$\boxed{5A-1}$ (a) Critical points occur where

$$x^2 - y^2 = 0 \quad \text{and} \quad x - xy = 0$$

Now $x^2 - y^2 = 0 \Rightarrow x = \pm y$

Also $x - xy = 0 \Rightarrow x(1-y) = 0$

$\Rightarrow x = 0$ or $y = 1$

$\therefore$ $x = 0$ and $y = 0$

OR $y = 1$ and $x = 1$

OR $y = 1$ and $x = -1$

$\therefore$ $(0,0)$, $(1,1)$ and $(-1,1)$ are the critical points

(b) Critical points occur where

$$1 - x + y = 0 \quad \text{and} \quad y + 2x^2 = 0$$

ie. $y = x - 1$

Then $0 = x - 1 + 2x^2$

ie $x = \frac{1}{2}$ or $x = -1$

But $x = \frac{1}{2} \Rightarrow y = -\frac{1}{2}$

and $x = -1 \Rightarrow y = -2$

$\therefore$ $(\frac{1}{2}, -\frac{1}{2})$ and $(-1,-2)$ are the critical points.

$\boxed{5A-2}$ (a) Let $y = x'$

Then $y' = x'' = -\mu(x^2-1)x' - x$

The autonomous equations are then

$$\begin{cases} x' = y \\ y' = -\mu(x^2-1)y - x \end{cases}$$

Critical points occur at

$$y = 0$$
$$-\mu(x^2-1)y - x = 0 \quad \text{ie. at } (0,0)$$

(b) Let $y = x'$

Then $y' = x'' = x' - 1 + x^2$

The autonomous equations are then

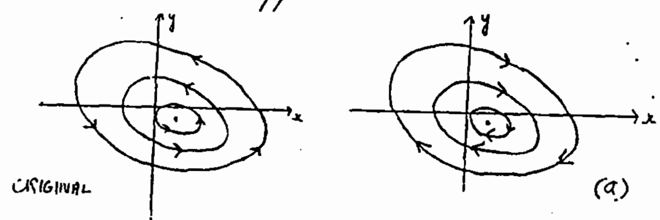
$$\begin{cases} x' = y \\ y' = y - 1 + x^2 \end{cases}$$

Critical points occur at

$$y = 0$$
$$y - 1 + x^2 = 0 \quad \therefore x^2 = 1 \quad \therefore x = \pm 1$$

So the critical points occur at $(1,0)$ and $(-1,0)$

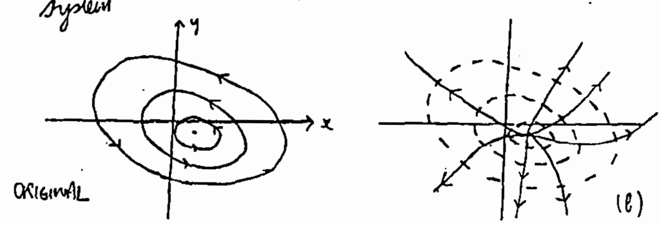
$\boxed{5A-3}$ (a) For this system the tangent vector $(-f(x,y), -g(x,y))$ to the trajectories is equal in magnitude and opposite in direction to the tangent vector $(f(x,y), g(x,y))$ to the original system. So the trajectories are the same but are traversed in the opposite direction



The critical points occur at

$$\left.\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}\right\} \text{ ie. the same for both systems}$$

$\boxed{5A-3}$ (b) For this system the tangent vector $(g(x,y), -f(x,y))$ to the trajectories is perpendicular to the tangent vector $(f(x,y), g(x,y))$ to the original system. So (b) represents the orthogonal trajectories of the original system



The critical points of (b) occur at

$$\left.\begin{aligned} g(x,y) &= 0 \\ -f(x,y) &= 0 \end{aligned}\right\} \text{ ie. the same as for the original system}$$

5A-4 a) Let $u = t - t_0$, let $\bar{x}(t) = x_1(t-t_0)$.

Then $x_1(t-t_0) = x_1(u)$ as a function of u
$= \bar{x}(t)$ as a function of t

[As an example: if $x_1 = t^2$, then $x_1(u) = u^2$;
and $\bar{x}(t) = t^2 - 2t_0 t + t_0^2$]

By hypothesis:

$$\frac{dx_1(t)}{dt} = f(x_1(t), y_1(t)) \qquad \frac{dy_1(t)}{dt} = g(x_1(t), y_1(t))$$

$\therefore$ changing letters formally:

$$\frac{dx_1(u)}{du} = f(x_1(u), y_1(u)) \qquad \frac{dy_1(u)}{du} = f(x_1(u), y_1(u)) \quad \circledast$$

But $\dfrac{d\bar{x}(t)}{dt} = \dfrac{dx_1(u)}{du}\cdot\dfrac{du}{dt} = \dfrac{dx_1(u)}{du}$; similarly $\dfrac{d\bar{y}(t)}{dt} = \dfrac{dy_1(u)}{du}$

Therefore, from $\circledast$ we get

$$\frac{d\bar{x}(t)}{dt} = f(\bar{x}(t), \bar{y}(t)) \qquad \frac{d\bar{y}(t)}{dt} = g(\bar{x}(t), \bar{y}(t)),$$

which shows that $\bar{x}(t), \bar{y}(t)$ is also a solution.

$\begin{cases}\bar{x}(t) \\ \bar{y}(t)\end{cases} = \begin{cases} x_1(t-t_0) \\ y_1(t-t_0)\end{cases}$ represent the same motion as $\begin{cases}x_1(t)\\ y_1(t)\end{cases}$, but occurring t_0 time-units later.

That is, $\begin{cases}\bar{x}(t_1+t_0) \\ \bar{y}(t_1+t_0)\end{cases} = \begin{cases}x_1(t_1)\\ y_1(t_1)\end{cases}$ so wherever $\begin{cases}x_1\\ y_1\end{cases}$ is at time t_1, $\begin{cases}\bar{x}\\ \bar{y}\end{cases}$ is there at time $t_1 + t_0$.

[This is the essential property of an autonomous system — the vector field does not change with time, so if we start at a given point t_0 seconds later, we follow the same ~~path~~ trajectory as before, but delayed by t_0 seconds.]

(b) Let $\begin{pmatrix} x_1(t) \\ y_1(t)\end{pmatrix}$ $\begin{pmatrix} x_2(t) \\ y_2(t)\end{pmatrix}$ be two trajectories which intersect at (a, b)

i.e. $\begin{pmatrix} x_1(t_0) \\ y_1(t_0)\end{pmatrix} = \begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} x_2(t_1) \\ y_2(t_1)\end{pmatrix}$ some t_0, t_1.

By part (a) $\begin{pmatrix} \bar{x}_2(t) \\ \bar{y}_2(t)\end{pmatrix} \equiv \begin{pmatrix} x_2(t - t_0 + t_1) \\ y_2(t - t_0 + t_1)\end{pmatrix}$

is also a solution to the ODE

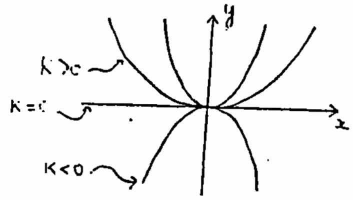
But $\begin{pmatrix} \bar{x}_2(t_0) \\ \bar{y}_2(t_0)\end{pmatrix} = \begin{pmatrix} x_2(t_1) \\ y_2(t_1)\end{pmatrix} = \begin{pmatrix} a \\ b\end{pmatrix}$

Thus by the uniqueness theorem

$$\begin{pmatrix} x_1(t) \\ y_1(t)\end{pmatrix} = \begin{pmatrix} \bar{x}_2(t) \\ \bar{y}_2(t)\end{pmatrix} = \begin{pmatrix} x_2(t - t_0 + t_1) \\ y_2(t - t_0 + t_1)\end{pmatrix} \text{ for all } t$$

i.e. $\begin{pmatrix} x_1(t) \\ y_1(t)\end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t)\end{pmatrix}$ are the same trajectory and differ at most by a change in parameter.

5B-1 (a) $\dfrac{y'}{x'} = \dfrac{dy}{dx} = \dfrac{-2y}{-x}$



$\therefore \dfrac{dy}{y} = 2\dfrac{dx}{x}$

$\therefore y = Kx^2$

(b) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t)\end{pmatrix}$ and $M = \begin{pmatrix} -1 & 0 \\ 0 & -2\end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$. This has solution

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

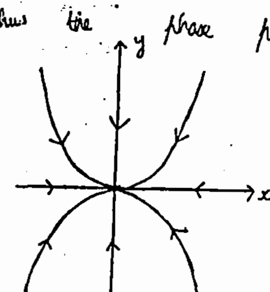
where λ_1 and λ_2 are the (distinct) eigenvalues of M with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$

Here $\lambda_1 = -1$, $\lambda_2 = -2$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1\end{pmatrix}$

Then $\begin{pmatrix} x(t) \\ y(t)\end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-2t}\end{pmatrix}$ All trajectories $\longrightarrow (0,0)$ as $t \to +\infty$

Thus the phase picture is:



The new trajectories are $\begin{cases} x = 0 \\ y = c_2 e^{-2t}\end{cases}$ $(c > 0, < 0, = 0)$

i.e. the positive and negative y-axis, and the trivial trajectory $\vec{x}(t) = 0$ (the origin)

c) As the picture shows, 3 trajectories are need to cover a typical solution curve from part (a): $\searrow$, $\swarrow$, and $\cdot$ (the origin).

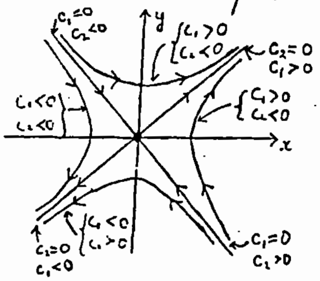
(d) This system may be obtained from the original by replacing t by $-t$. Thus we have the same trajectories but with the direction of the arrows reversed.

5B-2

a) $\frac{dy/dt}{dx/dt} = \frac{x}{y}$ $\therefore \frac{dy}{dx} = \frac{x}{y}$ soln: $y^2 - x^2 = c$ hyperbolas shown

b) $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

Thus the phase picture is.



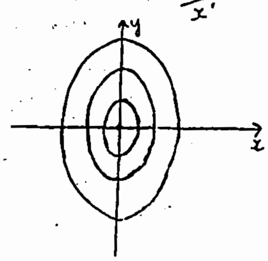
The only "new" trajectory is the trivial one $\vec{x}(t) = 0$ (the origin)

c) In general, each solution curve (part (a)) is covered by one trajectory. However, the two lines \ and / each require 3 trajectories to cover them.

(d) The system $\begin{cases} x' = -y \\ y' = -x \end{cases}$ has the same trajectories as the original system except the arrows are reversed.

5B-3

(a) $\frac{y'}{x'} = \frac{dy}{dx} = \frac{-2x}{y}$



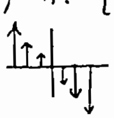
$\therefore \; y\,dy = -2x\,dx$

$\therefore \; \frac{y^2}{2} + x^2 = C$

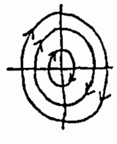
These are ellipses (for $C > 0$) and the point $(0,0)$ (for $C = 0$)

(b) For example, along the x-axis ($y=0$), the tangent vectors ~~are~~ at $(x_0, 0)$ is: $\begin{cases} x' = 0 \\ y' = -2x_0 \end{cases}$, i.e., $(0, -2x_0)$

Thus the field is



So the direction of motion along the ellipses is clockwise.



5B-4

(a) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

Thus $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The system has a critical point at $(0,0)$ which is a saddle point

The general solution is

$$\vec{x}(t) = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t}$$

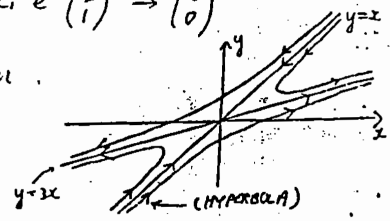
For $C_1 = 0$ and as $t \to \infty$

$$\vec{x}(t) = C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Also for $C_2 = 0$ and $t \to -\infty$

$$\vec{x}(t) = C_1 e^{t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the behaviour near the saddle point looks like



(b) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$.

M has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The system has an unstable node at $(0,0)$

The general solution is

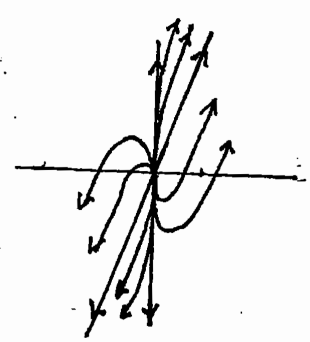
$$\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

So as $t \to -\infty$ all trajectories $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus the behaviour near the node looks like:

For $t \approx -\infty$, $C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$ is dominant term, $\therefore$ solns are near the y-axis

For $t \approx \infty$, $C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$ dominates so solns are parallel to $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$



5B-4

(c) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = -4$, $\lambda_2 = -1$

with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

The system has an asympt. stable node at $(0,0)$

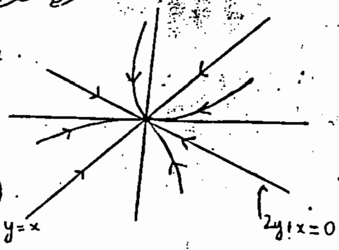
The general solution is

$$\vec{x}(t) = c_1 \vec{v}_1 e^{-4t} + c_2 \vec{v}_2 e^{-t}$$

As $t \to \infty$ all trajectories $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

~~$x(t) = c_1 e^{-4t} + 2c_2 e^{-t}$~~
~~$y(t) = c_1 e^{-4t} - c_2 e^{-t}$ not trivial curve!~~

The behaviour near the node looks like.



For ($t \approx -\infty$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$ dominates so sol'ns are parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
For $t \approx \infty$, $\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}$ dominates, so sol'ns are $\underbrace{\text{close to}}_{\text{"like"}}$ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.)

(d) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

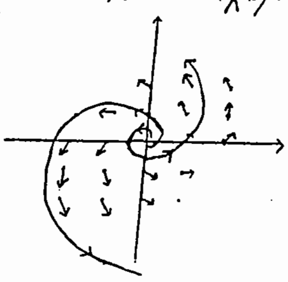
Then $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = 1 + i\sqrt{2}$, $\lambda_2 = 1 - i\sqrt{2}$

The system then has an unstable spiral around $(0,0)$.

Then $y = 0$
$x' = x$
$\therefore$ x is increasing where the spiral cuts the +ve x-axis

As we e^t behaviour the spiral is outwards from the origin



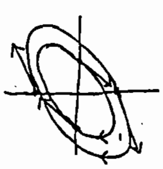
e) $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \vec{x}$

Eigenvalues are $\pm i$ (pure imaginary), so the system is a stable center.

(The curves are ellipses, since $\frac{dy}{dx} = \frac{-2x-y}{x+y}$ which integrates easily after cross-multiplying to $2x^2 + 2xy + y^2 = c$.)

Direction of motion:

For example, at $(1,0)$, the vector field is $x' = 1$, $y' = -2$. So motion is counterclockwise.



(a few other vectors are shown, inaccurately drawn...)

5B-5

(a) Let $y = x'$

Then, assuming $m \neq 0$,

$$y' = x'' = -\tfrac{c}{m}x' - \tfrac{k}{m}x$$

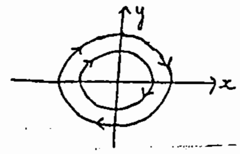
$\therefore$ The system is thus $\begin{cases} x' = y \\ y' = -\frac{k}{m}x - \frac{c}{m}y \end{cases}$

(b) The eigenvalues of $M = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$ are $\lambda_\pm = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

(i) $c = 0 \Rightarrow \lambda_\pm = \pm i\sqrt{\frac{k}{m}}$

Thus there is a stable centre at $(0,0)$.

