10. 18.03 PDE Exercises

10A. Heat Equation; Separation of Variables

10A-1 Solve the boundary value problem for the temperature of a bar of length 1 following the steps below.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1,\ t > 0 \tag{10A-1.1}$$

$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{10A-1.2}$$

$$u(x,0) = x \qquad 0 < x < 1 \tag{10A-1.3}$$

(i) Separation of variables. Find all solutions of the form $u(x,t) = v(x)w(t)$ to (10A-1.1) and (10A-1.2) (**not** (10A-1.3)). Write the list of possible answers in the form

$$u_k(x,t) = v_k(x)w_k(t)$$

Note that your answer is ambiguous up to a multiple, so we just pick the simplest $v_k(x)$ and the simplest $w_k(t)$ (so that $w_k(0) = 1$). With $w_k(0) = 1$, we see that

$$u_k(x,0) = v_k(x)$$

Thus, we have succeeded in solving our problem when the initial condition is $v_k(x)$.

(ii) Write the initial condition (10A-1.3) as a sum of $v_k(x)$ — a Fourier series.

$$x = \sum b_k v_k(x), \quad 0 < x < 1.$$

Hints: How should you extend the function x outside the range $0 < x < 1$? What is the period? What is the parity (odd/even)? Graph the extended function. Once you have figured out what it is, you will be able to find the series in your notes.

(iii) Superposition principle. Write the solution to (10A-1.1), (10A-1.2), and (10A-1.3) in the form

$$u(x,t) = b_1 u_1(x,t) + b_2 u_2(x,t) + \cdots ,$$

with explicit formulas for b_k and u_k.

(iv) Find $u(1/2, 1)$ to one significant figure.

10A-2 Use the same steps as in 10A-1 to solve the boundary value problem for the temperature of a bar of length 1:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x} \qquad 0 < x < 1, t > 0 \tag{10A-2.1}$$

$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{10A-2.2}$$

$$u(x,0) = 1 \qquad 0 < x < 1 \tag{10A-2.3}$$

10A-3 Consider the boundary value problem with inhomogeneous boundary condition give by:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x} \qquad 0 < x < 1, t > 0 \tag{10A-3.1}$$

$$u(0,t) = 1 \quad u(1,t) = 0 \qquad t > 0 \tag{10A-3.2}$$

$$u(x,0) = 1 \qquad 0 < x < 1 \tag{10A-3.3}$$

(a) In temperature problems a steady state solution u_{st} is constant in time:

$$\frac{\partial u}{\partial t} = 0$$

It follows that $u_{st} = U(x)$, a function depending only on x. Find the steady state solution $u_{st}(x,t) = U(x)$ to (10A-3.1) and (10A-3.2).

(b) Find the partial differential equation, endpoint, and initial conditions satisfied by $\tilde{u}(x,t) = u(x,t) - U(x)$. Then write down the formula for $\tilde{u}$. [Hint: We already know how to solve the problem with zero boundary conditions.]

(c) Superposition principle. Now that we have found $\tilde{u}$ and U, what is u?

(d) Estimate, to two significant figures, the time T it takes for the solution to be within 1% of its steady state value at the midpoint $x = 1/2$. In other words, find T so that

$$|u(1/2,t) - U(1/2)| \leq \frac{1}{100}U(1/2) \quad \text{for } t \geq T.$$

10B. Wave Equation

10B-1 (a) Find the normal modes of the wave equation on $0 \leq x \leq \pi/2$, $t \geq 0$ given by

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u, \quad u(0,t) = u(\pi/2,t) = 0, t > 0$$

(b) If the solution in part (a) represents a vibrating string, then what frequencies will you hear if it is plucked?

(c) If the length of the string is longer/shorter what happens to the sound?

(d) When you tighten the string of a musical instrument such as a guitar, piano, or cello, the note gets higher. What has changed in the differential equation?

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises