9. 18.03 Linear Algebra Exercises Solutions

9A. Matrix Multiplication, Rank, Echelon Form

9A-1. (i) No. The pivots have to occur in descending rows.
(ii) Yes. There's only one pivotal column, and it's as required.
(iii) Yes. There's only one pivotal column, and it's as required.
(iv) No. The pivots have to occur in consecutive rows.
(v) Yes.

9A-2 (i) $\begin{bmatrix} 4 \end{bmatrix} \sim \begin{bmatrix} 1 \end{bmatrix}$.
(ii) This is reduced echelon already.
(iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
(iv) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \sim$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (We'll soon see that this is not a surprise.)

9A-3 There are many answers. For example, $\begin{bmatrix} 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \end{bmatrix}$ work fine. Or $[\cos\theta \quad \sin\theta]$; etc.

9A-4 Here's several answers: $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix}$
The interesting thing to say is that the answer is any vector in the nullspace of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. The simplest solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

9A-5 (a) The obvious answer to this question is $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; for any matrix A with four columns, $A \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is the third column of A.

But there are other answers: Remember, the general solution is any particular solution plus the general solution to the homogeneous problem. The reduced echelon form of

A may be obtained by subracting the last row from the first row: $R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$R\mathbf{v} = \mathbf{0}$ has solutions which are multiples of $\begin{bmatrix} -3 \\ -2 \\ -1 \\ 1 \end{bmatrix}$. So for any t, $A \begin{bmatrix} 1-3t \\ 1-2t \\ 1-t \\ 1+t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) For any matrix with three rows, the product $[\,0 \quad 0 \quad 1\,]A$ is its third row. In this case, the rows are linearly independent—the only row vector $\mathbf{u}$ such that $\mathbf{u}A = \mathbf{0}$ is $\mathbf{u} = \mathbf{0}$—so there are no other solutions.

9A-6 (a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$. Rank 1, as always for row $\times$ column with both nonzero.

(b) $\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [2]$ Rank 1 just because it's nonzero!

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$ Rank 2: the columns are linearly independent (as are the rows).

9B. Column Space, Null Space, Independence, Basis, Dimension

9B-1 First of all, any four vectors in $\mathbb{R}^3$ are linearly dependent. The question here is how. The first thing to do is to make the 3×4 matrix with these vectors as columns. Now linear relations among the columns are given by the null space of this matrix, so we want to find a nonzero element in this null space. To find one, row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The null space does't change under these row operations, and the null space is the space of linear relations among the columns. In the reduced echelon form it's clear that the first three columns are linearly independent. (This is clear from the original matrix, too, because the first three columns form an upper triangular matrix with nonzero entries down the diagonal.) The first three variables are pivotal, and the last is free. Set the last one equal to 1 and solve for the first two to get a basis vector for the null space:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Check: the sum of the first, second, and fourth columns is 4 times the third.

9B-2 (a) Reduction steps: row exchange the top two rows, to get a pivot at upper right; then subtract twice the new top row from the bottom row.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last transformation uses the pivot in the second row to eliminate other entries in the second column. This is now reduced echelon.

Stop and observe: The third column is twice the second minus the first. That's true in the original matrix as well! The fourth column is the 3 times the second minus twice the first. That's also true in the original matrix!

Those linear relations can be expressed as matrix multiplications. The first two variables are pivot variables and the last two are free variables. Setting $x_3 = 1$ and $x_4 = 0$ and then the other way around gives the two vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$, and they form a basis for the null space of A. Any linearly independent pair of linear combinations of them is another basis for the null space.

Similarly for A^T:

$$A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two variables are again pivotal, and the third is free. Set it equal to 1 and solve for the pivotal variables to find the basis vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Any nonzero multiple of this vector is another basis for the null space.

(b) We need a particular solution to $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. You could form the augmented matrix, by adjoining $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a fifth column, and row reduce the result. Maybe it's easier to just look and see what's there. For example the difference of the first two columns is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: so $\mathbf{x_p} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ works fine. Then the general solution is

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Then we need a particular solution to $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Again, the difference of the first two columns works; so the general solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

9B-3 (a) This is easy to do directly: all such vectors are multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, so that by itself forms a basis of this subspace. But we can do this as directed as well. One way to express the conditions is to say that $x_1 = x_2$, and $x_2 = x_3$, and $x_3 = x_4$. These three equations can be represented by the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The fourth variable is free, and if we set it equal to 1 we find that the other three are 1 as well.

(b) A matrix representation of this relation is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x} = 0$. This is already in reduced echelon form! The first variable is pivotal, the last three are free. We find $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) Now there are two equations, represented by the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$. The first two variables are pivotal and the last two are free. A basis: $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Looking back at the original equations, this makes sense, doesn't it?

9B-4 (a) If $c \neq 0$, there are pivots on three rows and the rank is 3. So $c = 0$. The rank will still be 3 unless the third row is a linear combination of the first two. The first column can't occur (because of the nonzero entries on the left), so we must have $d = 2$

(b) The answer is the same, since dim(Null space)+dim(Column space)=width.

9C. Determinants and Inverses

9C-1 (a) $R(\alpha)R(\beta) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$
$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$
$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R(\alpha+\beta).$
Geometrically, $R(\theta)$ rotates vectors in the plane by θ radians counterclockwise.

(b) $\det R(\theta) = (\cos\theta)^2 + (\sin\theta)^2 = 1$. $R(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(-\theta)$.

9C-2 (a) $\det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1$; $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

(b) $\det \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = 1$ (by cross-hatch, or because the determinant of an upper-triangular matrix is the product of the diagonal entries). To find the inverse, row reduce

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

so the inverse is $\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2$ by crosshatch. So we expect a 2 in the denominator of the inverse.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

so the inverse is $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

The original matrix was symmetric. Is it an accident that the inverse is also symmetric?

(d) $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. The inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$.

9D. Eigenvalues and Eigenvectors

9D-1 (a) The eigenvalues of upper or lower triangular matrices are the diagonal entries: so for A we get 1 and 2. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is clearly an eigenvector for value 1, as is any multiple. For $\lambda = 2$, we want a vector in the null space of $A - \lambda I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple will do nicely.

For B, the eigenvalues are 3 and 4. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is clearly an eigenvector for $\lambda = 4$, as is any multiple. For $\lambda = 3$, we want a vector in the null space of $A - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any multiple will do nicely.

(b) $AB = \begin{bmatrix} 4 & 4 \\ 2 & 8 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - (\mathrm{tr} A)\lambda + \det A = \lambda^2 - 12\lambda + 24 = (\lambda - 6)^2 - (36 - 24)$ has roots $\lambda_{1,2} = 6 \pm \sqrt{12}$.

(c) If $A\mathbf{x} = \lambda\mathbf{x}$, then $(cA)\mathbf{x} = cA\mathbf{x} = c\lambda\mathbf{x}$, so $c\lambda$ is an eigenvalue of cA. If $c = 0$ then $cA = 0$ and its only eigenvalue is 0; otherwise, this argument is reversible, so the eigenvalues of cA are exactly c times the eigenvalues of A.

(d) $A + B = \begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - 10\lambda + 23 = (\lambda - 5)^2 - (25 - 23)$ and so $\lambda_{1,2} = 5 \pm \sqrt{2}$.

9D-2 (a) $\lambda^2 - 2\lambda + 1 = (1 - \lambda)^2$.

(b) $(1 - \lambda)^3$.

(c) $\det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = (-\lambda)^3 + 1 + 1 - (-\lambda) - (-\lambda) - (-\lambda) = -\lambda^3 + 3\lambda + 2$.

(d) $(1 - \lambda)(2 - \lambda)(3 - \lambda)(4 - \lambda)$.

9D-3 The eigenvalue equation is $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mathbf{v} = \lambda\mathbf{v}$. Let's write $\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$, where $\mathbf{x}$ has m components and $\mathbf{y}$ has n components. Then the eigenvalue equation is equivalent to the two equation $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{y} = \lambda\mathbf{y}$, for the same λ. λ is an eigenvalue if there is *nonzero* vector satisfying this equation. This means that *either* $\mathbf{x}$ *or* $\mathbf{y}$ must be nonzero. So one possiblity is that λ is an eigenvalue of A, $\mathbf{x}$ is a nonzero eigenvector for A and this eigenvalue, and $\mathbf{y} = \mathbf{0}$. Another possibility is that λ is an eigenvalue of B, $\mathbf{x} = \mathbf{0}$, and $\mathbf{y}$ is a nonzero eigenvector for B with this eigenvalue. Conclusion: The eigenvalues of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n$.

9D-4 The eigenvalue equation is $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$. This expands to $A\mathbf{y} = \lambda\mathbf{x}$ and $B\mathbf{x} = \lambda\mathbf{y}$. We want either $\mathbf{x}$ or $\mathbf{y}$ to be nonzero. We compute $AB\mathbf{x} = A(\lambda\mathbf{y}) = \lambda A\mathbf{y} = \lambda^2\mathbf{x}$ and $BA\mathbf{y} = B\lambda\mathbf{x}) = \lambda B\mathbf{x} = \lambda^2\mathbf{y}$. So if $\mathbf{x} \neq \mathbf{0}$, then λ^2 is an eigenvalue of AB, and if $\mathbf{y} \neq \mathbf{0}$ then λ^2 is an eigenvalue of BA. Enough said.

9E. Two Dimensional Linear Dynamics

9E-1 (a) Write $\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$, so $\dot{\mathbf{u}} = A\mathbf{u}$ with $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. (Attention to signs!)

(b) A is singular (columns are proportional) so one eigenvalue is 0. The other must be -2 because the trace is -2.

(c) 0 has eigenvectors given by the multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so one normal mode solution is the constant solution $\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple. This is the steady state, with the same number of people in both rooms. To find an eigenvector for the eigenvalue -2, form $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; so we have $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple. The corresponding normal mode is $\mathbf{u_2} = e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and the general solution is $\mathbf{u} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. With $\mathbf{u}(0) = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$, this gives the equations $a + b = 30, a - b = 10$, or $a = 20, b = 10$: so $\mathbf{u} = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The first term is the steady state; the second is a transient, and it decays rapidly to zero.

When $t = 1$ we have $v(1) = 20 + 10e^{-2} \sim 21.36$ and $w(1) = 20 - 10e^{-2} \sim 18.64$.

(d) When $t = \infty$ (so to speak), the transient has died away and the rooms have equalized at 20 each.

9E-2 (a) This is the companion matrix for the harmonic oscillator, as explained in LA.7, with $\omega = 1$. The basic solutions are $\mathbf{u_1}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\mathbf{u_2}(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, and the general solution is a linear combination of them. A good way to think of this is to remember that the second entry is the derivative of the first, and the first can be any sinusoidal function of angular frequency 1: so let's write it as $\mathbf{u} = \begin{bmatrix} A\cos(t-\phi) \\ -A\sin(t-\phi) \end{bmatrix} = A\begin{bmatrix} \cos(t-\phi) \\ \sin(t-\phi) \end{bmatrix}$.

For these to lie on the unit circle we should take $A = 1$. That's it: $\mathbf{u} = \begin{bmatrix} \cos(t-\phi) \\ \sin(t-\phi) \end{bmatrix}$ for any ϕ. ϕ is both phase lag and time lag, since $\omega = 1$. The solution goes through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t = \phi$.

(b) Now the characteristic polynomial is $p_A(\lambda) = \lambda^2 - 1$, so there are two real eigenvalues, 1 and -1. [The phase portrait is a saddle.] Eigenvectors for 1 are killed by $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and multiples. The matrix is symmetric, so the eigenvectors for -1 are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and multiples. So the normal modes are $\mathbf{u_1} = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u_2} = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The general solution is $\mathbf{u} = ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

In order for $\mathbf{u}(t_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some t_0, you must have $ae^{t_0} = 1$ and $be^{-t_0} = 0$, because the two eigenvectors are linearly independent. So $b = 0$, and a can be any positive number (because you can then solve for t_0): $ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $a > 0$. These are "ray" solutions.

9F. Normal Modes

9F-1 The characteristic polynomial is $p(s) = s^4 - c$. Its roots are the fourth roots of c. Write r for the real positive fourth root of $|c|$. If $c > 0$, the fourth roots of c are $\pm r$ and $\pm ir$. There are four exponential solutions, but only the ones with imaginary exponent give rise to sinusoidal solutions. They are the sinusoidal functions of angular frequency r. If $c < 0$, the fourth roots of c are $r\frac{\pm 1 \pm i}{\sqrt{2}}$. None of these is purely imaginary, so there are no (nonzero) sinusoidal solutions. When $c = 0$ the equation is $\frac{d^4x}{dt^4} = 0$. The solutions are the polynomials of degree at most 3. The constant solutions are sinusoidal.

9G. Diagonalization, Orthogonal Matrices

9G-1 The null space is (width)-(rank)=9 dimensional, so there are 9 linearly independent eigenvectors for the eigenvalue 0. There's just one more eigenvalue, which can't be zero because the sum of the eigenvalues is the trace, 5. It has to be 5.

9G-2 (a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ has eigenvalues 1 and 3. 1 has nonzero eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (or any nonzero multiple; by inspection or computation). For 3, $A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ so we have nonzero eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or any nonzero multiple). With these choices, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has eigenvectors 0 (because it's singular) and 4 (because its trace is 4; or by computation). An eigenvector for 0 is a vector in the null space, so e.g. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple will do. For 4, $A - 4I = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$ so we have nonzero eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (or any nonzero multiple). With these choices, $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

(b) $A^3 = S\Lambda^3 S^{-1}$; to be explicit, $\Lambda^3 = \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}$.

$A^{-1} = S\Lambda^{-1} S^{-1}$; to be explicit, $\Lambda^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$.

9G-3 [The (i, j) entry in $A^T A$ is the dot product of the ith and jth columns of A. The columns form an orthonormal set exactly when this matrix of dot products is the identity matrix.]

$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3)$, so eigenvalues -1 and -3 and $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$. Eigenvectors for 1 are killed by $A - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any nonzero multiple. We could find an eigenvector for 3 similarly, or just remember that eigenvectors for distinct eigenvalues of a symmetric matrix are orthogonal, and write down $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Each of these vectors has length $\sqrt{2}$, so an orthogonal S is given by $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(There are seven other correct answers; one could list the eigenvalues in the opposite order, and change the signs of the eigenvectors.)

9H. Decoupling

9H-1 The rabbit matrix is $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. We found eigenvalues $2, 5$ and eigenvector matrix $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, with inverse $S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. The decoupling coordinates are the entries in $\mathbf{y}$ such that $S\mathbf{y} = \mathbf{x}$, or $\mathbf{y} = S^{-1}\mathbf{x} = \frac{1}{3}\begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$: $y_1 = 2x_1 - x_2$ (for eigenvalue 2) and $y_2 = x_1 + x_2$ (for eigenvalue 5), or any nonzero multiples of these, do the trick. $y_1(t) = e^{2t}y_1(0)$, $y_2(t) = e^{5t}y_2(0)$. $y_2(t)$ is the sum of the populations, and it grows exponentially with rate 5, just as if there was no hedge.

9I. Matrix Exponential

9I-1 By time invariance the answer will be the same for any t_0; for example we could take $t_0 = 0$; so $\begin{bmatrix} x(t_0+1) \\ y(t_0+1) \end{bmatrix} = e^A \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix}$. Then we need to compute the exponential matrix. We recalled the diagonalization of the rabbit matrix above, so $e^{At} = Se^{\Lambda t}S^{-1}$, and $e^A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix} \dfrac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

9I-2 $p_A(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1$ has roots $-1 \pm i$. Eigenvectors for $\lambda = -1+i$ are killed by $A - (-1+i)I = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$; for example $\begin{bmatrix} -1 \\ 1-i \end{bmatrix}$. The exponential solutions are $e^{(-1+i)t}\begin{bmatrix} -1 \\ 1-i \end{bmatrix}$ and its complex conjugate, so we get real solutions as real and imaginary parts, which we put into the columns in a fundamental matrix:

$$\Phi(t) = e^{-t}\begin{bmatrix} -\cos t & -\sin t \\ \cos t + \sin t & -\cos t + \sin t \end{bmatrix}$$

$\Phi(0) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$, $\Phi(0)^{-1} = -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

$$e^{At} = \Phi(t)\Phi(0)^{-1} = e^{-t}\begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{bmatrix}$$

9J. Inhomogeneous Solutions

9J-1

(a) As always, substitution of $\mathbf{u_p} = e^{2t}\mathbf{v}$ gives the exponential response formula

$$\mathbf{v} = \begin{bmatrix} -4 & -5 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 10 \\ -7 \end{bmatrix} \Rightarrow \mathbf{u_p} = -\frac{1}{5}e^{2t}\begin{bmatrix} 10 \\ -7 \end{bmatrix}.$$

(b) Complex replacement and the trial solution $\mathbf{z_p} = e^{2it}\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix} 2i-1 & 0 \\ -1 & 2i-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{-2-6i} \begin{bmatrix} 2i-2 \\ 1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -4-8i \\ -1+3i \end{bmatrix}$$

So

$$\begin{aligned} \mathbf{z_p} &= \frac{1}{20}(\cos(2t) + i\sin(2t)) \begin{bmatrix} -4-8i \\ -1+3i \end{bmatrix} \\ &= \frac{1}{20}\left(\begin{bmatrix} -4\cos(2t) + 8\sin(2t) \\ -\cos(2t) - 3\sin(2t) \end{bmatrix} + i \begin{bmatrix} -4\sin(2t) - 8\cos(2t) \\ -\sin(2t) + 3\cos(2t) \end{bmatrix} \right) \end{aligned}$$

Therefore

$$\mathbf{u_p} = \operatorname{Re}(\mathbf{z_p}) = \frac{1}{20} \begin{bmatrix} -4\cos(2t) + 8\sin(2t) \\ -\cos(2t) - 3\sin(2t) \end{bmatrix}$$

9J-2 We want to compute a particular solution to $\dot{\mathbf{x}} = A\mathbf{x} + \begin{bmatrix} -e^t \\ 0 \end{bmatrix}$, using $\mathbf{x}(t) = \Phi(t)\int \Phi(t)^{-1} \begin{bmatrix} -e^t \\ 0 \end{bmatrix} dt$.

$\Phi(t) = Se^{\Lambda t}$, so $\Phi(t)^{-1} = e^{-\Lambda t}S^{-1}$. $S^{-1}\begin{bmatrix} -e^t \\ 0 \end{bmatrix} = -\frac{1}{3}e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\Phi(t)^{-1}\begin{bmatrix} -e^t \\ 0 \end{bmatrix} = e^{-\Lambda t}S^{-1}\begin{bmatrix} -e^t \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-5t} \end{bmatrix} e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 2e^{-t} \\ e^{-4t} \end{bmatrix}$.

$\int \Phi(t)^{-1}\begin{bmatrix} -e^t \\ 0 \end{bmatrix} dt = \frac{1}{3}\begin{bmatrix} 2e^{-t} \\ \frac{1}{4}e^{-4t} \end{bmatrix}$.

$\mathbf{x}(t) = \Phi(t)\frac{1}{3}\begin{bmatrix} 2e^{-t} \\ \frac{1}{4}e^{-4t} \end{bmatrix} = \frac{1}{3}S\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ \frac{1}{4}e^{-4t} \end{bmatrix} = \frac{1}{12}e^t S \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \frac{e^t}{12}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \frac{e^t}{4}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

We still need to get the initial condition right. We need a solution to the homogeneous equation with inital value $\frac{1}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Luckily, this is an eigenvector, for value 5, so the relevant solution is $\frac{e^{5t}}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and the solution we seek is $\frac{e^t}{4}\begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{e^{5t}}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises