7. Fourier Series

Based on exercises in Chap. 8, Edwards and Penney, Elementary Differential Equations

7A. Fourier Series

7A-1. Find the smallest period for each of the following periodic functions:

a) $\sin \pi t/3$ b) $|\sin t|$ c) $\cos^2 3t$

7A-2. Find the Fourier series of the function $f(t)$ of period 2π which is given over the interval $-\pi < t \le \pi$ by

a) $f(t) = \begin{cases} 0, & -\pi < t \le 0; \\ 1, & 0 < t \le \pi \end{cases}$ b) $f(t) = \begin{cases} -t, & -\pi < t < 0; \\ t, & 0 \le t \le \pi \end{cases}$

7A-3. Give another proof of the orthogonality relations $\displaystyle\int_{-\pi}^{\pi} \cos mt \, \cos nt \, dt = \begin{cases} 0, & m \ne n; \\ \pi, & m = n \; . \end{cases}$

by using the trigonometric identity: $\cos A \cos B = \frac{1}{2}\big(\cos(A+B) + \cos(A-B)\big)$.

7A-4. Suppose that $f(t)$ has period P. Show that $\int_I f(t)\, dt$ has the same value over any interval I of length P, as follows:

a) Show that for any a, we have $\displaystyle\int_P^{a+P} f(t)\, dt = \int_0^a f(t)\, dt$. (Make a change of variable.)

b) From part (a), deduce that $\displaystyle\int_a^{a+P} f(t)\, dt = \int_0^P f(t)\, dt$.

7B. Even and Odd Series; Boundary-value Problems

7B-1. a) Find the Fourier cosine series of the function $1-t$ over the interval $0 < t < 1$, and then draw over the interval $[-2, 2]$ the graph of the function $f(t)$ which is the sum of this Fourier cosine series.

b) Answer the same question for the Fourier sine series of $1-t$ over the interval $(0,1)$.

7B-2. Find a formal solution as a Fourier series, for these boundary-value problems (you can use any Fourier series derived in the book's Examples):

a) $x'' + 2x = 1$, $x(0) = x(\pi) = 0$;
b) $x'' + 2x = t$, $x'(0) = x'(\pi) = 0$ (use a cosine series)

7B-3. Assume $a > 0$; show that $\displaystyle\int_{-a}^{0} f(t)\, dt = \pm \int_0^a f(t)\, dt$, according to whether $f(t)$ is respectively an even function or an odd function.

7B-4. The Fourier series of the function $f(t)$ having period 2, and for which $f(t) = t^2$ for $0 < t < 2$, is

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_1^\infty \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_1^\infty \frac{\sin n\pi t}{n} \; .$$

Differentiate this series term-by-term, and show that the resulting series does not converge to $f'(t)$.

7C. Applications to resonant frequencies

7C-1. For each spring-mass system, find whether pure resonance occurs, without actually calculating the solution.

a) $2x'' + 10x = F(t)$; $F(t) = 1$ on $(0, 1)$, $F(t)$ is odd, and of period 2;
b) $x'' + 4\pi^2 x = F(t)$; $F(t) = 2t$ on $(0, 1)$, $F(t)$ is odd, and of period 2;
c) $x'' + 9x = F(t)$; $F(t) = 1$ on $(0, \pi)$, $F(t)$ is odd, and of period 2π.

7C-2. Find a periodic solution as a Fourier series to $x'' + 3x = F(t)$, where $F(t) = 2t$ on $(0, \pi)$, $F(t)$ is odd, and has period 2π.

7C-3. For the following two lightly damped spring-mass systems, by considering the form of the Fourier series solution and the frequency of the corresponding undamped system, determine what term of the Fourier series solution should dominate — i.e., have the biggest amplitude.

a) $2x'' + .1x' + 18x = F(t)$; $F(t)$ is as in 7C-2.
b) $3x'' + x' + 30x = F(t)$; $F(t) = t - t^2$ on $(0, 1)$, $F(t)$ is odd, with period 2.

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