Physically we'd expect this as putting $c = 0$ ($m, k > 0$) in the ODE gives the SHM equation. Then x and x' are periodic with period $2\pi\sqrt{\frac{m}{k}}$. Thus we expect periodic trajectories in phase space



Here $c^2 - 4km < 0$

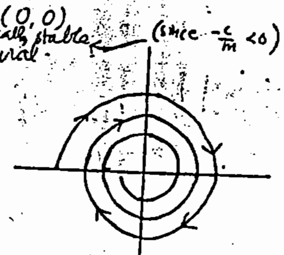
(ii) $\sqrt{c^2 - 4km} = 2i\sqrt{km}\left(1 - \frac{c^2}{4km}\right)^{\frac{1}{2}}$.

or, neglecting c, $\approx 2i\sqrt{km}$

Then $\lambda_\pm = -\frac{c}{m} \pm i\sqrt{\frac{k}{m}}$ (eigenvalues)

The behaviour near $(0,0)$ is that of an asymptotically stable spiral (since $-\frac{c}{m} < 0$)

The "radius" of the spiral decays as $t \to \infty$ like $e^{-\frac{c}{m}t}$ i.e. very slowly indeed!



Physically we have lightly damped harmonic motion e.g. a particle at the end of a spring oscillating in air. The motion is almost simple harmonic but the amplitude of oscillation decays slowly with time.

(iii) No!

When $c^2 - 4km \geq 0$, then as $k, m \geq 0$ we see $\sqrt{c^2 - 4km} \leq |c|$

Thus adding or subtracting $\sqrt{c^2 - 4km}$ to $-c$ cannot change its sign.

i.e. when the λ's are real, ~~either~~ they're both ~~positive or~~ ~~both~~ negative (since $c \geq 0$ always).

$\boxed{5C-1}$

$x' = x - y + xy$
$y' = 3x - 2y - xy$

linearization is (at $(0,0)$): $x' = x - y$, $y' = 3x - 2y$

$\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$ char eqn: $m^2 + m + 1 = 0$, $m = \dfrac{-1 \pm \sqrt{-3}}{2}$ $\therefore$ asymp. stable spiral

$\boxed{5C-2}$

$x' = x + 2x^2 - y^2$
$y' = x - 2y + x^3$

lin'zn: $x' = x$, $y' = x - 2y$ $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$

eigenvalues are $1, -2$ (since mx. is Δular) $\therefore$ unstable saddle

$\boxed{5C-3}$

$x' = 2x + y + xy^3$
$y' = x - 2y - xy$

lin'zn: $x' = 2x + y$, $y' = x - 2y$ $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$m^2 - 5 = 0$
$m = \pm\sqrt{5}$ unstable saddle

$\boxed{5C-4}$

$x' = 1 - y$
$y' = x^2 - y^2$

critical pts: $1 - y = 0$ $\therefore y = 1$; $x^2 - y^2 = 0$ $\therefore x = \pm 1$ $(1,1)$ and $(-1,1)$.

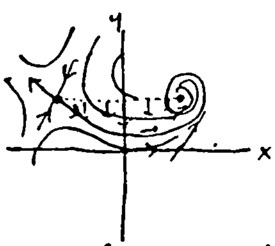
At $(1,1)$: since the in general Jac. matrix (of partial derivs) is $\begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$, the lin'zn is $\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$m^2 + 2m + 2 = 0$
$m = -1 \pm \dfrac{\sqrt{-4}}{2} = -1 \pm i$ $\therefore$ asym. stable spiral.

At $(-1,1)$: lin'zn is (again using Jacobian:) $\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ $\therefore m^2 + 2m - 2 = 0$, $m = -1 \pm \sqrt{3} = -2.73, +.73$

$\therefore$ unstable saddle. Eigenvectors: $-m\alpha_1 - \alpha_2 = 0$ $\therefore \begin{bmatrix} 1 \\ -m \end{bmatrix}$ $\therefore \begin{bmatrix} 1 \\ -.73 \end{bmatrix}, \begin{bmatrix} 1 \\ 2.73 \end{bmatrix}$

Using this info:



(Along dotted line, $y=1$, a few dir. field vectors are drawn, using the original system: $x' = 0$, $y' = x^2 - 1$

A few other vectors are drawn in to help the sketch

$\boxed{5C-5}$

This one is work, but instructive: think of x, y as 2 population which mutually ~~eat~~ destroy each other: $x - x^2$, $3y - 2y^2$ represent their "natural" growth laws, the $-xy$ terms their mutual destruction. [Like two hostile tribes, non-cannibalistic].

$x' = x - x^2 - xy$
$y' = 3y - 2y^2 - xy$

Critical points: $x(1 - x - y) = 0$
$y(3 - 2y - x) = 0$

From equation 1, either $x = 0$, or $1 - x - y = 0$.

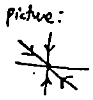
If $x = 0$, eqn 2 says: $y = 0$ or $y = 3/2$

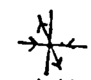
If $1 - x - y = 0$, eqn 2 says:
either $y = 0$ (in which case $1 - x = 0$, $x = 1$)
or $3 - 2y - x = 0$ (in which case we solve the 2 eqns: $1 - x - y = 0$, $3 - 2y - x = 0$ getting $y = 2$, $x = -1$

Summary: critical points are
$(0,0)$, $(0, 3/2)$, $(1, 0)$, $(-1, 2)$.

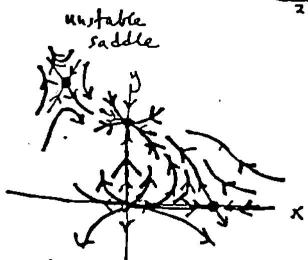
Now we determine their types: Jacobian matrix: $\begin{bmatrix} 1 - 2x - y & -x \\ -y & -x + 3 - 4y \end{bmatrix}$

$(0,0)$: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$  unstable node.

$(0, 3/2)$: $\begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$ eigenvs: $-1/2, -3$; vectors: $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ picture: 
asymp. stable node

$(1, 0)$ $\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ eigenvs: $-1, 2$; $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}$  unstable saddle

$(-1, 2)$ $\begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$ $m^2 + 3m - 2 = 0$, $m = \dfrac{-3 \pm \sqrt{17}}{2}$; $m \approx 1/2$, $m \approx -7/2$; $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$



the fat lines are impressionistic pieces of solution curves. Note there is no mutual coexistence! The tribe y always wins, (unless there is none of it to start with), essentially because of its stronger growth rate.

$\boxed{5D-1}$

a) Putting right-side of equations in (2) $= 0$ gives (assume $x \neq 0, y \neq 0$)

$$-\frac{x}{y} = 1 - x^2 - y^2 = \frac{y}{x} \qquad \therefore -x^2 = y^2$$

$$\text{so } x^2 + y^2 = 0 \quad \therefore x = 0,\ y = 0$$

(contradiction)

b) $(\cos t, \sin t)$ satisfies the system (just substitute); trajectory is the unit circle.

c) Equation (3) shows that if $R > 1$, the direction field points in towards the unit ⊙, and (along circle radius R) if $R < 1$, it points out towards the unit circle. Thus every solution curve is always getting closer to the unit ⊙.

$\boxed{5D-2}$

a) Bendixson criterion:

$$\operatorname{div}(f,g) = (1+3x^2) + (1+3y^2) > 0$$

$\therefore$ no limit cycle in xy-plane

b) System has no critical points, since $x^2+y^2=0 \Rightarrow x=0,\ y=0$, and this does not make $1+x-y=0$.

$\therefore$ no limit cycles.

c) System has no critical points if $x < -1$, $\therefore$ no limit cycles in this region.

[To see this: $x^2 - y^2 = 0 \Rightarrow y = \pm x$

$$2x + x^2 + y^2 = 0 \text{ and } y = \pm x \Rightarrow 2x + 2x^2 = 0 \quad \therefore x = 0, -1$$

thus critical pts. are $(0,0), (-1,1), (-1,-1)$.]

d) Bendixson's criterion:

$$\operatorname{div}(f,g) = a + 2bx - 2cy + 2cy - 2bx = a$$

$\therefore$ no limit cycles in xy-plane if $a \neq 0$.

$\boxed{5D-3}$

The system (7) is

$$x' = y$$
$$y' = -v(x) - u(x)y$$

a) By Bendixson's criterion,

$\operatorname{div}(f,g) = 0 - u(x) < 0$ for all x, y if $u(x) > 0$.

$\therefore$ no periodic solution.

b) $v(x) > 0 \Rightarrow$ system has no critical point [at a critical point, $y = 0$, $\therefore v(x) = 0$]

$\therefore$ no periodic solution.

$\boxed{5D-5}$ (like 5D-1)

$\boxed{5E-1}$ a) linearization is

$$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{bmatrix}1 & -4\\ 2 & -1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{at } (0,0).$$

Char. eqn: $\lambda^2 + 7 = 0$

$(0,0)$ is a center.

For non-lin. system, $(0,0)$ could be a center; or, unstable or asymptotically stable spiral.

b) linearization is

$$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}3 & -1\\ -6 & +2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{at } (0,0)$$

char. eqn: $\lambda^2 - 5\lambda = 0$, $\lambda = 0, 5$

$\therefore$ $(0,0)$ is not isolated — it is one of a line of critical points, all unstable:  ← line of crit pts

For non-linear system, picture could stay like this; or turn into an unstable node or saddle.

$\boxed{5E-2}$ a) $x' = y$ $\quad J = \begin{bmatrix}0 & 1\\ 1-2x & 0\end{bmatrix}$

$y' = x(1-x)$

Crit. pts: $(0,0)$, $(1,0)$

At $\underline{(0,0)}$, $J = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\lambda^2 - 1 = 0$

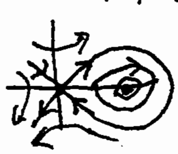
$\lambda = 1$, $\vec{\alpha} = \begin{pmatrix}1\\1\end{pmatrix}$; $\lambda = -1$, $\vec{\alpha} = \begin{pmatrix}1\\-1\end{pmatrix}$.

This is an <u>unstable saddle</u>.

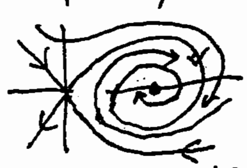
At $(1,0)$, $J = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $\lambda = \pm i$

This is a <u>center</u>, <u>clockwise motion</u>.

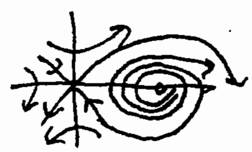
For non-linear system, three possibilities:



$(1,0)$ center



asymp. stable spiral



unstable spiral

$\boxed{5E-2}$ b) $x' = x^2 - x + y$

$y' = -y x^2 - y$

Crit. pts: $\begin{cases} x^2 - x - y = 0 \\ -y(x^2+1) = 0 \end{cases}$ $\quad \therefore y = 0$, $x = 0, 1$

Two crit. pts: $(0,0)$, $(1,0)$.

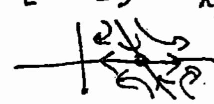
$$J = \begin{bmatrix}2x-1 & 1\\ -2xy & -x^2-1\end{bmatrix}$$

At $(0,0)$: $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$ $\quad \lambda = -1$, $\vec{\alpha} = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad$ repeated incomplete eigenvalue asy. stable node

Picture: 

At $(1,0)$: $\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}$ $\quad \lambda_1 = 1$, $\vec{\alpha}_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$\lambda_2 = -2$, $\vec{\alpha}_2 = \begin{pmatrix}1\\-3\end{pmatrix}$

Picture:  unstable saddle.

For non-linear system, two possibilities

$(0,0)$ becomes an asym. stable node

$(0,0)$ becomes an asym. stable spiral

$\boxed{5E-3}$ The new system is

$$x' = \tfrac{5}{4}ax - pxy,$$

$$y' = -by + qxy$$

whose critical pt is $\left(\frac{b}{q}, \frac{5a/4}{p}\right)$.

Crit. pt. for the orig. system is: $\left(\frac{b}{q}, \frac{a}{p}\right)$.

So the effect is to leave the flower population the same, but to increase the <u>borer</u> population by 25%.

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises