10. 18.03 PDE Exercises Solutions

10A. Heat Equation; Separation of Variables

10A-1 (i) Trying a solution $u(x,t) = v(x)w(t)$ leads to separated solutions $u_k(x,t) = v_k(x)w_k(t)$ where $v_k(x) = \sin(\pi k x)$, and $w_k(t) = e^{-2\pi^2 k^2 t}$, and $k = 0, 1, 2, \ldots$.

(ii) $u(x,0) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\sin(\pi k x)$

(iii) $u(x,t) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\sin(\pi k x)e^{-\pi^2 k^2 t}$

(iv) $u(\frac{1}{2}, 1) \approx .00032$

10A-2 (i) Separated solutions $u_k(x,t) = v_k(x)w_k(t)$ where $v_k(x) = \sin(\pi k x)$, and $w_k(t) = e^{-2\pi^2 k^2 t}$ (Note factor of 2 from (10A-2.1)) and $k = 0, 1, 2, \ldots$

(ii) $u(x,0) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(\pi k x)}{k}$

(iii) $u(x,t) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(\pi k x)}{k}e^{-2\pi^2 k^2 t}$

10A-3 (a) $u_{st}(x,t) = U(x) = 1 - x$

(b) Since the heat equation is linear $\tilde{u}$ satisfies the PDE (10A-3.1). At the boundary ($x = 0$ and $x = 1$) we have $\tilde{u}(0,t) = u(0,1) - u_{st}(0,t) = 1 - 1 = 0$. Likewise $\tilde{u}(1,t) = 0$. That is, $\tilde{u}$ is a solution to the heat equation with homogeneous boundary conditions in 10A-1. The initial condition is $\tilde{u}(x,0) = x$. We found the coefficients for this in 10A-1.

$$\tilde{u}(x,t) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\sin(\pi k x)e^{-2\pi^2 k^2 t}.$$

(c)

$$u(x,t) = U(x) + \tilde{u}(x,t) = 1 - x + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\sin(\pi k x)e^{-2\pi^2 k^2 t}.$$

(d) $u(x,t) - U(x) = \tilde{u}(x,t)$ The term in the sum for $\tilde{u}$ that decays the slowest is when $k = 1$. Therefore we need $\frac{2}{\pi}e^{-2\pi^2 T} = .01U(1/2) = .005$. Solving we get $T = .246$

10B. Wave Equation

10B-1 (a) Separating variables, we look for a solutions of the form $u(x,t) = v(x)w(t)$, which leads to $v''(x) = \lambda v(x)$ with $v(0) = v(\pi/2) = 0$, and hence

$$v_k(x) = \sin(2kx)$$

Consequently, $\ddot{w}_k = -(2k)^2 c^2 w_k$, whic implies

$$w_k(t) = A\cos(2ckt) + B\sin(2ckt)$$

The normal modes are

$$u_k(x,t) = \sin(2kx)(A\cos(2ckt) + B\sin(2ckt)),$$

where A and B must be specified by an initial position and velocity of the string.

(b) The main note (from the mode u_1) has frequency $\dfrac{2c}{2\pi} = \dfrac{c}{\pi}$. You will also hear the higher harmonics at the frequencies $\dfrac{ck}{\pi}$, $k = 2, 3, \ldots$. (The sound waves induced by the vibrating string depend on the frequency in t of the modes.)

(c) Longer strings have lower frequencies, lower notes, and shorter strings have higher frequencies, higher notes. If the length of the string is L, then the equations $v''(x) = \lambda v(x)$, $v(0) = v(L) = 0$ lead to solutions $v_k(x) = \sin(k\pi x/L)$. (In part (a), $L = \pi/2$.) The associated angular frequencies in the t variable are $kc\pi/L$, so the larger L, the smaller $kc\pi/L$ and the lower the note. Thus c is inversely proportional to the length of the string.

(d) When you tighten the string, the notes get higher, and the frequency you hear is increased. (Tightening the string increases the tension in the string and increases the spring constant, which corresponds to c. The frequencies of the sounds are directly proportional to c.)

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises