

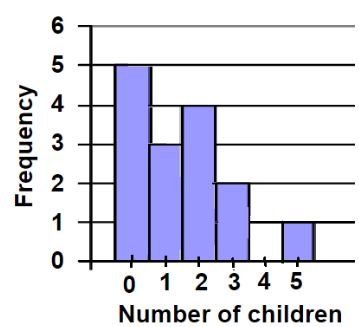
1. For each scenario, state the population and the sample.
   a. A person collects the gas prices at 25 gas stations in Phoenix, AZ.
   b. A study was conducted of schools across the U.S. about whether they require school uniforms. Two hundred ninety-six schools gave their response to the question, "*Does your school require school uniforms?*"
2. In each scenario, determine whether the data would best be described as *quantitative data* or *qualitative data*:
   a. heights of trees in a forest, in feet
   b. calories in various types of candy bars sold at a convenience store
   c. dominant hand of each of your classmates
   d. favorite sport
3. A study to determine the opinion of Maryland voters about the use of marijuana for medical purposes is being conducted. Identify the sampling method described as *simple random sample, stratified sample, cluster sample, systematic sample*, *convenience sample* or *voluntary response sample.*
   a. The researchers attend a festival in a town in Maryland and ask all the voters they can what their opinions are.
   b. The researchers divide Maryland voters into groups based on the person's race, and then take random samples from each group.
   c. The researchers get a list of all Maryland voters and call the 50th person on the list. Then they call every 1,000th person after the 50th person.
   d. The researchers call every voter in each of 10 Maryland zip codes that were randomly chosen.
   e. The researchers run an advertisement on Maryland television stations directing voters to give their opinion on a website.
   f. The researchers call only those with voter ID numbers generated randomly by a computer.
4. The circle graph shows the results of a survey that asked 60 football fans their favorite snack while watching the game.



   a. How many of the fans chose wings?
   b. How many more fans chose nachos than chose popcorn?
   c. What is the measure of the central angle in the graph for the portion of the fans who chose chips? *Round to the nearest whole degree as necessary.*
5. Tiffany has a game spinner with several colors. She spun the spinner several times and recorded the color she got after each spin. The data are shown.

red	red	yellow	green	red
blue	yellow	yellow	red	blue
yellow	yellow	red	yellow	red
green	yellow	yellow	red	yellow
red	red	red	yellow	green

   a. Make a frequency table to summarize the colors that Tiffany spun on the spinner. Include a column for frequency and relative frequency in the table.
   b. Use the frequency table to draw a bar graph of the data.
   c. Use the frequency table to find the size of the angle you should draw for each color in a pie chart. *Round to the nearest whole degree.*
   d. Give the mode(s) of this data set.
6. A group of adults were asked how many children they have in their families. The bar graph shows the responses.



   a. How many adults where questioned?
   b. What percent of the adults questioned had 0 children?
   c. What is the mode of this data set?

7. A group of adults where asked how many cars they had in their household. The data are shown.

1	4	2	2	0	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

a. Construct an ungrouped frequency table for data.
b. Based on the frequency table, draw a histogram of the data.

8. The SAT test scores for Michigan students in reading for several years are given in the table below (College Board: Michigan, 2012).

SAT Scores for Michigan Students

496	500	502	504	505	507	509
497	500	503	504	505	507	512
499	500	503	505	505	508	515
499	500	504	505	506	508	518
499	501	504	505	507	509	522
500	502	504	505	507	509	525

a. Create a grouped frequency table using 495-499 as the first class. Show frequencies, relative frequencies, and class marks.
b. Based on the frequency table, create a histogram of the data.

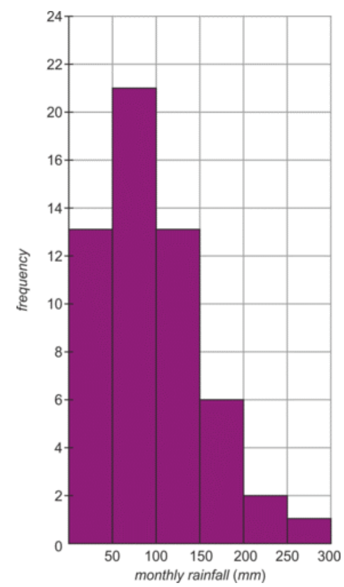
9. The frequency table shows the ages of the passengers on a train. Use the frequency table to answer the questions.

Ages of Passengers	Frequency
0 – 14	6
15 – 29	11
30 – 44	15
45 – 59	8
60 – 74	2
75 – 89	1

a. How many passengers were on the train?
b. What class width was used to group the data?
c. What is the modal class?
d. What is the class mark of the last class?
e. If an additional class were added to the end of the table, what would be its lower and upper limits?

10. Monthly rainfall (in millimeters) for Beaver Creek, Oregon, was collected over several months. Use the histogram of the data to answer the questions.

a. What class width was used to create the histogram?
b. What is the modal class?
c. In how many of these months was the total rainfall 150 mm or more?
d. What is the relative frequency of the first class (0-49 mm)?



11. A group of diners were asked how much they typically pay for lunch. Their responses were

$7 $10 $9 $8 $7 $6 $7 $5 $13

Find each of these descriptive measures (by hand) to the nearest cent.

a. mean
b. median
c. 5-number summary
d. range
e. standard deviation
f. midrange

12. A math student took five tests and has a mean of 85 points. He took another test and scored 100 points. What is his mean score now?

13. The frequency table shows the number of home runs for the past ten baseball games of the season. Find the mean and median number of home runs.

number of home runs	frequency
4	1
5	2
6	0
7	2
8	4
9	1

14. The city gas mileage (in mpg) for small four-wheel-drive pick-up trucks are given in the table below.

Gas Mileages of Small Four-wheel-drive Trucks

17	18	17	14	18	16
14	14	15	17	18	18
14	16	16	14	14	16

Use a calculator (where possible) to find each statistic. *Round to the nearest hundredth (two decimal places) where necessary.*

a. mean
b. median
c. standard deviation
d. five-number summary
e. IQR
f. range
g. midrange

15. The following statistics represent scores on an exam.

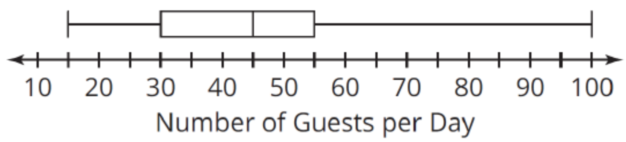
Mode = 78 points Q1 = 67 points Mean = 82 points Q3 = 84 points Median = 80 points 90^{th} percentile = 96 points

a. What was the most common grade?
b. Half of the students have grades higher than what score?
c. About what percent of students have grades higher than 84 points?
d. About what percent of students scored higher than 67 points?
e. If there were 30 students who took the exam, how many scored higher than 96 points?

16. Professor Smith teaches two science classes, one in the morning and the other in the afternoon. After grading the unit test for the two classes, she calculated that the morning class scored a mean of 75 points and a standard deviation of 5.7 points. The afternoon class scored a mean of 75 points and a standard deviation of 12.5 points. Which of these conclusions is correct? *Choose one:*

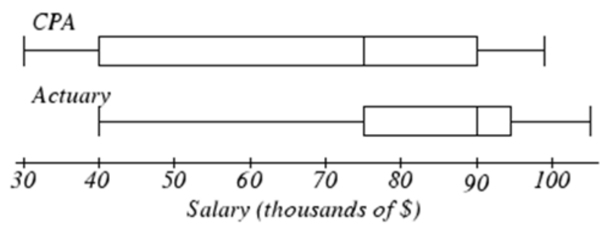
   a. There were more students who took the test in the afternoon class than in the morning class.
   b. Typically, afternoon students did better than morning students on the test.
   c. There is more variation in the test scores in the afternoon class than in the morning class.
   d. The test scores in the afternoon class are more consistent to each other than those in the morning class.

17. The box plot represents the number of guests each day at an island resort during the last tourist season. Use it to answer the questions.



   a. What is the median number of guests each day at this resort?
   b. On what percent of the days did the resort have 45 or more guests?
   c. On what percent of the days did the resort have 55 or fewer guests?
   d. What is the IQR for the number of guests at this resort?
   e. If this box-and-whisker plot shows data for the past 136 days, on how many of the days did the resort have 30 or fewer guests?

18. The box plot below compares salaries of actuaries and CPAs.



   a. What salary is at the 75^{th} percentile for CPA's?
   b. Which profession has the higher median?
   c. Which profession has salaries that are more varied?
   d. Kendra makes the median salary for an actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?
   e. What percent of actuaries makes more than the median salary of a CPA?

19. The mean systolic blood pressure of people in the U.S. is 124 with standard deviation of 16. Assume that systolic blood pressure follows a normal distribution. Draw a normal curve, label it, and use the *Empirical Rule* to answer the questions.

   a. What percent of the people in the U.S. have systolic blood pressure between 92 and 156?
   b. What percent of the people in the U.S. have systolic blood pressure between 108 and 124?
   c. What percent of the people in the U.S. have systolic blood pressure below 108?
   d. What percent of the people in the U.S. have systolic blood pressure above 156?

20. The mean height of men in the U.S. is 69.1 inches with a standard deviation of 2.9 inches. Assume that height follows a normal distribution. Use a TI calculator to answer the questions. Round parts a-c to the nearest whole percent. Round parts d and e to the nearest tenth of an inch.

   a. What percent of men in the U.S. are shorter than 68 inches?
   b. What percent of men in the U.S. are taller than 75 inches?
   c. What percent of men in the U.S. are between 66 and 72 inches tall?
   d. How tall are the shortest 20% of U.S. men in the U.S.?
   e. How tall is a man if he is one of the tallest 5% men in the U.S?

CHAPTER OVERVIEW

5: Personal Finance

5.1: Simple Interest

Money is not free to borrow! When you borrow money, the cost you pay to borrow that money is called **interest.** The amount of money borrowed is known as **principal** or **present value**, P.

At some point in the future, you pay back the amount you borrowed originally as well as the interest, I, that was charged. The total amount paid back is known as **future value**, F. We can summarize this with the formula $F = P + I$.

The same happens when you make an investment. You deposit a certain amount of money (the present value, P) that accumulates interest I over time. In the future you withdraw the accumulated amount -- both your original principal as well the as interest earned. Again, $F = P + I$.

The interest you pay or earn can be calculated in different ways. In this section we consider *simple interest*. Simple interest is interest that is calculated only on the principal, P. This means you are paying or earning the same amount of interest every period. An example of simple interest is when someone purchases a U.S. Treasury Bond.

Simple Interest

Simple interest is interest that is only paid on the principal:

$$\text{simple interest} = \text{principal} \times \text{rate} \times \text{time}$$

Example 1

Sue borrows $2,000 at 5% annual simple interest from her bank. How much does she owe after five years?

Solution

Each year, Sue will pay 5% (or 0.05) of the principal that she borrowed as the cost for borrowing the money. The balance that Sue owes grows constantly by $100 each year because each year Sue is paying interest only the original $2,000 she borrowed.

Accumulation of Simple Interest Using a Table

Year	Interest Owed	Total Balance Owed
1	$\$2000 \times 0.05 \times 1 = \$100$	$\$2000 + \$100 = \$2100$
2	$\$2000 \times 0.05 \times 1 = \$100$	$\$2100 + \$100 = \$2200$
3	$\$2000 \times 0.05 \times 1 = \$100$	$\$2200 + \$100 = \$2300$
4	$\$2000 \times 0.05 \times 1 = \$100$	$\$2300 + \$100 = \$2400$
5	$\$2000 \times 0.05 \times 1 = \$100$	$\$2400 + \$100 = \$2500$

After 5 years, Sue owes $2,500. The total accumulated interest during these 5 years was $\$2000 \times 0.05 \times 5 = \$500$.

This leads us to the simple interest formulas.

Simple Interest Formulas

$$I = Prt \;\; F = P + I \;\; F = P(1 + rt)$$

where,

- I is the interest accumulated over t years
- P is the present value, or principal
- r is the annual percentage rate (APR) written as a decimal
- t is number of years
- F is the future value

Example 2

Chad got a student loan for $10,000 at 8% annual simple interest. How much does he owe after one year? How much interest will he pay for that one year?

Solution

In this scenario, $P=\$10,000$, $r=0.08$, and $t=1 \text{ year}$. We substitute these values into the third formula to find the future value:

$$F=P(1+rt)$$

$$F=10,000(1+0.08\times 1)=\$10,800$$

Chad owes $10,800 after one year. Using the second formula, he will pay $F-P=\$10,800-\$10,000=\$800$ in interest.

Notice that you could have also used the first formula to find the amount Chad paid in interest: $I=Prt=10,000\times 0.08\times 1=\$800$.

Try it Now 1

You borrowed $6,000 from a family member at an APR of 6% for 5 years to start a business. How much do you need to repay after 5 years?

Answer

$7,800.00

The formulas for I and F only work if time t and interest rate r are expressed using the same time units. For example, if interest rate is expressed as an *annual* rate then time t must be expressed in years. But, if the interest rate is 2% *per month* then time t should be expressed in months.

Typically, interest rate is given as an annual rate unless otherwise stated. But, sometimes the interest rate can be given monthly or even daily.

If time t and interest rate r are expressed in different time units, convert them to the same time units. It is always easier to convert t to the units in which r is given. We all know that number of days varies from to month and from year to year. To simplify this matter, it is common to use the following correspondence in financial math calculations:

- 1 month = 30 days
- 1 year = 360 days
- 1 year = 52 weeks
- I year = 12 month

The following two examples illustrate conversion of time units.

Example 3

Aysha borrowed $4,000 from the bank for a term of 6 months at a 6.5% annual interest rate. How much did Aysha have to pay for the use of the money? How much did she repay to the bank on the due date of the loan?

Solution

In this scenario, the 6.5% interest rate is given as an annual (yearly) rate and time is given as 6 months. Therefore, time must be converted to years to match the unit on the rate: $t=6 \text{ months}=\frac{6}{12}=0.5 \text{ year}$.

Then, we can find the accumulated interest to answer the first question using $P=\$4,000$, $r=0.065$, and $t=0.5$:

$$I=Prt=4,000\times 0.065\times 0.5=\$130$$

Aysha repays the future value $F=P+I=\$4,000+\$130=\$4,130$.

Notice that you could also find the future value using the formula: $F=10,000(1+0.065\times 0.5)=\$4,130$.

Try it Now 2

Suppose a loan is taken where $P = \$587$, $r = 6.05\%$ annually, and $t = 90$ days. Find the accumulated interest I.

Answer

$I = \$8.88$

Example 4

Anna pawned her ring for $300. To get her ring back in 40 days she needs to pay back $355. What is the annual interest rate charged by the pawn shop?

Solution

In this scenario, the future value F is known, but the interest rate r is unknown. We have $P = \$300$, $F = \$355$, and $t = 40$ days.

First, find the amount of accumulated interest that Anna will pay: $I = F – P = \$355 - \$300 = \$55$.

Because we are looking for the annual interest rate, convert time to years: $t = 40$ days = $= \frac{40}{360} = \frac{1}{9}$ year.

Solve the equation to find the rate r:

$I = Prt$

$55 = 300 \cdot r \cdot \frac{1}{9}$

$55 \cdot 9 = 300 \cdot r$

$495 = 300 \cdot r$

$r = \frac{495}{300} = 1.65 = 165\%$

Try it Now 3

Find P when $r = 2.1\%$ annually, $t = 135$ days, and $I = \$37.80$.

Answer

$P = \$4,800$

5.2: Compound Interest

Most banks, loans, and credit cards charge *compound interest*, not simple interest. Compound interest is both calculated on the original principal and on all interest that has been added to the original principal. You can think of this as "interest on your interest."

Compound Interest

Compound interest is interest that is paid or earned on the principal as well as on the interest that has accrued.

Interest can be compounded at various time intervals. Interest on a mortgage or auto loan is usually compounded monthly. Interest on a savings account may be compounded quarterly (four times a year). Interest on a credit card may be compounded weekly or even daily. Here are some common compounding types that you may encounter in financial problems and in your everyday life.

Common Compounding Periods

Compounding type	Number of compounding periods per year
Annually	1
Semiannually	2
Quarterly	4
Monthly	12
Daily	365

Consider the following example that illustrates how interest is compounded.

Example 1

We deposit $1,000 in a bank account offering 3% interest compounded monthly. Let's explore how our money will grow.

Solution

The 3% interest rate is an annual percentage rate (APR) – the total interest rate to be paid during the year. Because interest is being paid monthly, we will earn only $\frac{3\%}{12}=0.25\%=0.0025$ per month.

- In the first month, $P=\$1,000$ and $r=0.0025$, so

  $I=\$1,000(0.0025)=\$2.50$

  $F=\$1,000+\$2.50=\$1,002.50$

  At the end of the first month, we will earn $2.50 in interest, raising our account balance to $1,002.50.

- In the second month, $P=\$1,002.50$ and $r=0.0025$, so

  $I=\$1,002.50(0.0025)=\$2.51$ (rounded)

  $F=\$1,002.50+\$2.51=\$1,005.51$

  Notice that at the end of the second month we will earn slightly more interest than in the first month. This is because we earned interest not only on the original $1,000 principal, but we also earned interest on the $2.50 of interest accumulated during the first month. This is the key advantage that compounding of interest gives us.

The table shows how the interest accumulates and is compounded for a complete year (12 months):

Compound Interest Using a Table

Month	Starting Balance	Interest Earned	Ending Balance
1	$1,000.00	$1,000.00 × 0.0025 = $2.50	$1,000.00 + $2.50 = $1,002.50
2	$1,002.50	$1,002.50 × 0.0025 = $2.51	$1,002.50 + $2.51 = $1,005.01
3	$1,005.01	$1,005.01 × 0.0025 = $2.51	$1,005.01 + $2.51 = $1,007.52

Month	Starting Balance	Interest Earned	Ending Balance
4	$1,007.52	$1,007.52 × 0.0025 = $2.52	$1,007.52 + $2.52 = $1,010.04
5	$1,010.04	$1,010.05 × 0.0025 = $2.53	$1,010.05 + $2.53 = $1,012.57
6	$1,012.57	$1,012.57 × 0.0025 = $2.53	$1,012.57 + $2.53 = $1,015.10
7	$1,015.10	$1,015.10 × 0.0025 = $2.54	$1,015.10 + $2.54 = $1,017.64
8	$1,017.64	$1,017.64 × 0.0025 = $2.54	$1,017.64 + $2.54 = $1,020.18
9	$1,020.18	$1,020.18 × 0.0025 = $2.55	$1,020.18 + $2.55 = $1,022.73
10	$1,022.73	$1,022.73 × 0.0025 = $2.56	$1,022.73 + $2.56 = $1,025.29
11	$1,025.29	$1,025.29 × 0.0025 = $2.56	$1,025.29 + $2.56 = $1,027.85
12	$1,027.85	$1,027.85× 0.0025 = $2.57	$1,027.85 + $2.57 = $1,030.42

So, after one year of compounding the interest monthly, the balance in the account is $1,030.42.

The calculations of interest and the new balance month-after-month can get tedious and long. To make these calculations more efficient, observe the pattern in the table: The future value after m months, F_m, can be presented using the future value from the previous month, F_{m-1}:

$$F_m = F_{m-1} \cdot (1 + \tfrac{0.03}{12})$$

This observation leads to the standard compound interest formula.

Compound Interest Formula

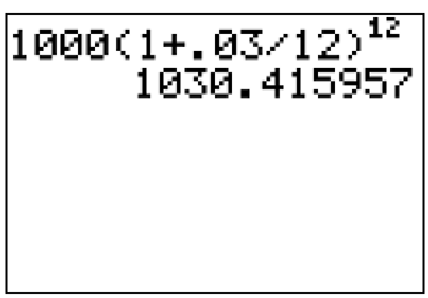
$$F = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

where,

- F is future value
- P is present value, or principal
- r is the annual percentage rate (APR) written as a decimal
- n is the number of compounding periods per year
- t is number of years

Using the formula derived above, let's solve Example 1 again where $P = \$1,000$, $r = 0.03$ annually, $n = 12$ times, and $t = 1$ year. A screenshot of entering the calculation on a TI calculator is also shown.

$$F = 1,000\left(1 + \tfrac{0.03}{12}\right)^{12 \cdot 1}$$



The accumulated balance after 12 months is $1,030.42 and agrees with the result in the table.

So is there a significant difference between compounded interest and simple interest? The next example will investigate this question.

✔ Example 2

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate than a savings or checking account, but you cannot access your investment for a specified length of time. Suppose you deposit $3,000 into a CD paying 6% interest compounded monthly. After 20 years, how will the value of the CD compare to the value of another type of account that pays 6% simple interest?

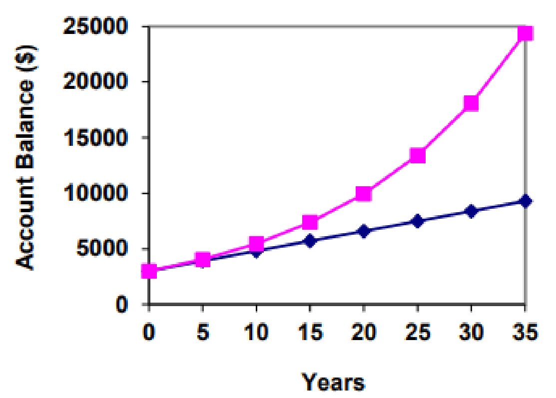
Solution

In this example,

$$\begin{array}{ll} P=\$3,000 & \text{the initial deposit} \\ r=0.06 & 6\% \text{ annual rate} \\ n=12 & 12 \text{ months in } 1 \text{ year} \\ t=20 & \text{since we're looking for how much we'll have after } 20 \text{ years} \end{array}$$

The table and graph compares an account's balance using 6% simple interest to the CD balance using 6% interest compounded monthly every 5 years throughout 35 years.

Years	Value of Account Paying 6% Simple Interest ($15 per month)	Value of CD Paying 6% Interest Compounded Monthly (0.5% per month)
5	$\$3,900$	$\$4,046.55$
10	$\$4,800$	$\$5,458.19$
15	$\$5,700$	$\$7,362.28$
20	$\$6,600$	$\$9,930.61$
25	$\$7,500$	$\$13,394.91$
30	$\$8,400$	$\$18,067.73$
35	$\$9,300$	$\$24,370.65$



The graph shows there is not a great deal of difference between the two types of interest in the first few years. But as time increases, the value of the CD where interest is compounded increases at a faster and faster pace and at a much faster rate than the value of the account paying simple interest.

Using the formulas for simple interest and compound interest, we can find the future value F of the two different accounts in $t=20$ years with an initial deposit of $P=\$3,000$ and an interest rate of $r=6\%$.

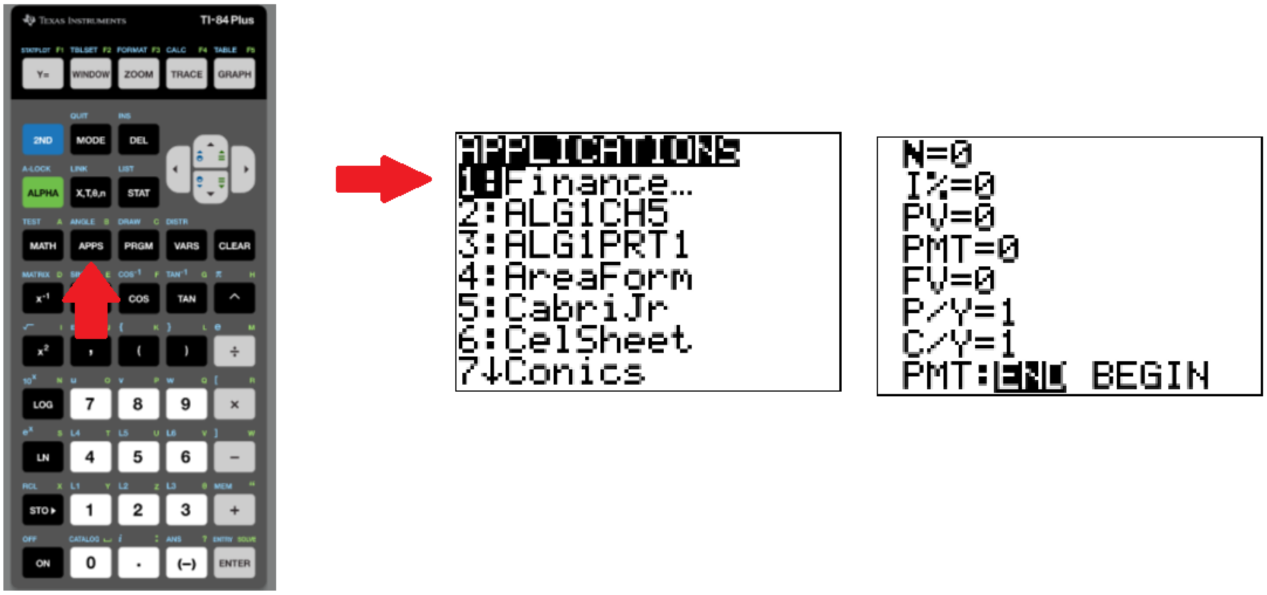
Simple Interest: $F=P(1+rt)=3,000(1+0.06\cdot 20)=\$6,660$

Compounded Interest: $F=P\left(1+\frac{r}{n}\right)^{nt}=3,000\left(1+\frac{0.06}{12}\right)^{12\cdot 20}=\$9,930.61$

When interest is compounded, the amount of money in the CD is $\$9,930.61-\$6,600.00=\$3,330.61$ more than if simple interest had accrued.

One can always use any scientific calculator to find value of F using the compound interest formula. However, computations become much more algebraically intense when solving for the principal P, for the rate r, or for the time t. Therefore, it is recommended to use the *Time Value Money Solver (TVM Solver)* app available on a TI calculator to solve all problems related to compounded interest.

To access the *TVM Solver*, press the **APPS** key and then choose **Finance...**.

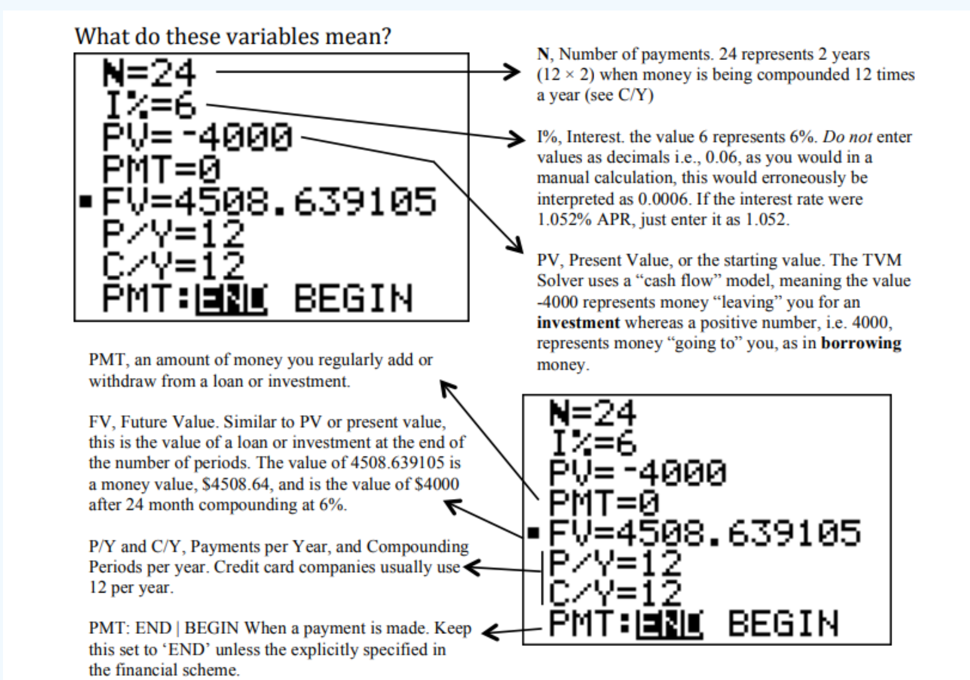


Let us consider how to use the *TVM Solver* settings on following example.

✔ Example 3

Find the total amount accumulated and the total interest earned if \$4,000 is invested for 2 years at a 6% compounded monthly.

Solution



When solving for a value using the *TVM Solver* app, you will need to enter all the given values, but skip over the value you are solving for (in this case FV.) Once you have entered all given values, go back to the value you want to find and press the **ALPHA** button and then **ENTER**. A square will appear next to FV to show that this value was calculated using the given information.

According to the *TVM Solver*, the total amount in the account will accumulate to $4,508.64. This means that the account has earned $\$4,508.64 - \$4,000 = \$508.64$ in interest during the 2 years.

For the following examples, we will continue to make use of the *TVM Solver*.

Example 4

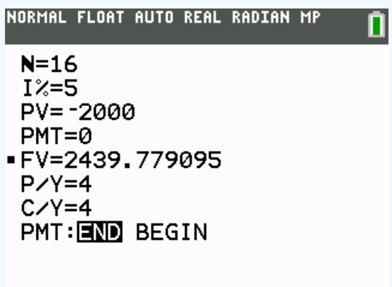
Find the total amount accumulated and the total interest earned when $2,000 is invested for 4 years at 5% interest compounded quarterly (four times each year.)

Solution

For this scenario,

- $N = 16$: The number of times the interest is compounded is $4 \text{ years } \times 4 \text{ times per year } = 16$.
- $I\% = 5$: Remember to leave this as a percentage. Do not change to a decimal.
- $PV = -2000$: Recall that $'-'$ represents money leaving you and put into an account.

Completing the remaining values as in Example 3, we then solve for future value FV by moving the cursor back to the value for FV and pressing **ALPHA** and then **ENTER.** The result is shown on the calculator screen.



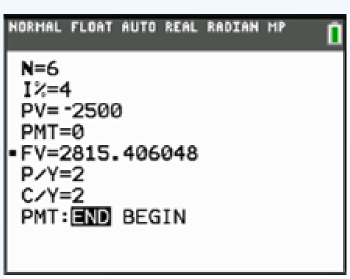
The accumulated amount in the account will be $\$2,439.78$. The total interest earned during the 2 years will be $\$2,439.78 – \$2,000.00 = \$439.78$

Try it Now 1

Use the *TVM Solver* app to solve:

Find the total amount accumulated and the total interest earned when $2,500 is invested for 3 years at 4% interest compounded semi-annually (twice a year).

Answer



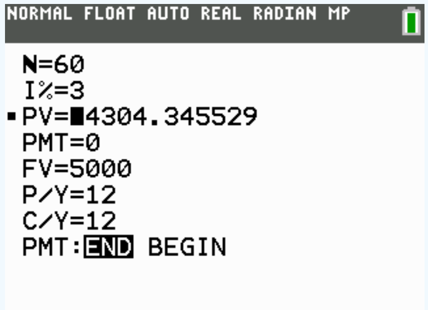
The accumulated amount in the account will be $\$2,815.41.$ The total interest earned during the 3 years will be $\$2,815.41 – \$2,500.00 = \$315.41$.

✓ Example 5

How much do you need to invest into a 5-year CD (certificate of deposit) that pays 3% interest compounded monthly so that it will be worth $5,000 at the end of the term?

Solution

In this problem, the future value is given, $FV = \$5,000$, and we need to solve for the present value, PV. We also are given that $N = 5 \text{ years } \times 12 \text{ times per year } = 60 \text{ times }$ and $I = 3\%$.



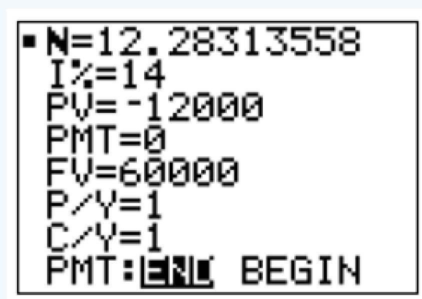
As the *TVM Solver* calculates, you need to invest $4,304.35 in order to reach $5,000 in 5 years.

✓ Example 6

A rare violin has an estimated value of $12,000. Its value history indicates that it appreciates at 14% per year. How many years will it take to appreciate to $60,000?

Solution

Here we need to solve for N. Set $I = 14\%$, $PV = -12,000$ and $FV = 60,000$.



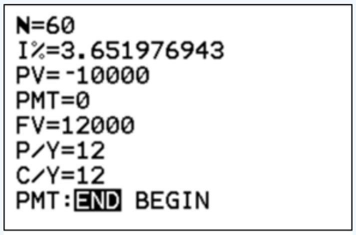
The value $N = 12.28313558$ is in years so figuring the decimal 0.28 as part of a 52-week year gives $0.28 \times 52 = 14.56 \text{ weeks}$. The final answer is about 12 years and 14 weeks.

? Try it Now 2

Use the *TVM Solver* app to solve:

You investing \$10,000 for 5 years into account where the interest is compounded monthly. What interest rate will you need to for the account to accumulate to \$12,000 during that time?

Answer



$I \approx 3.65\%$

In summary, the *TVM Solver* app on the TI calculator is similar to the financial math functions in Microsoft *Excel* . Using the *TVM Solver* application for compound interest problems allows us to solve for any parameter (PV, FV, I, N, and so on) without having to solve a logarithmic or exponential equation. While the *TVM Solver* is useful for solving compound interest problems, it is not useful for solving simple interest problems.

The next two sections will use the *TVM Solver* app to solve annuity and loan repayment problems.

5.3: Savings Annuities

Most of us aren't able to put a large sum of money in the bank today for retirement in the future. Instead, we save for the future by depositing a smaller amount of money from each paycheck into a type of account called an **annuity**. Then, upon retirement, we begin to withdraw from the annuity for income.

In this section we will consider one type of way to save for retirement or some other large purchase called a *savings annuity*. A typical example of this type of annuity account is a retirement plan or an IRA plan into which someone makes a regular deposit from each paycheck. This type of annuity is also known as a *savings plan*. In the next section we will look at how the funds in an annuity will be paid out at retirement.

Savings Annuity

A **savings annuity** is used to save money by depositing equal amounts periodically so the money deposited and the interest accumulates to some value for future use.

Mathematical formulas for a savings annuity are given below. As in the last section, we will emphasize using the *TVM Solver* application but will calculate each example with a formula as well.

Savings Annuity Formulas

$$A=\frac{d\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\left(\frac{r}{n}\right)}$$

$$d=\frac{A\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}$$

where

- A is the balance in the account after t years
- d is the regular deposit amount (or payment) each month, quarter, year, etc.
- r is the annual interest rate (APR) in decimal form
- n is the number of compounding periods in one year
- t is the number of years

If the compounding frequency is not stated, assume there is the same number of compounds in a year as there are deposits made in a year.

- If you make your deposits every year, then use yearly compounding: $n=1$.
- If you make your deposits every quarter, then use quarterly compounding: $n=4$.
- If you make your deposits every month, then use monthly compounding: $n=12$.
- If you make your deposits every week, then use weekly compounding: $n=52$.
- And so on.

Example 1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA at 6% interest, what is the balance in your account after 20 years? How much interest will you have earned? What percentage of the balance is interest?

Solution

To use a formula to solve this example, use the formula solved for A since we want to know the balance at some point in the future. Here, we know

$d = \$100$ the monthly deposit
$r = 0.06$ 6% annual rate
$n = 12$ since we're doing monthly deposits, we'll compound monthly
$t = 20$ we want the amount after 20 years

Putting these values into the equation:

$$A = \frac{d\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)} = \frac{100\left[\left(1+\frac{0.06}{12}\right)^{20\cdot 12} - 1\right]}{\left(\frac{0.06}{12}\right)} = \frac{100\left[(1.005)^{240} - 1\right]}{(0.005)} = \$46,204.09$$

The account will grow to $46,204.09 after 20 years.

To find the amount of interest earned, we first need the total of all deposits that are made during the 20 years: $\$100 \times 20 \text{ years } \times 12 \text{ months per year } = \$24,000$. The difference between the balance and the total deposits made is $\$46,204.09 - \$24,000 = \$22,204.09$.

Therefore, the total amount of interest earned is $22,204.09.

To find the percentage of the balance that is interest, divide the interest earned by the total balance: $\frac{\$22,204.09}{\$46,204.09} \approx 0.48056 \approx 48.1\%$.

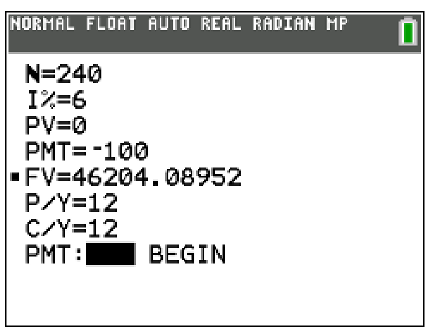
After 20 years, 48.1% of the balance is from accrued interest.

We can also use the *TVM Solver* on the TI calculator to solve this problem. Recall to access the *TVM Solver* press the **APPS** key and then choose the **Finance...** option from the menu. Similar to using the *TVM Solver* to determine the future value for a compound interest problem, we will solve for FV.

N is the number of times a deposit is made and interest is compounded during the entire period. The big difference is that now PMT will show amount of the regular deposit. The PMT amount must include a $'-'$ sign in front so that it identifies the direction of cash flow as away from you.

- $N = 240$: The number of deposits and times compounding is done is $20 \text{ years } \times 12 \text{ months } = 240$.
- $I\% = 6$: Remember to leave this as a percentage. Do not change to a decimal.
- $PV = 0$: There is no money in the account before the first deposit is made.
- $PMT = -100$: This is the amount deposited into the account with the negative sign showing cash flow away from you.
- $P/Y = C/Y = 12$ because deposits are made and the interest is compounded 12 times per year.

Solve for future value FV by moving the cursor back to the value for FV and pressing **ALPHA** and then **ENTER**.

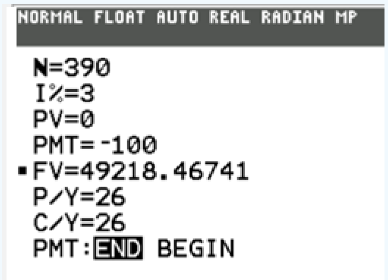


The *TVM Solver* agrees with the result using the formula in Example 1. The balance of the IRA in 20 years will accumulate to $46,204.09.

? Try it Now 1

Professor Mirtova saves for retirement by depositing $100 from each of her biweekly paychecks into a savings annuity with an annual interest rate of 3%. She has been doing this for the past 15 years she has worked at PGCC. How much money has accumulated in her account during this period? How much money did the professor deposit into her annuity during this period? How much interest did Professor Mirtova earn?

Answer



The balance in the account is now \$49,212.47. Professor Mirtova has deposited a total of \$39,000. The amount of interest earned is \$10,212.47.

Sometimes, a "lump sum" may be deposited into a savings plan initially and then followed by regular smaller deposits. The next example will illustrate this type of scenario.

✓ Example 2

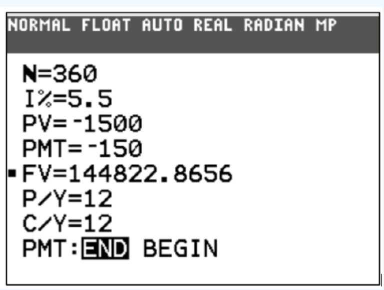
You want to jumpstart your savings for retirement by depositing \$1,500 from your tax refund into an account. You will continue to deposit \$150 every month into the same account. The savings account earns 5.5% interest compounded monthly. How much will be in the account after 30 years?

Solution

To solve this example with a formula, separately use the compound interest formula from Section 5.2 to find future value of the initial \$1,500 deposit and then use the annuity savings formula from this section to find future value of the deposits made during the next 30 years. Finally, add these two future value amounts to find the value of the account.

Solving the problem is much less complicated using the *TVM Solver* and can be done in one step.

- $N = 360$: The number of deposits and times compounding is done is $30 \text{ years } \times 12 \text{ months } = 360$.
- $I\% = 5.5$: Remember to leave this as a percentage. Do not change to a decimal.
- $PV = -1,500$: This is the amount that is initially deposited in the account from the tax refund. Note the negative sign represents cash flow away from you.
- $PMT = -150$: This is the amount deposited into the account with the negative sign showing cash flow away from you.
- $P/Y = C/Y = 12$ because deposits are made and the interest is compounded 12 times per year.

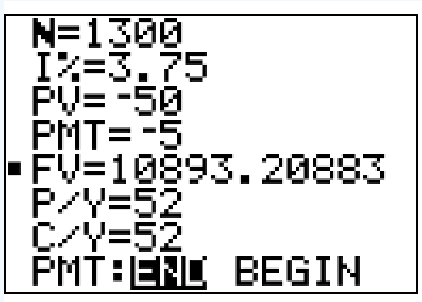


Solving for future value FV, the account will grow to \$144,822.87 after 30 years.

? Try it Now 2

You deposit \$50 into an account and plan to invest \$5 every week into the same account. The account pays interest of 3.75% annually. If you keep up these same deposits for 25 years, how much will accumulate in the account? What is total of the deposits? What interest has accrued?

Answer



The balance in the account is $10,893.21. The total of the deposits is $6,550. The interest that has accrued is $4,343.21.

Another typical application of a savings annuity is to find the amount that should be deposited, d, each term to reach a desired future value. This is especially important in financial planning for retirement or saving to make a large purchase. For example, if you know how much money you need for retirement, how much should you be saving each month?

✔ Example 3

You want to have a half million dollars in your retirement account when you retire in 30 years. Your account earns 8% interest. How much should you deposit each month to meet your retirement goal?

Solution

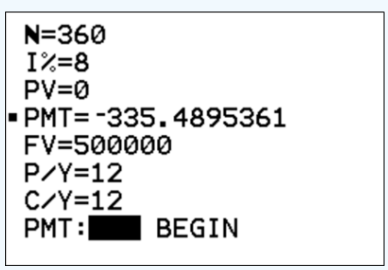
To solve this with a formula, use the second formula solved for d, the regular deposit (payment) amount, where

$A = \$500,000$	the retirement goal (future value)
$r = 0.08$	8% annual rate
$n = 12$	since we're doing monthly deposits, we'll compound monthly
$t = 30$	we want the amount after 30 years

$$d = \frac{A\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]} = \frac{500,000\left(\frac{0.08}{12}\right)}{\left[\left(1+\frac{0.08}{12}\right)^{12\cdot 30}-1\right]} \approx \$335.49$$

You will need to deposit $335.49 each month to have $500,000 in 30 years if the account earns 8% interest.

To find the answer using the *TVM Solver,* we solve for PMT when $FV = \$500,000$ and $N = 30 \times 12 = 360$.



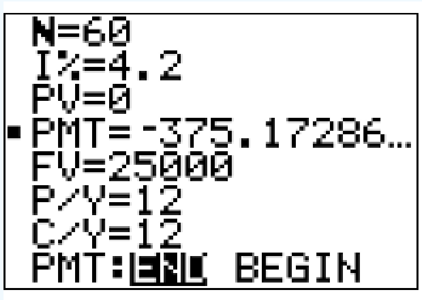
Again, monthly deposits of $335.49 are needed for 30 years to accumulate a half million dollars for retirement

? Try it Now 3

During the next 5 years you will save for the down payment on a house. The houses you are considering cost around $250,000. To approve your mortgage, the bank requires a 10% down payment. Find the down payment amount. Then, find how much you must deposit monthly into a savings annuity that earns 4.2% interest so you will have the required down payment in 5 years.

Answer

The required down payment is $25,000.



To save for this down payment, monthly deposits of $375.17 must be made into the savings annuity.

5.4: Payout Annuities and Loans

In the last section you learned about *savings annuities*. In a savings annuity, you accumulate money by making many deposits on a regular schedule and the total of these deposits grows by accumulating interest.

In this section, we will learn about a second type of annuity called a *payout annuity*. A payout annuity begins as a large amount of money that is then "paid out" through many withdrawals of the same amount of money on a regular schedule. As the withdrawals are paid out, the remaining money in the account continues to earn interest. After a certain amount of time paying out the withdrawals, the account will end up empty.

While a savings annuity can be used to save for retirement, a payout annuity is used as income once reaching retirement. The same payout annuity formulas can be also used to describe paying off conventional loans in which you borrow a certain amount of money and then pay it off periodically by making equal payments. Examples include car loans and home mortgages.

Payout Annuities

Mathematical formulas for a payout annuity are given below. As in the previous sections, we will emphasize using the *TVM Solver* application on the TI calculator but will also demonstrate several examples with a formula.

Payout Annuity Formulas

$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}$$

$$d=\frac{A\cdot\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

where

- A is the balance in the account at the beginning (the starting amount or principal)
- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- r is the annual interest rate (APR) in decimal form
- n is the number of compounding periods in one year
- t is the number of years the withdrawals will continue

In these formulas we assume that withdrawals happen on the same schedule as interest is compounded.

One common scenario involving a payout annuity is to determine the amount you will need to save by the time you retire so that you can withdraw a certain monthly or yearly payment from your savings for several years.

Example 1

After retiring, you want to be able to take \$1,000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$$\begin{array}{ll} d=\$1,000 & \text{the monthly withdrawal} \\ r=0.06 & 6\%\text{ annual rate} \\ n=12 & \text{since we're doing monthly withdrawals, we'll compound monthly} \\ t=20 & \text{we are taking withdrawals for 20 years} \end{array}$$

We're looking for how much money needs to be in the account at the beginning, A.

Putting these values into the first formula:

$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}=\frac{1,000\left[1-\left(1+\frac{0.06}{12}\right)^{-12\cdot 20}\right]}{\left(\frac{0.06}{12}\right)}=\frac{1,000\left[1-(1.005)^{-240}\right]}{0.005}\approx 139,580.77$$

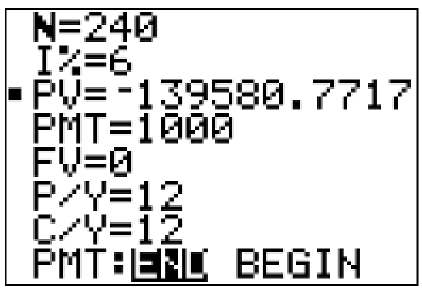
You will need to have $139,580.77 in the account when you retire.

Notice that you withdrew a total of $240,000 ($1,000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case, the interest earned is $\$240,000-\$139,580.77=\$100,419.29$.

We can also use the *TVM Solver* to solve payout annuities. Recall to access the *TVM Solver* press the **APPS** key and then choose the **Finance...** option from the menu. We need to solve for PV where

- $N=240$: The number of withdrawals and times compounding is done is $20 \text{ years } \times 12 \text{ months } = 240$.
- $I\%=6$: Remember to leave this as a percentage. Do not change to a decimal.
- $FV=0$: There is no money in the account after the last withdrawal is made.
- $PMT=1000$: The withdrawal amount is positive because you will receive money from the annuity. The direction of the cash flow is "coming to you."
- $P/Y=C/Y=12$ because deposits are made and the interest is compounded 12 times per year.

Solve for present value PV by moving the cursor back to the value for PV and pressing **ALPHA** and then **ENTER**.

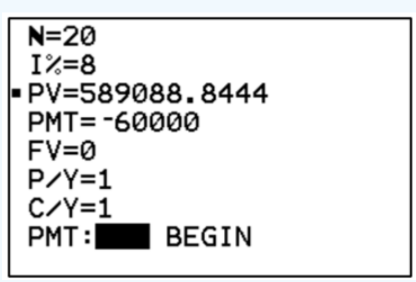


Here, the value of PV is negative and indicates that this is the amount that you have deposited into the annuity account. The cash flow of this amount is "away from you."

? Try it Now 1

Brandon believes he will need $60,000 for living expenses after he retires. How much money should Brandon have at retirement so he can withdraw this amount each year for 20 years from an annuity earning 8% interest compounded annually?

Answer



$589,088.84

Once reaching retirement with a certain amount in your account, you might want to know how much you can withdraw each month. The next example will determine the amount of the regular withdrawal when the beginning amount in the account is known.

Example 2

Denise has $500,000 in her account when she retires. She wants to take monthly withdrawals from the account during the next 30 years. Her retirement account earns 8% interest. How much will Denise be able to withdraw each month?

Solution

In this example, we're solving for d so use the second formula for a payout annuity with the following values:

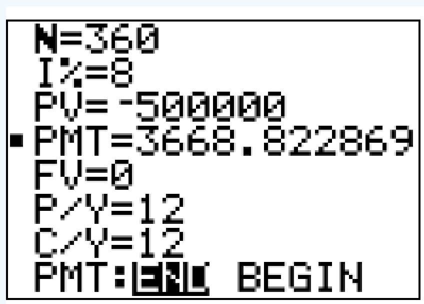
$r = 0.08$	8% annual rate
$n = 12$	since we're doing monthly withdrawals
$t = 30$	since we are taking withdrawals for 30 years
$A = \$500,000$	we are beginning with $\$500,000$

$$d = \frac{A \cdot \left(\frac{r}{n}\right)}{1 - \left(1 + \frac{r}{n}\right)^{-nt}} = \frac{500,000 \cdot \left(\frac{0.08}{12}\right)}{1 - \left(1 + \frac{0.08}{12}\right)^{-12 \cdot 30}} \approx 3668.82$$

Denise can withdraw $3,668.82 each month for 30 years.

Solving for PMT with the *TVM Solver* requires that

- $N = 360$: The number of monthly payments made is 30 years $\times$ 12 months $= 360$.
- $I\% = 8$: Remember to leave this as a percentage. Do not change to a decimal.
- $PV = -500,000$: This is the amount Denise has invested into the annuity. The direction of the cash flow is "going away from Denise."
- $FV = 0$: Nothing will be owed at the end of the loan.
- $P/Y = C/Y = 12$: Payments are made and the interest is compounded monthly (12 times per year.)

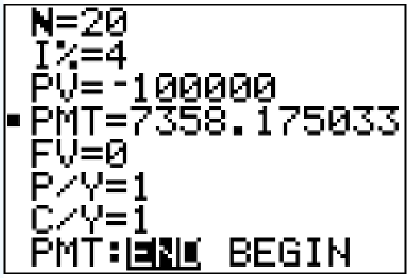


Solving for PMT, Denise can expect to receive $3,668.82 monthly for 30 years.

Try it Now 2

A university has received $100,000 from a donor who specifies that it is to be used for annual scholarships for the next 20 years. If the university invests the donation into an annuity that pays 4% interest, how much can the university give in scholarships each year?

Answer



$7,358.18

Paying Off Loans

Now you will learn about **conventional loans** (also called *amortized loans* or *installment loans*). Examples of these types of loans include auto loans and home mortgages. One great thing about solving problems about loans is that they use exactly the same formulas as the payout annuity. This is because we begin with an amount we owe and make regular payments that reduce the balance until it is $0.

Example 3

You can afford $200 per month as a car payment. If you get an auto loan at 3% interest for 60 months (5 years), how much can you afford to pay for the car? In other words, what loan amount can you pay off with $200 per month?

Solution

In this example, re're looking for the starting value of the loan, A.

$$\begin{array}{ll} d=\$200 & \text{the monthly loan payment} \\ r=0.03 & 3\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly payments, we'll compound monthly} \\ t=5 & \text{we're making monthly payments for 5 years} \end{array}$$

Putting these values into the first formula:

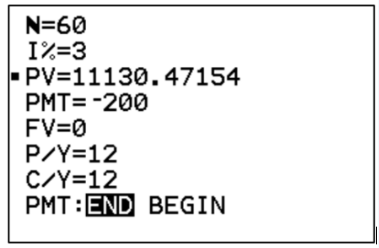
$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)} = \frac{200\left[1-\left(1+\frac{0.03}{12}\right)^{-12\cdot 5}\right]}{\left(\frac{0.03}{12}\right)} = \frac{200\left[1-(1.0025)^{-60}\right]}{0.0025} \approx 11,130.47$$

You can afford a loan in the amount of $11,130.47.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, the amount of interest you're paying to finance the loan is $\$12,000-\$11,130.47=\$869.53$.

To confirm this answer, solve for PV with the *TVM Solver:*

- $N=60$: The number of monthly payments made is $5 \text{ years } \times 12 \text{ months } = 60$.
- $I\%=3$: Remember to leave this as a percentage. Do not change to a decimal.
- $PMT=-200$: The amount of monthly payment is negative. You are making this payment to the loan company and the direction of the cash flow is "going away from you."
- $FV=0$: Nothing will be owed at the end of the loan.
- $P/Y=C/Y=12$: Payments are made and the interest is compounded monthly (12 times per year.)



One of the most common loans people have is a home mortgage. A **home mortgage** is a long-term loan in which the property is pledged as security for payment of the difference between the sales price and the down payment.

Example 4

You want to take out a $140,000 mortgage for a new home. The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example, we're looking for d because we want to know the payment amount. Use the second formula for a payout annuity with the following values:

$$\begin{array}{ll} r=0.06 & \text{6\% annual rate} \\ n=12 & \text{since we're doing monthly payments, we'll compound monthly} \\ t=30 & \text{we're making monthly payments for 30 years} \\ A=\$140,000 & \text{the starting loan amount} \end{array}$$

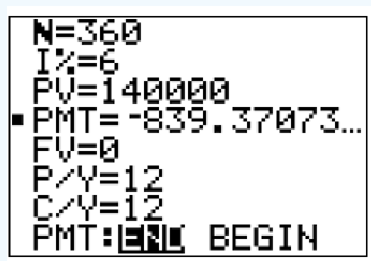
$$d=\frac{A\cdot\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}} = \frac{140,000\cdot\left(\frac{0.06}{12}\right)}{1-\left(1+\frac{0.06}{12}\right)^{-12\cdot 30}} = \frac{140,000\cdot(0.005)}{1-(1.005)^{-240}} \approx 839.37$$

You will make payments of $839.37 per month for 30 years.

You're paying a total of $302,173.20 to the loan company ($839.37 per month for 360 months.) Over the life of the loan, you will pay interest in the amount of $\$302,173.20-\$140,000=\$162,173.20$.

Solving for PMT with the *TVM Solver:*

- $N=360$: The number of monthly payments made is $30 \text{ years } \times 12 \text{ months } =360$.
- $I\%=6$: Remember to leave this as a percentage. Do not change to a decimal.
- $PV=140,000$: The amount of mortgage is positive because the loan company has given you this money. The direction of the cash flow is "coming to you."
- $FV=0$: Nothing will be owed at the end of the loan.
- $P/Y=C/Y=12$: Payments are made and the interest is compounded monthly (12 times per year.)

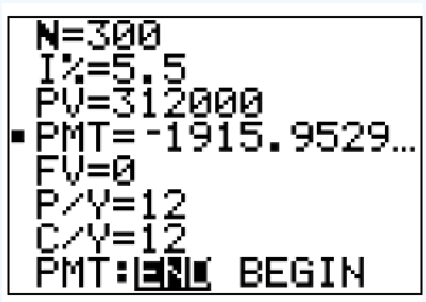


The monthly payment is displayed as a negative amount on the *TVM Solver* because the direction of cash flow is "going away from you." That is, you are paying this amount to the loan company.

Try it Now 4

In order to buy a house Janine is taking a mortgage of $312,000. Her lender will finance it for 25 years at an annual interest rate of 5.5%. How much will Janine's monthly payment be and how much interest will she pay over the life of the loan?

Answer



Janine's monthly payment will be $1,915.95, and she will pay $262,785 in interest.

Example 5

Liz is buying a house that costs $180,000. Her bank requires a 15% down payment. The mortgage rate is 6% for a 30 year loan. To obtain this interest rate, her bank requires an upfront payment of "2 points" at the time of closing. What is Liz's monthly mortgage payment? What is the total amount that Liz actually pays for her house?

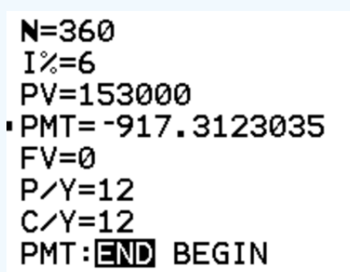
Solution

First we need to find Liz's required down payment: $\$180,000 \times 0.15 = \$27,000$.

Liz must pay the down payment amount on her own, so Liz's mortgage will be in the amount of $\$180,000 - \$27,000 = \$153,000$.

The bank also requires Liz to pay "2 points" upfront. *Points* refer to "percentage points" of the mortgage: 2 points amount to 2% of the mortgage or $\$153,000 \times 0.02 = \$3,060.00$.

Now, we find the monthly mortgage payment using *TVM Solver* settings similar to those in Example 4.



Therefore, Liz's monthly mortgage payment will be $917.31.

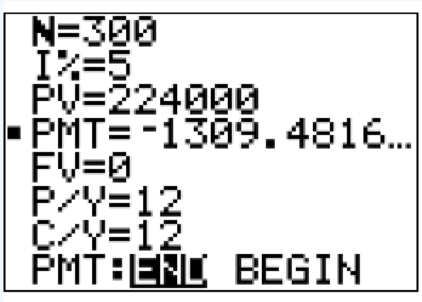
The total amount Liz pays for the house includes the *down payment* + *points* + *monthly payments for 30 years:* $\$27,000 + \$3,060 + (\$917.31 \times 12 \times 30) = \$360,291.60$.

Try it Now 5

Charles buys a house that costs $280,000. His bank requires a 20% down payment. The mortgage rate is 5% when 1 point is paid upfront at the time of closing. The term is 25 years.

Find Charles' down payment, mortgage amount, monthly mortgage payment, and total house cost over the 25 years.

Answer



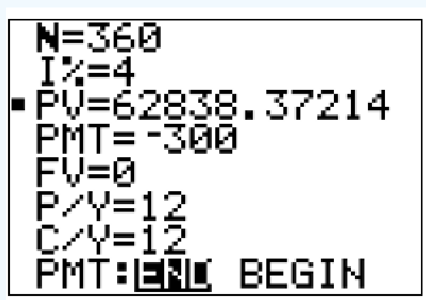
The down payment is \$56,000. The mortgage amount is \$224,000. The monthly payment is \$1,309.48. Charles will pay a total of \$451,084 to buy the house.

✔ Example 6

You want to buy a house, but how much can you afford to pay for it? The highest monthly payment you can afford is \$300. You are sure that you can secure a 30-year mortgage from your credit union at 4% interest, but a condition of the mortgage requires at least a 10% down payment. What is the price of the house you can afford to buy?

Solution

First, find the amount of a mortgage you can pay off with a monthly payment of \$300 every month for 30 years. Using the TVM Solver, we solve for PV:



Your mortgage amount can be as high as \$62,838.37.

Your down payment is 10% of the sales price which means the mortgage amount is 90% of the sales price. To find the sales price you can afford, we can set up a proportion and solve for the sale price:

$$\frac{\text{mortgage amount}}{\text{sales price}} = \frac{\$62,838.37}{\text{sales price}} = \frac{90\%}{100\%}$$

Solve this equation by cross multiplying to obtain: $90\%(\text{sales price}) = 100\%(\$62,838.37)$

Finally, divide by 90% to get $\frac{\$62,838.37}{90\%} = \$69,820.41$

5.5: Review Exercises

1. You borrow $500 for a trip at 11% annual simple interest for two years.
    a. Find the interest you will pay on the loan.
    b. How much will you have to pay the bank at the end of the two years?
2. A U.S. Treasury Bond earns 5.2% annual simple interest. If Suzy earns $4,160 in interest after 10 years, how much was Suzy's bond worth originally?
3. Darren deposited $1,500 into a savings account and earned $180 of simple interest at a 4% annual interest rate. How long was Darren's money in the account?
4. Find the annual simple interest rate on an account if you invest $1,000 and it is worth $1,200 three years later. *Round to the nearest hundredth of a percent*.
5. Derek borrowed $1,500 from his parents at a 5.55% annual simple interest rate for 18 months. How much will Derek pay back to his parents?
6. A payday loan company offers a $500 loan for 60 days. It charges you a fee of a $50 fee for the loan. Considering this fee is really simple interest, what is the annual interest rate charged by the company?

Solve these problems, using the *TVM Solver* app where appropriate.

7. You deposit $300 in an account earning 5% interest compounded annually. How much will be in the account in 10 years?
8. You deposit $2,000 in an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 20 years?
    b. How much interest will you earn during this time?
9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. Suppose you need $1,250 to purchase a new TV in 3 years. You have $1,000 to deposit into a savings account now. What annual interest rate is needed if the interest is compounded monthly? *Round to the nearest hundredth of a percent.*
11. Amira deposited $2,000 into a savings account earning 4.59% APR compounded monthly. How many years will it take her original deposit to grow to $5,000?
12. You deposit $200 every month for 30 years into an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 30 years?
    b. How much total money will you have put into the account?
    c. How much interest will you have earned during those 30 years?
13. At age 30, Suzy starts an IRA to save for retirement. She wants to have $750,000 by the time she retires in 35 years at age 65. If she can count on an APR of 6%, how much should she be putting in the IRA each month?
14. Business Enterprises is saving up some money to expand. The company deposits $25,000 into an account initially and then begins depositing $5,000 every quarter thereafter. The account earns an APR of 7.2% compounded quarterly. How much money will the company have in the account after 5 years?
15. You would like to have $20,000 to spend on a new car in five years. You open a savings account with an APR of 4.85%. How much must you deposit each week to reach this goal?
16. You want to be able to withdraw $30,000 each year from an annuity for 25 years. Your account earns 8% interest annually.
    a. How much do you need in your account at the beginning of the 25 years?
    b. How much total money will you withdraw from the account over 25 years?
    c. How much of the money you withdraw is interest that you earned?
17. Loren already knows that he will have $1,000,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide in income each month?
18. Chris has saved $200,000 for retirement, and it is in an account earning 6% interest. She plans to withdraw $3,000 every month. How many months will her retirement fund last?
19. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years) with no money down. What will your monthly payments be?

20. You have a credit card balance of $1,245 to pay off. The interest rate on the balance is 10% compounded monthly. You decide to pay $50 each month towards the debt and will no longer charge anything to the account.
    a. To the nearest whole month, how long will it take to pay off the credit card?
    b. How much will you pay the credit card company during this time?
    c. What part of your total payments will be the interest?
21. Lynn bought a $200,000 condo. Her down payment for the condo was 20% and she financed the rest through a 8% mortgage for 30 years.
    a. What was the amount of Lynn's down payment?
    b. How much did Lynn borrow for the mortgage?
    c. What is Lynn's monthly payment?
    d. How much will Lynn pay in all over the 30 years?
    e. How much does Lynn pay in interest over the 30 years?
    f. If Lynn could find a better mortgage rate of 6.5%, how much cheaper would her monthly payment be?
22. Marie can afford a $250 per month car payment. She’s found a 5-year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?

CHAPTER OVERVIEW

6: Voting and Apportionment

6.1: Voting Methods

Every couple of years or so, voters go to the polls to cast ballots for their choices for mayor, governor, senator, president, and so on. Then, election officials count the ballots and declare a winner. But how do the election officials determine who the winner is?

If there are only two candidates, then there is no problem figuring out the winner. The candidate with more than 50% of the votes wins. This is known as the *majority*. So the candidate with the majority of the votes is the winner.

Majority Rule

Majority rule means that the candidate or choice receiving more than 50% of all the votes is the winner.

But what happens if there are three candidates, no one receives more than 50%, and so no candidate receives a majority? The answer to that question depends on where you live.

Some places decide that the person with the most votes wins, even if they don't have a majority. There are problems with this in that someone could be liked by 35% of the people, but is disliked by 65% of the people. So you have a winner that the majority doesn't like. Other places conduct *runoff elections* where the top two candidates have to run again, and then the winner is chosen from the runoff election. There are some problems with this method. First, it is very costly for the candidates and the election office to hold a second election. Second, you don't know if you will have the same voters voting in the second election, and so the preferences of the voters in the first election may not be taken into account.

Preference Voting

What can be done to have a better election that has someone liked by more voters yet doesn't require a runoff election? A ballot method that can fix this problem is known as a *preference ballot*. In this method, voters choose not only their favorite candidate but also their second favorite, their third favorite, and so on. You also may hear this referred to as *ranked choice voting*.

Preference Ballots

Ballots in which voters choose not only their favorite candidate, but voters actually order all of the candidates from their most favorite down to their least favorite.

To understand how a preference ballot works and how to determine the winner, we will look at an example.

Example 1A

Suppose an election is held to determine which bag of candy will be opened. The choices (candidates) are *Hershey's Miniatures* (M), *Nestle Crunch* (C), and *Mars' Snickers* (S). Each voter is asked to complete a ballot by marking their first, second, and third place choices.

Here are the ballots cast by the 18 voters.

Ballots Cast for the Candy Election

Voter	Anne	Bob	Chloe	Dylan	Eli	Fred
1st choice	C	M	C	M	S	S
2nd choice	S	S	M	C	M	M
3rd choice	M	C	S	S	C	C

Voter	George	Hiza	Isha	Jacy	Kalb	Lan
1st choice	S	S	S	M	C	M
2nd choice	M	M	M	C	M	C
3rd choice	C	C	C	S	S	S

Voter	Makya	Nadira	Ochen	Paki	Quinn	Riley
1st choice	S	S	C	C	S	S
2nd choice	M	M	M	M	M	M
3rd choice	C	C	S	S	C	C

Now we must count the ballots. It isn't as simple as just counting how many voters like each choice of candy. We have to look at how many liked the candidate in first place, second place, and third place. So there needs to be a better way to organize the results. This is known as a *preference schedule.*

Preference Schedule

A preference schedule is a table used to organize the results of all the preference ballots in an election.

Example 1B

Create a preference schedule for the ballots in the the Candy Election.

Solution

We are interested in counting how many voters chose each ordering. Looking at the preference ballots from Example 1A, you can see that

- Dylan, Jacy, and Lan all voted for the same order: M, then C, then S.
- Bob is the only voter with the order: M, then S, then C.
- Chloe, Kalb, Ochen, and Paki chose the order C, then M, then S.
- Anne is the only voter who voted C, then S, then M.
- The other 9 voters selected the order S, then M, then C.
- No voter selcted the order S, then C, then M, so this will not be included in the preference table.

We can summarize and present this information in the preference schedule.

Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

Methods of Determining a Winner

Once voters have voted, it is time to determine a winner. There are several different methods that be used to determine a winner of an election. The ones we will discuss here are

- Plurality Method,
- Borda Count method,
- Plurality with Elimination Method, and
- Pairwise Comparisons Method.

Each of these methods will be illustrated using the Candy Election ballots.

Plurality Method

The easiest, and the most common method in the United States, is the *Plurality Method*. In this method, the choice with the most first-preference votes is declared the winner. Only the first choice matters in the vote counting, and all other preferences are ignored.

Plurality Method

The choice with the most first-place votes is declared the winner.

Example 1C

Use the preference schedule from the Candy Election to find the winner using the Plurality Method.

Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

Solution

From the preference schedule you can see that

- $3+1=4$ people choose Hershey's Miniatures as their first choice,
- $4+1=5$ people picked Nestle Crunch as their first choice, and
- 9 people picked Snickers as their first choice.

So, Snickers wins with the most first-place votes, although Snickers does not have the majority. A candy would need more than 50% of the 18 votes, or *more than* 9 votes, to win a majority.

There is a problem with the Plurality Method. Notice that 9 people picked Snickers as their first choice, yet 7 chose it as their third choice. Thus, 9 people may be happy if the Snickers bag is opened, 7 voters will not be happy at all. So let's look at another way to determine the winner that might be more pleasing to the entire group of voters.

Borda Count Method

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770. In this method, points are assigned to candidates based on their ranking. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Borda Count Method

Each place on a preference ballot is assigned points. Last place receives one point, next-to-last place receives two points, and so on. Thus, if there are N candidates, then first-place receives N points. Now, multiply the point value for each place by the number of voters at the top of the column of the preference schedule to find the points each candidate wins in a column. Lastly, total up all the points for each candidate. The candidate with the most points wins.

Example 1D

Use the preference schedule from the Candy Election to find the winner using the Borda Count Method.

Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

Solution

Third choice receives one point, second choice receives two points, and first choice receives three points.

- There were 3 voters who chose the order M, C, S. So, M receives $3 \cdot 3 = 9$ points for first place, C receives $3 \cdot 2 = 6$ points, and S receives $3 \cdot 1 = 3$ points for those ballots.
- There was 1 voter who chose the order M, S, C. So, M receives $1 \cdot 3 = 3$ points for first place, C receives $1 \cdot 2 = 2$ points, and S receives $1 \cdot 1 = 1$ points for those ballots.
- There were 4 voters who chose the order C, M, S. So, C receives $4 \cdot 3 = 12$ points for first place, M receives $4 \cdot 2 = 8$ points, and S receives $4 \cdot 1 = 4$ points for those ballots.
- The same process is conducted for the other columns. The table below summarizes the points that each candy received.

Preference Schedule of the Candy Election with Borda Count Points

Number of voters	3	1	4	1	9
1st choice 3 points	M $3 \cdot 3 = 9$	M $1 \cdot 3 = 3$	C $4 \cdot 3 = 12$	C $1 \cdot 3 = 3$	S $9 \cdot 3 = 27$
2nd choice 2 points	C $3 \cdot 2 = 6$	S $1 \cdot 2 = 2$	M $4 \cdot 2 = 8$	S $1 \cdot 2 = 2$	M $9 \cdot 2 = 18$
3rd choice 1 point	S $3 \cdot 1 = 3$	C $1 \cdot 1 = 1$	S $4 \cdot 1 = 1$	M $1 \cdot 1 = 1$	C $9 \cdot 1 = 9$

Adding up these points gives,

$$\text{M: } 9+3+8+1+18=39 \quad \text{C: } 6+1+12+3+9=31 \quad \text{S: } 3+2+4+2+27=38$$

Thus, Hershey's Miniatures wins using the Borda Count Method with 39 points.

So who is the winner? With one method, Snicker's wins. With another method, Hershey's Miniatures wins. It all depends on which method you use. Therefore, you need to decide which method to use before you run the election.

Plurality with Elimination Method

The *Plurality with Elimination Method* should only be used when the Plurality Method winner does not have a majority. The choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%). This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election. A version of the Plurality with Elimination Method is used by the International Olympic Committee to select host nations.

Plurality with Elimination Method

Eliminate the candidate with the least amount of 1st place votes and re-distribute their votes amongst the other candidates according to their 2nd place preference. Repeat this process until a winner is decided. At any time during this process if a candidate has a majority of first-place votes, then that candidate is the winner.

Example 1E

Use the preference schedule from the Candy Election to find the winner using the Plurality with Elimination Method.

Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

Solution

This isn't the most exciting example, since there are only three candidates, but the process is the same whether there are three or many more.

First, look at how many first-place votes there are: M has $3+1=4$ votes, C has $4+1=5$, and S has 9 votes. There are a total of 18 votes and majority would be 10 votes. No candidate has received a majority, but M has the fewest number of votes so M is eliminated from the preference schedule.

Preference Schedule for the Candy Election with M Eliminated

Number of voters	3	1	4	1	9
1st choice	~~M~~	~~M~~	C	C	S
2nd choice	C	S	~~M~~	S	~~M~~
3rd choice	S	C	S	~~M~~	C

After M is removed from the election and the next preferences are used, the preference schedule becomes

Preference Schedule for the Candy Election with M Eliminated

Number of voters	3	1	4	1	9
1st choice	C	S	C	C	S
2nd choice	S	C	S	S	C

And then we can combine all the same preferences down into a simpler table:

Preference Schedule for the Candy Election Condensed

Number of voters	8	10
1st choice	C	S
2nd choice	S	C

So C has 8 first-place votes, and S has 10 first-place votes. S has achieved a majority. Thus, Snicker's wins using the Plurality with Elimination Method.

Pairwise Comparisons Method

The final method we will examine is the *Pairwise Comparisons Method*. In this method, each pair of candidates is compared, using all preferences to determine which of the two is "more preferred." The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $1/2$ point. After all pairwise comparisons are made, the candidate with the most points is declared the winner.

Pairwise Comparisons Method

Compare each candidate to the other candidates in one-on-one match-ups. Give the winner of each pairwise comparison a point. Give each candidate $1/2$ point if there is a tie. The candidate with the most points wins.

Example 1F

Use the preference schedule from the Candy Election to find the winner using the Pairwise Comparisons Method.

Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M

3rd choice	S	C	S	M	C

Solution

Taking only two candidates at a time, there are 3 one-on-one matchups possible: M vs. C, M vs. S, and S vs. C.

If you only have an election between M and C, then M wins the 3 votes in the first column, the 1 vote in the second column, and the 9 votes in the last column. C receives the rest of the votes. That means that M has $3+1+9=13$ votes while C has $4+1=5$ votes. So, M wins when compared to C. M gets 1 point.

If you only compare M and S (the next one-on-one match-up), then M wins the 3 votes in the first column, the 1 vote in the second column, and the 4 votes in the third column. S receives the rest of the votes. That means that M has $3+1+4=8$ votes while S has $1+9=10$ votes. So, S wins when compared to M. S gets 1 point.

Finally, if you compare C and S (the last one-on-one-match-up), then C wins the 3 votes in the first column, the 4 votes in the third column, and the 1 vote in the fourth column. S receives the rest of the votes. That means that C has $3+4+1=8$ votes while S has $1+9=10$ votes. So, S wins when compared to C. S gets 1 point.

The table summarizes the results.

Summary of One-on-One Match-Ups for the Candy Election

Match-Up 1	Match-Up 2	Match-Up 3
M vs. C	M vs. S	S vs. C
13 to 5	8 to 10	10 to 8
Winner of Match-Up 1: M	Winner of Match-Up 2: S	Winner of Match-Up 3: S

- M: 1 point
- S: 2 points
- C: 0 points

Thus, Snickers wins the election using the Pairwise Comparisons Method.

The problem with this method is that many overall elections (not just the one-on-one match-ups) will end in a tie, so you need to have a tie-breaker method designated before beginning the tabulation of the ballots. Another problem is that if there are more than three candidates, the number of pairwise comparisons that need to be analyzed becomes unwieldy. So, how many pairwise comparisons are there?

In Example 1F, there were 3 one-on-one comparisons when there were 3 candidates. You may think that means the number of pairwise comparisons is the same as the number of candidates, but that is not correct. Let's see if we can come up with a way of knowing how many one-on-one matchups will be needed from the number of candidates.

Suppose you have 4 candidates called A, B, C, and D. A is to be matched up with B, C, and D (3 comparisons). B is to be compared with C and D, but has already been compared with A (2 comparisons). C needs to be compared with D, but has already been compared with A and B (1 more comparison). Therefore, the total number of one-on-one match-ups is $3+2+1=6$ comparisons that need to be made with 4 candidates.

What about 5 or 6, or more candidates? Looking at 5 candidates, the first candidate needs to be matched-up with 4 other candidates, the second candidate needs to be matched-up with 3 other candidates, the third candidate needs to be matched-up with 2 other candidates, and the fourth candidate needs to only be matched-up with the last candidate for 1 more match-up. Thus, the total is $4+3+2+1=10$ pairwise comparisons when there are 5 candidates. For 6 candidates, you would have $5+4+3+2+1=15$ pairwise comparisons to do. For small numbers of candidates, it isn't hard to add these numbers up, but for large numbers of candidates there is a shortcut for adding numbers together.

It turns out that the following formula is true: $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$. So, if there are n candidates, then there are $\frac{n(n-1)}{2}$ pairwise comparisons that can be made. For example, in an election with 10 candidates, there are $9+8+7+6+5+4+3+2+1=\frac{10(10-1)}{2}=\frac{10(9)}{2}=45$ pairwise comparisons.

As you can see from, the winner of an election can change depending on which voting method is used. The remainder of this section provides more examples using each of the four voting methods we have discussed.

✓ Example 2

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Ten voters' individual ballots are shown below. Summarize the results into a preference schedule and decide the winner using the Plurality Method.

AOH, AHO, OHA, HAO, AHO, OHA, HAO, OHA, HAO, AHO

Solution

- 1 voter has the preference AOH,
- 3 voters have the preference AHO,
- 3 voters have the preference OHA, and
- 3 voters have the preference HAO.

The preference schedule for this election is

	1	3	3	3
1^{st} choice	A	A	O	H
2^{nd} choice	O	H	H	A
3^{rd} choice	H	O	A	O

For the plurality method, we only care about the first-choice options. Totaling them up,

- A: 1+3 = 4 first-choice votes,
- O: 3 first-choice votes, and
- H: 3 first-choice votes.

Anaheim is the winner using the Plurality Method. Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality but not a majority of the votes. Anaheim did not receive a majority of the vote because 4 is not more than 50% of 10 votes.

? Try it Now 1

Three candidates are running in an election for County Executive: Goings (G), McCarthy (M), and Bunney (B). The preference schedule is shown below. Which candidate wins using the Plurality Method? Does the winning candidate receive a majority of first place votes?

	54	24	70	22	119
1^{st} choice	G	G	M	M	B
2^{nd} choice	M	B	G	B	M
3^{rd} choice	B	M	B	G	G

Answer

Using the Plurality method,

- G gets $54 + 24 = 78$ first-choice votes,
- M gets $70 + 22 = 92$ first-choice votes, and
- B gets 119 first-choice votes.

Bunney (B) wins under the Plurality Method. There are a total of $78 + 92 + 119 = 289$ votes, but 119 is only about 41% of the votes. So, Bunney did not receive a majority.

Example 3

A group of mathematicians are getting together for a conference. The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown to the right. The votes for where to hold the conference are summarized in the preference schedule.

Where should the conference be held if the decision is made using the Borda Count Method?

	51	25	10	14
1^{st} choice	Seattle	Tacoma	Puyallup	Olympia
2^{nd} choice	Tacoma	Puyallup	Tacoma	Tacoma
3^{rd} choice	Olympia	Olympia	Olympia	Puyallup
4^{th} choice	Puyallup	Seattle	Seattle	Seattle

Solution

For each of the 51 ballots in the first column of the preference table, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate points awarded. We proceed through the remaining 25, 10, and 14 ballots in the table.

	51	25	10	14
$1^{\text{st choice}}$ 4 points	Seattle $4 \cdot 51 = 204$	Tacoma $4 \cdot 25 = 100$	Puyallup $4 \cdot 10 = 40$	Olympia $4 \cdot 14 = 56$
$2^{\text{nd choice}}$ 3 points	Tacoma $3 \cdot 51 = 153$	Puyallup $3 \cdot 25 = 75$	Tacoma $3 \cdot 10 = 30$	Tacoma $3 \cdot 14 = 42$
3^{rd} choice 2 points	Olympia $2 \cdot 51 = 102$	Olympia $2 \cdot 25 = 50$	Olympia $2 \cdot 10 = 20$	Puyallup $2 \cdot 14 = 28$
4^{th} choice 1 point	Puyallup $1 \cdot 51 = 51$	Seattle $1 \cdot 25 = 25$	Seattle $1 \cdot 10 = 10$	Seattle $1 \cdot 14 = 14$

Adding up the points for each city,

- Seattle: $204 + 25 + 10 + 14 = 253$ points,
- Tacoma: $153 + 100 + 30 + 42 = 325$ points,
- Puyallup: $51 + 75 + 40 + 28 = 194$ points, and
- Olympia: $102 + 50 + 20 + 56 = 228$ points.

Under the Borda Count Method, Tacoma is the winner of this vote. Note that, however, if this election were decided using the Plurality Method, Seattle would win with the most votes.

Try it Now 2

Refer to the election in *Try It Now 1*. The preference schedule is shown below. Which candidate wins using the Borda Count Method?

	54	24	70	22	119
1^{st} choice	G	G	M	M	B
2^{nd} choice	M	B	G	B	M
3^{rd} choice	B	M	B	G	G

Answer

Borda points are calculated as shown in the table.

	54	24	70	22	119
1^{st} choice 3 points	G $54 \cdot 3 = 162$	G $24 \cdot 3 = 72$	M $70 \cdot 3 = 210$	M $22 \cdot 3 = 66$	B $119 \cdot 3 = 357$
2^{nd} choice 2 points	M $54 \cdot 2 = 108$	B $24 \cdot 2 = 48$	G $70 \cdot 2 = 140$	B $22 \cdot 2 = 44$	M $119 \cdot 2 = 238$
3^{rd} choice 1 point	B $54 \cdot 1 = 54$	M $24 \cdot 1 = 24$	B $70 \cdot 1 = 70$	G $22 \cdot 1 = 22$	G $119 \cdot 1 = 119$

- G: $162 + 72 + 140 + 22 + 119 = 515$ points
- M: $108 + 24 + 210 + 66 + 238 = 646$ points
- B: $54 + 48 + 70 + 44 + 357 = 573$ points

McCarthy (M) is the winner using Borda Count.

Example 5

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans, called A, B, C, D, and E here for simplicity. The choice of slogan will be made using the Plurality with Elimination Method.

	3	4	4	6	2	1
1^{st} choice	B	C	B	D	B	E
2^{nd} choice	C	A	D	C	E	A
3^{rd} choice	A	D	C	A	A	D
4^{th} choice	D	B	A	E	C	B
5^{th} choice	E	E	E	B	D	C

Solution

If this was a plurality election, slogan B would win with 9 first-choice votes, compared to 0 for slogan A, 4 for slogan C, 6 for slogan D, and 1 for slogan E. There are total of $3 + 4 + 4 + 6 + 2 + 1 = 20$ votes. A majority would be 11 votes. No choice yet has a majority, so we proceed to elimination rounds.

Round 1: We make the first elimination. Choice A has the fewest first-place votes (0), so we remove that choice from the preference table.

	3	4	4	6	2	1
1^{st} choice	B	C	B	D	B	E
2^{nd} choice	C		D	C	E	
3^{rd} choice		D	C			D
4^{th} choice	D	B		E	C	B
5^{th} choice	E	E	E	B	D	C

We then shift everyone's choices up to fill the gaps.

	3	4	4	6	2	1
1^{st} choice	B	C	B	D	B	E
2^{nd} choice	C	D	D	C	E	D
3^{rd} choice	D	B	C	E	C	B
4^{th} choice	E	E	E	B	D	C

After the first elimination, slogan B has 9 first-choice votes, compared to 4 for slogan C, 6 for slogan D, and 1 for slogan E. There is still no choice with a majority, so we eliminate again.

Round 2: We make our second elimination. Choice E has the fewest first-place votes, so we remove that choice, shifting everyone's options to fill the gaps.

	3	4	4	6	2	1
1^{st} choice	B	C	B	D	B	D
2^{nd} choice	C	D	D	C	C	B
3^{rd} choice	D	B	C	B	D	C

Now slogan B has 9 first-choice votes, slogan C has 4 votes, and slogan D has 7 votes. Still no choice has reached a majority of 11, so we eliminate again.

Round 3: We make our third elimination. Slogan C has the fewest votes and is eliminated.

	5	4	4	6	1
1^{st} choice	B	D	B	D	D
2^{nd} choice	D	B	D	B	B

Now, slogan B has 9 first-place votes while slogan D has 11. Slogan D has now gained a majority and is declared the winner using Plurality with Elimination.

Try it Now 3

Refer to the election in *Try It Now 1*. The preference schedule is shown below. Which candidate wins using the Plurality with Elimination Method?

	54	24	70	22	119
1^{st} choice	G	G	M	M	B
2^{nd} choice	M	B	G	B	M
3^{rd} choice	B	M	B	G	G

Answer

Initial Votes: G has 78 first-place votes, M has 92 first-place votes, and B has 119 first place votes.

G is eliminated, leaving only M and B.

	54	24	70	22	119
1^{st} choice	M	B	M	M	B
2^{nd} choice	B	M	B	B	M

Now, M has 146 votes and B has 143 votes. Candidate M is the winner and has gained a majority over Candidate B.

Example 6

A club is holding elections for president and will use the Pairwise Comparisons Method to decide the winner. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule for the election is shown below.

	14	10	8	4	1
1^{st} choice	A	C	D	B	C
2^{nd} choice	B	B	C	D	D
3^{rd} choice	C	D	B	C	B
4^{th} choice	D	A	A	A	A

Solution

With 4 candidates, there are $3+2+1=6$, or $\frac{4(3)}{2}$, comparisons to make:

A vs B: 14 prefer A, and $10+8+4+1=23$ prefer B	B gets 1 point
A vs C: 14 prefer A, and $10+8+4+1=23$ prefer C	C gets 1 point
A vs D: 14 prefer A, and $10+8+4+1=23$ prefer D	D gets 1 point
B vs C: $14+4=18$ prefer B, and $10+8+1=19$ prefer C	C gets 1 point
B vs D: $14+10+4=28$ prefer B, and $8+1=9$ prefer D	B gets 1 point
C vs D: $14+10+1=25$ prefer C, and $8+4=12$ prefer D	C gets 1 point

So, A gets 0 points, B gets 2 points, C gets 3 points, and D gets 1 point.

Using the Pairwise Comparison Method, we declare C as the winner.

❓ Try it Now 4

Refer to the election in *Try It Now 1*. The preference schedule is shown below. Which candidate wins using the Pairwise Comparisons Method?

	54	24	70	22	119
1^{st} choice	G	G	M	M	B
2^{nd} choice	M	B	G	B	M
3^{rd} choice	B	M	B	G	G

Answer

G vs M: 78 prefer G, and 211 prefer M	M gets 1 point
G vs B: 148 prefer G, and 141 prefer B	G gets 1 point
M vs B: 146 prefer M, and 143 prefer B	M gets 1 point

So, G gets 1 point, M gets 2 points, and B gets 0 points.

Using the Pairwise Comparison Method, we declare M as the winner.

6.2: Apportionment Methods

The college has hired 9 computer technicians and will assign them to monitor the 4 computer labs on campus. However, each computer lab contains a different number of computer workstations. It sounds like a fairly simple job to allocate the 9 technicians but 9 is not divisible by 4. And even if the technicians could be evenly divided by 4, the computer labs are different sizes so the workload would not be the same for each lab. The question is how to assign 9 technicians in a fair way that takes into consideration the different sizes of the labs. *Apportionment methods* can help us come up with an equitable solution to allocate the technicians.

Apportionment

Apportionment is the the problem of dividing up a fixed number of items or resources among groups of different sizes.

Basic Concepts of Apportionment

The apportionment methods we will look at in this chapter were all created as a way to divide the seats in the U.S. House of Representatives among the states based on the size of the population for each state. The terminology we use in apportionment reflects this history. An important concept is that the number of *seats* a state has in Congress is proportional to the population of the state. In other words, states with large populations get lots of seats and states with small populations only get a few seats.

However, apportionment problems do not need to occur only within the context of dividing seats in Congress among the states. Apportionment can be used in other scenarios where a certain number of items need to be divided among groups of different sizes. For example, we might wish to fairly allocate 20 new employees among 3 hotels based on how many rooms each of the hotels have.

The first step in any apportionment problem is to calculate the *standard divisor*. This is the ratio of the total population of all the groups to the number of items to be shared. When apportioning the seats in the U.S. House of Representatives today, the total population is the population of the 50 states in the United States, and the number of items to be shared is 435 seats.

Standard Divisor

The **standard divisor** (SD) for a population is the quotient of the total population divided by the number of items to be allocated (seats.)

$$\mathrm{SD}=\frac{\text{total population}}{\#\text{ of items}}$$

The next step is to find the *standard quota* for each group in the population. This is the exact number of items that should be allocated to each group. If we were apportioning representatives to the states, this would be the exact number of seats that each state should get if decimal values were possible.

Standard Quota

A **standard quota** (SQ) for a group is the quotient of the size of the group (state population) divided by the standard divisor.

$$\mathrm{SQ}=\frac{\text{group population}}{\text{standard divisor}}$$

Example 1A

Hamiltonia is a small (fictional) country consisting of six states. It is governed by a senate with 25 members. The number of senators for each state is proportional to the population of the state. The following table shows the population of each state as of the last census.

Find the standard divisor and the standard quotas for each of the states of Hamiltonia.

Population of the States of Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

Solution

The standard divisor is the total population divided by the number of items (or seats) to be allocated: $SD = \frac{\text{total population}}{\#\text{ of items}} = \frac{237{,}000}{25} = 9,480$. This tells us that each seat in the senate should correspond to a group of 9,480 people.

Next, we need to find how many of these groups of 9,480 people belong to each state. To do this, find the standard quota for each state:

- Alpha: $SQ = \frac{\text{group population}}{\text{standard divisor}} = \frac{24{,}000}{9480} = 2.532$ If fractional seats were possible, Alpha would get 2.532 seats.
- Beta: $SQ = \frac{\text{group population}}{\text{standard divisor}} = \frac{56{,}000}{9480} = 5.907$. If fractional seats were possible, Beta would get 5.907 seats.
- Similar calculations for the other states have been done and placed in the table.

Standard Quotas for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001

Notice that the sum of the standard quotas is 25.001, the total number of seats. This is a good way to check your arithmetic. (Do not worry about the 0.001. That is due to rounding and is negligible.)

The standard quota for each state is usually a decimal number, but in real life the number of seats allocated to each state must be a whole number. For example, it would be impossible to give Alpha 2.532 senators. We could give this state 2 senators or 3 senators, but not 2.532. Rounding off the standard quota by the usual method of rounding does not always work. Sometimes the total number of seats allocated is too high and other times it is too low. In Example 1A, the total number of seats allocated would be $3+6+3+2+7+5=26$ if we used the usual rounding rule. But remember, there are only 25 seats to allocate.

When we round off the standard quota for a state, the result should be the whole number just below the standard quota. This value is called the **lower quota.** In the extremely rare case that the standard quota is a whole number, use the standard quota for the lower quota.

✔ Example 1B

Use the results from Example 1A to find the lower quotas for each of the states in Hamiltonia.

Solution

Round the each standard quota down to find the lower quota.

Lower Quotas for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001
Lower Quota	2	5	2	1	6	4	20

The total of the lower quotas is 20 (below the number of seats to be allocated.)

This primary issue in apportioning is what to do with the "left-overs." As you can see from Example 1A, if we round the standard quota according to normal rounding rules, we will assign more seats than we have. As you can see from Example 1B, if we give every state its lower quota, there will be 5 seats unassigned to states.

Apportionment Methods and the United States Congress

The U.S. Constitution requires that the seats for the House of Representatives be apportioned among the states every ten years based on the sizes of the populations. Through U.S. history, the number of states has changed, and the population has grown. The number of seats in the House has also changed many times. In 1941, the number of seats in the House was fixed at 435 and remains this number today.

Since 1792, five different apportionment methods have been proposed, and four of these methods have been used to apportion the seats in the House of Representatives. In many situations the five methods give the same results. However, in some situations, the results depend on the method used. We will look at only two basic apportionment methods in this section: *Hamilton's method* and *Jefferson's method*.

Hamilton’s Method

Alexander Hamilton proposed the first apportionment method to be approved by Congress. Unfortunately for Hamilton, President Washington vetoed its selection. This veto was the first presidential veto utilized in the new U.S. government. A different method proposed by Thomas Jefferson was used instead for the next 50 years. Later, Hamilton’s method was used off and on between 1852 and 1901.

Summary of Hamilton's Method

1. Use the standard divisor to find the standard quota for each state.
2. Temporarily allocate to each state its lower quota of seats. At this point, there should be some seats that were not allocated.
3. Starting with the state that has the largest fractional part and working toward the state with the smallest fractional part, allocate one additional seat to each state until all the seats have been allocated.

Let's illustrate Hamilton's method of apportionment using the six states in the country of Hamiltonia from Example 1A and Example 1B.

Example 1C

Use Hamilton’s method to finish the allocation of seats in Hamiltonia.

Solution

We've already completed Step 1 and Step 2. The numbers in parentheses show the ranking of the fractional parts of the standard quotas from largest to smallest. For example, Zeta’s standard quota, 4.958, has the largest fractional part, 0.958, and is marked (1).

Fractional Parts for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532 (6)	5.907 (3)	2.954 (2)	1.793 (5)	6.857 (4)	4.958 (1)	25.001
Lower Quota	2	5	2	1	6	4	20

We find the sum of the lower quotas to determine how many seats still need to be allocated. Twenty of the 25 seats have been allocated so there are 5 remaining seats. Allocate the seats, in order from largest fractional part to smallest fractional part until the 5 remaining seats have been allocated. The leftover seats go to Zeta, Gamma, Beta, Epsilon, and Delta. The additional and final allocations of the seats are shown in the table

Final Allocation for Hamiltonia Seats Using Hamilton’s Method

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001
Lower Quota	2	5	2	1	6	4	20
Final Allocation	2	5 + 1 = 6	2 + 1 = 3	1 + 1 = 2	6 + 1 = 7	4 + 1 = 5	25

Overall, Alpha gets 2 senators, Beta gets 6 senators, Gamma gets 3 senators, Delta gets 2 senators, Epsilon gets 7 senators, and Zeta gets 5 senators.

While apportionment has its roots in allocating seats of Congress to the states, the same principles and methods also can be used to allocate other items and resources.

Example 2

In the city of Adamstown, 42 new firefighters have just completed their training. They are to be assigned to the five firehouses in town in a manner proportional to the population in each fire district. The populations are listed in the following table.

Population of Fire Districts in Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465

Apportion the new firefighters to the fire houses using Hamilton's Method.

Solution

The standard divisor is $\text{SD} = \frac{\text{total population}}{\#\text{ of items}} = \frac{69{,}465}{42} \approx 1654$.

Start by dividing each district's population by the standard divisor and rounding each standard quota down. Order the fractional parts of the standard quotas from largest to smallest. As shown in the table, the sum of the lower quotas is 41 so there is one firefighter left over that must be assigned to a fire house. Assign this firefighter to District D since D has the largest fractional part.

Allocation of Firefighters Using Hamilton's Method

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
Standard Quota	15.121 (3)	5.296 (2)	7.007 (5)	5.456 (1)	9.117 (4)	41.998
Lower Quota	15	5	7	5	9	41
Final Allocation	15	5	7	5 + 1 = 6	9	42

The final allocation of the new firefighters is 15 for A, 5 for B, 7 for C, 6 for D, and 9 for E.

Try it Now 1

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

Population Rhode Island's Counties

County	Bristol	Kent	Newport	Providence	Washington	Total
Population	49,875	166,158	82,888	626,667	126,979	1,052,567

Answer

The standard divisor is $\frac{1{,}052{,}567}{75} \approx 14,034$. The table shows the standard quotas, lower quota, and additional allocations that must be done to allocate all 75 seats.

Allocation of Seats Using Hamilton's Method

County	Bristol	Kent	Newport	Providence	Washington	Total
Population	49,875	166,158	82,888	626,667	126,979	1,052,567
Standard Quota	3.554 (4)	11.840 (2)	5.906 (1)	44.653 (3)	9.048 (5)	75.001

County	Bristol	Kent	Newport	Providence	Washington	Total
Lower Quota	3	11	5	44	9	72
Final Allocation	3	11 + 1 = 12	5 + 1 = 6	44 + 1 = 45	9	75

We need 75 representatives and we only have 72 using the lower quotas, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport (1), Kent (2), and Providence (3).

Jefferson's Method

Jefferson's method was the first method used to apportion the seats in the U.S. House of Representatives in 1792. It was used through 1832.

Jefferson's method divides all populations by a modified divisor and then rounds the results down to the lower quota. Sometimes the total number of seats will be too large and other times it will be too small. We keep guessing modified divisors until the method assigns the correct total number of seats. Dividing by a larger modified divisor will make each quota smaller so the sum of the lower quotas will be smaller. It is easy to remember which way to go. If the sum is too large, make the divisor larger. If the sum is too small, make the divisor smaller. All the quotas are rounded down so using the standard divisor will give a sum that is too small. Our guess for the first modified divisor should be a number smaller than the standard divisor.

Summary of Jefferson's Method

1. Find the standard divisor.
2. Pick a modified divisor, d, that is slightly less than the standard divisor.
3. Divide each state's population by d to get its modified quota.
4. Round each modified quota down to its lower quota.
5. Find the sum of the lower quotas.
6. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too large, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps 3 through 6 until the correct number of seats are apportioned.

Let's illustrate Jefferson's method of apportionment using the six states in the country of Hamiltonia from Example 1A.

Example 4

Use Jefferson's method to allocate the 25 seats of the senate in Hamiltonia.

Population of the States of Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

Solution

From Example 1A, we know the standard divisor is 9,480 and the sum of the lower quotas is 20. In Jefferson's method the standard divisor will always give us a sum that is too small so we begin by making the standard divisor smaller. There is no formula for this: just guess something. Let's try the modified divisor, $d = 9,000$.

Modified Quota for $d = 9,000$

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
$d = 9,000$	2.667	6.222	3.111	1.889	7.222	5.222	
Lower Quota	2	6	3	1	7	5	24

The sum of 24 is too small so we need to try again by making the modified divisor smaller. Let's try $d = 8,000$.

Modified Quota for $d = 8,000$

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
$d = 9{,}000$	2.667	6.222	3.111	1.889	7.222	5.222	
Lower Quota	2	6	3	1	7	5	24
$d = 8{,}000$	3.000	7.000	3.500	2.125	8.125	5.875	
Lower Quota	3	7	3	2	8	5	28

This time the sum of 28 is too big. Try again making the modified divisor larger. We know the divisor must be between 8,000 and 9,000 so let's try 8,500.

Modified Quota for $d = 8,500$

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
$d = 9{,}000$	2.667	6.222	3.111	1.889	7.222	5.222	
Lower Quota	2	6	3	1	7	5	24
$d = 8{,}000$	3.000	7.000	3.500	2.125	8.125	5.875	
Lower Quota	3	7	3	2	8	5	28
$d = 8{,}500$	2.824	6.588	3.294	2.000	7.647	5.529	
Lower Quota	2	6	3	2	7	5	25

This time the sum is 25 so we are done. Alpha gets 2 senators, Beta gets 6 senators, Gamma gets 3 senators, Delta gets 2 senators, Epsilon gets 7 senators, and Zeta gets 5 senators.

Note: This is the same result as we got using Hamilton's method in Example 1C. However, the two methods do not always give the same result.

Example 5

A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed in the below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject using Jefferson's method.

Number of Students Enrolled

Subject	Math	English	Chemistry	Biology	Total
Enrollment	330	265	130	70	795

Solution

First find the standard divisor: $\frac{795}{15} = 53$. We begin by choosing a modified divisor that is smaller than the standard divisor.

Guess #1: $d = 50$. The sum of 14 is still too small to so make the modified divisor d smaller.

Guess #2: $d = 45$. The sum is 15 so we are done.

Apportionment of Tutors Using Jefferson's Method

Subject	Math	English	Chemistry	Biology	Total
Enrollment	330	265	130	70	795
$d = 50$	6.6	5.3	2.6	1.4	
Lower Quota	6	5	2	1	14

Subject	Math	English	Chemistry	Biology	Total
$d = 45$	7.333	5.889	2.889	1.556	
Lower Quota	7	5	2	1	15

The college should hire 7 math tutors, 5 English tutors, 2 chemistry tutors, and 1 biology tutor.

? Try it Now 2

The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information in two ways: first using Hamilton's method and then Jefferson's method.

Customers per Shift

Shift	Morning	Midday	Afternoon	Evening	Total
Customers	95	305	435	515	1,350

Answer

The standard divisor is $\frac{1{,}350}{20} = 67.5$. The table shows the standard quotas, lower quotas, ranking of fractional amounts, and additional allocations that must be done to allocate all 20 salespersons using Hamilton's Method.

Apportionment of Salespersons Using Hamilton's Method

Shift	Morning	Midday	Afternoon	Evening	Total
Customers	95	305	435	515	1,350
Standard Quota	1.407 (4)	4.519 (2)	6.444 (3)	7.630 (1)	20.000
Lower Quota	1	4	6	7	18
Final Allocation	1	4 + 1 = 5	6	7 + 1 = 8	20

Using Hamilton's method, the apportionment of salespeople should be 1 for morning, 5 for midday, 6 for afternoon, and 8 for evening.

Apportionment of Salespersons Using Jefferson's Method

Shift	Morning	Midday	Afternoon	Evening	Total
Customers	95	305	435	515	1,350
Guess #1: $d = 65$	1.462	4.692	6.692	7.923	
Lower Quota	1	4	6	7	18
Guess #2: $d = 60$	1.583	5.083	7.25	8.583	
Lower Quota	1	5	7	8	21
Guess #3: $d = 62$	1.532	4.919	7.016	8.306	
Lower Quota	1	4	7	8	20

Using Jefferson's method, the apportionment of salespeople should be 1 for morning, 4 for midday, 7 for afternoon, and 8 for evening.

- **9.1: Apportionment - Jefferson’s, Adam’s, and Webster’s Methods** by Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, & Kim Sonier has no license indicated. Original source: https://www.coconino.edu/open-source-textbooks#college-mathematics-for-everyday-life-by-inigo-jameson-kozak-lanzetta-and-sonier.
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6.3: Review Exercises

1. An organization recently made a decision about which company to use to redesign its website and host its members' information. The Board of Directors will vote using preference ballots ranking their first choice to last choice of the following companies: Allied Web Design (A), Ingenuity Incorporated (I), and Yeehaw Web Trends (Y). The individual ballots are shown below. Create a preference schedule summarizing these results.

AIY, YIA, YAI, AIY, YIA, IAY, IYA, IAY, YAI, YIA, AYI, YIA, YAI

2. A group needs to decide where their next conference will be held. The choices are Kansas City (K), Lafayette (L), and Minneapolis (M). The individual ballots are shown below. Create a preference schedule summarizing these results.

KLM, LMK, MLK, LMK, MKL, KLM, KML, LMK, MKL, MKL, MLK, MLK

3. A book club holds a vote to figure out what book they should read next. They are picking from three different books. The books are labeled A, B, and C, and the preference schedule for the vote is below.

Preference Table for Book Election

Number of voters	12	9	8	5	10
1st choice	A	B	B	C	C
2nd choice	C	A	C	A	B
3rd choice	B	C	A	B	A

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

4. An election is held for a new vice president at a college. There are three candidates (A, B, C), and the faculty rank which candidate they like the most. The preference ballot is below.

Preference Table for Vice President Election

Number of voters	8	10	12	9	4	1
1st choice	A	A	B	B	C	C
2nd choice	B	C	A	C	A	B
3rd choice	C	B	C	A	B	A

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

5. A city election for a city council seat was held between 4 candidates, Martorana (M), Jervey (J), Riddell (R), and Hanrahan (H). The preference schedule for this election is below.

Preference Table for City Council Election

Number of voters	60	73	84	25	110
1st choice	M	M	H	J	J
2nd choice	R	H	R	R	M
3rd choice	H	R	M	M	R
4th choice	J	J	J	H	H

a. How many voters voted in the election?
b. How many votes are needed for a majority?
c. Find the winner using the Plurality Method.
d. Find the winner using the Borda Count Method.
e. Find the winner using the Plurality with Elimination Method.
f. Find the winner using the Pairwise Comparisons Method.

6. A local advocacy group asks members of the community to vote on which project they want the group to put its efforts behind. The projects are green spaces (G), city energy code (E), water conservation (W), and promoting local business (P). The preference schedule for this vote is below.

Preference Table for Community Projects Election

Number of voters	12	57	23	34	13	18	22	39
1st choice	W	W	G	G	E	E	P	P
2nd choice	G	P	W	E	G	P	W	E
3rd choice	E	E	E	W	P	W	G	W
4th choice	P	G	P	P	W	G	E	G

a. How many voters voted in the election?
b. How many votes are needed for a majority?
c. Find the winner using the Plurality Method.
d. Find the winner using the Borda Count Method.
e. Find the winner using the Plurality with Elimination Method.
f. Find the winner using the Pairwise Comparisons Method.

7. A solar system consisting of five planets has a governing council of 135 members who are apportioned proportional to the populations of the planets. The population for each planet is listed in the following table.

Populations of Five Planets

Planet	Ajax	Borax	Calax	Delphi	Eljix	Total
Population	183,000	576,000	274,000	749,000	243,000	2,025,000

a. For each planet, find the standard quota, the upper quota and the lower quota. Give your answers in a table.
b. Use Hamilton’s method to apportion the 135 council members.
c. Use Jefferson's method to apportion the 135 council members

8. The country named Erau has five states and a total of 200 seats available in its House of Representatives. The number of seats that each state receives is proportional to the population of that state. The populations of the states are given in the table below.

Populations of States in Erau

State	1	2	3	4	5	Total
Population	3,500,000	1,200,000	530,000	999,000	771,000	7,000,000

   a. For each state, find the standard quota, the upper quota and the lower quota. Give your answers in a table.
   b. Use Hamilton’s method to apportion the 200 seats.
   c. Use Jefferson's method to apportion the 200 seats.

9. Last year a city had three school districts:
   - North, with a population of 5,200 children
   - South, with a population of 10,600 children,
   - West, with a population of 15,100 children.

   a. Use Hamilton’s method to apportion 50 speech therapists among the districts using the populations for last year.
   b. This year, the city took over another school district. The new East district has a population of 9,500 children. If the number of speech therapists is increased by 15 to accommodate the new district, use Hamilton’s method to apportion the 65 speech therapists.

10. Three people invest in a treasure dive, each investing the amount listed below. The dive results in finding 36 gold coins.

    Alice: \$7,600 Ben: \$5,900 Carlos: \$1,400

    a. Use Hamilton’s method to apportion the 36 coins.
    b. Use Jefferson's method to apportion the 36 coins.

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