

# Poles, Amplitude Response, Connection to the Exponential Input Theorem

## 1. Introduction

For our standard linear constant coefficient system

$$p(D)x = f \tag{1}$$

the transfer function is  $W(s) = 1/p(s)$ . In this case, what we will call the poles of  $W(s)$  are simply the zeros of the characteristic polynomial  $p(s)$  (also known as the characteristic roots). We have had lots of experience using these roots in this course and know they give important information about the system. The reason we talk about the poles of the transfer function instead of just sticking with the characteristic roots is that the system (1) is a special case of a **Linear Time Invariant** (LTI) system and all LTI systems have a transfer function, while the characteristic polynomial is defined only for systems described by constant coefficient linear ODE's like (1).

We have seen (Notes S) that the stability of the system (1) is determined by the roots of the characteristic polynomial. We saw as well that the amplitude response of the system to a sinusoidal input of frequency  $\omega$  is also determined by the characteristic polynomial, namely formula FR (2) says the gain is

$$g(\omega) = \frac{1}{|p(i\omega)|}$$

The Laplace transform gives us another view of a signal by transforming it from a function of  $t$ , say  $f(t)$ , to a function  $F(s)$  of the complex frequency  $s$ .

A key object from this point of view is the transfer function. For the system (1), if we consider  $f(t)$  to be the input and  $x(t)$  to be the output, then the transfer function is  $W(s) = 1/p(s)$ , which is again determined by the characteristic polynomial.

We will now learn about poles and the pole diagram of an LTI system. This ties together the notions of stability, amplitude response and transfer function, all in one diagram in the complex  $s$ -plane. The pole diagram gives us a way to visualize systems which makes many of their important properties clear at a glance; in particular, and remarkably, the pole diagram

1. shows whether the system stable;
2. shows whether the unforced system is oscillatory;
3. shows the exponential rate at which the unforced system returns to equilibrium (for stable systems); and
4. gives a rough picture of the amplitude response and practical resonances of the system.

For these reasons the pole diagram is a standard tool used by engineers in understanding and designing systems.

We conclude by reminding you that every LTI system has a transfer function. Everything we learn in this session will apply to such systems, including those not modeled by DE's of the form (1)

## 2. Definition of Poles

**2.1. Rational Functions** A rational function is a ratio of polynomials  $q(s)/p(s)$ .

**Examples.** The following are all rational functions.  $(s^2 + 1)/(s^3 + 3s + 1)$ ,  $1/(ms^2 + bs + k)$ ,  $s^2 + 1 + (s^2 + 1)/1$ .

If the numerator  $q(s)$  and the denominator  $p(s)$  have no roots in common, then the rational function  $q(s)/p(s)$  is in **reduced form**

**Example.** The three functions in the example above are all in reduced form.

**Example.**  $(s - 2)/(s^2 - 4)$  is not in reduced form, because  $s = 2$  is a root of both numerator and denominator. We can rewrite this in reduced form as

$$\frac{s - 2}{s^2 - 4} = \frac{s - 2}{(s - 2)(s + 2)} = \frac{1}{s + 2}.$$

**2.2. Poles** For a rational function in reduced form the **poles** are the values of  $s$  where the denominator is equal to zero; or, in other words, the points where the rational function is not defined. We allow the poles to be complex numbers here.

**Examples.** a) The function  $1/(s^2 + 8s + 7)$  has poles at  $s = -1$  and  $s = -7$ .

b) The function  $(s - 2)/(s^2 - 4) = 1/(s + 2)$  has only one pole,  $s = -2$ .

c) The function  $1/(s^2 + 4)$  has poles at  $s = \pm 2i$ .

d) The function  $s^2 + 1$  has no poles.

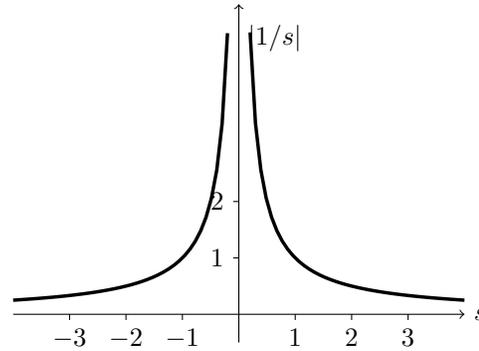
e) The function  $1/(s^2 + 8s + 7)(s^2 + 4)$  has poles at  $-1, -7, \pm 2i$ . (Notice that this function is the product of the functions in (a) and (c) and that its poles are the union of poles from (a) and (c).)

**Remark.** For ODE's with system function of the form  $1/p(s)$ , the poles are just the roots of  $p(s)$ . These are the familiar characteristic roots, which are important as we have seen.

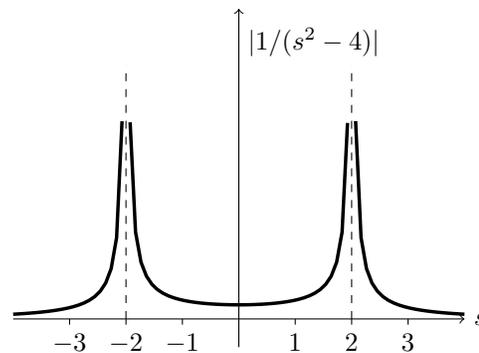
## 2.3. Graphs Near Poles

We start by considering the function  $F_1(s) = \frac{1}{s}$ . This is well defined for every complex  $s$  except  $s = 0$ . To visualize  $F_1(s)$  we might try to graph it. However it will be simpler, and yet still show everything we need, if we graph  $|F_1(s)|$  instead.

To start really simply, let's just graph  $|F_1(s)| = \frac{1}{|s|}$  for  $s$  real (rather than complex).

Figure 1: Graph of  $\frac{1}{|s|}$  for  $s$  real.

Now let's do the same thing for  $F_2(s) = 1/(s^2 - 4)$ . The roots of the denominator are  $s = \pm 2$ , so the graph of  $|F_2(s)| = \frac{1}{|s^2 - 4|}$  has vertical asymptotes at  $s = \pm 2$ .

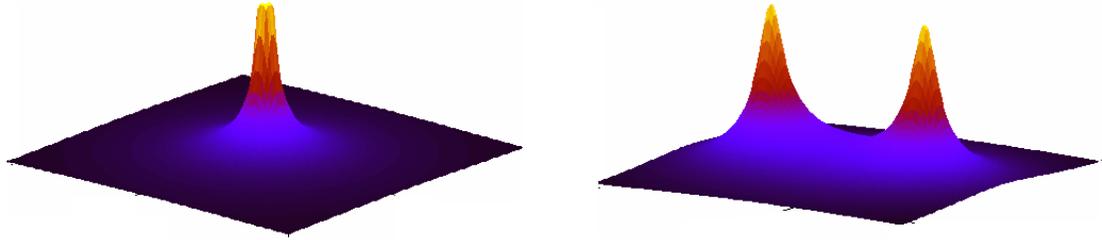
Figure 2: Graph of  $\frac{1}{|s^2 - 4|}$  for  $s$  real.

As noted, the vertical asymptotes occur at values of  $s$  where the denominator of our function is 0. These are what we defined as the poles.

- $F_1(s) = \frac{1}{s}$  has a single pole at  $s = 0$ .
- $F_2(s) = \frac{1}{s^2 - 4}$  has two poles, one each at  $s = \pm 2$ .

Looking at Figures 1 and 2 you might be reminded of a tent. The poles of the tent are exactly the vertical asymptotes which sit at the poles of the function.

Let's now try to graph  $|F_1(s)|$  and  $|F_2(s)|$  when we allow  $s$  to be complex. If  $s = a + ib$  then  $F_1(s)$  depends on two variables  $a$  and  $b$ , so the graph requires three dimensions: two for  $a$  and  $b$ , and one more (the vertical axis) for the value of  $|F_1(s)|$ . The graphs are shown in Figure 3 below. They are 3D versions of the graphs above in Figures 1 and 2. At each pole there is a conical shape rising to infinity, and far from the poles the function fall off to 0.

Figure 3: The graphs of  $|1/s|$  and  $1/|s^2 - 4|$ .

Roughly speaking, the poles tell you the shape of the graph of a function  $|F(s)|$ : it is *large near the poles*. In the typical pole diagrams seen in practice, the  $|F(s)|$  is also small far away from the poles.

#### 2.4. Poles and Exponential Growth Rate

If  $a > 0$ , the exponential function  $f_1(t) = e^{at}$  grows rapidly to infinity as  $t \rightarrow \infty$ . Likewise the function  $f_2(t) = e^{at} \sin bt$  is oscillatory with the amplitude of the oscillations growing exponentially to infinity as  $t \rightarrow \infty$ . In both cases we call  $a$  the *exponential growth rate* of the function.

The formal definition is the following

**Definition:** The **exponential growth rate** of a function  $f(t)$  is the smallest value  $a$  such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{e^{bt}} = 0 \quad \text{for all } b > a. \quad (2)$$

In words, this says  $f(t)$  grows slower than any exponential with growth rate larger than  $a$ .

##### Examples.

1.  $e^{2t}$  has exponential growth rate 2.
2.  $e^{-2t}$  has exponential growth rate -2. A negative growth rate means that the function is *decaying* exponentially to zero as  $t \rightarrow \infty$ .
3.  $f(t) = 1$  has exponential growth rate 0.
4.  $\cos t$  has exponential growth rate 0. This follows because  $\lim_{t \rightarrow \infty} \frac{\cos t}{e^{bt}} = 0$  for all positive  $b$ .
5.  $f(t) = t$  has exponential growth rate 0. This may be surprising because  $f(t)$  grows to infinity. But it grows linearly, which is slower than any positive exponential growth rate.
6.  $f(t) = e^{t^2}$  does not have an exponential growth rate since it grows faster than any exponential.

#### Poles and Exponential Growth Rate

We have the following theorem connecting poles and exponential growth rate.

**Theorem:** The exponential growth rate of the function  $f(t)$  is the largest real part of all the poles of its Laplace transform  $F(s)$ .

**Examples.** We'll check the theorem in a few cases.

1.  $f(t) = e^{3t}$  clearly has exponential growth rate equal to 3. Its Laplace transform is  $1/(s - 3)$  which has a single pole at  $s = 3$ , and this agrees with the exponential growth rate of  $f(t)$ .

2. Let  $f(t) = t$ , then  $F(s) = 1/s^2$ .  $F(s)$  has one pole at  $s = 0$ . This matches the exponential growth rate zero found in (5) from the previous set of examples.
3. Consider the function  $f(t) = 3e^{2t} + 5e^t + 7e^{-8t}$ . The Laplace transform is  $F(s) = 3/(s - 2) + 5/(s - 1) + 7/(s + 8)$ , which has poles at  $s = 2, 1, -8$ . The largest of these is 2. (Don't be fooled by the absolute value of -8, since  $2 > -8$ , the largest pole is 2.) Thus, the exponential growth rate is 2. We can also see this directly from the formula for the function. It is clear that the  $3e^{2t}$  term determines the growth rate since it is the dominant term as  $t \rightarrow \infty$ .
4. Consider the function  $f(t) = e^{-t} \cos 2t + 3e^{-2t}$ . The Laplace transform is  $F(s) = \frac{s}{(s+1)^2+4} + \frac{3}{s+2}$ . This has poles  $s = -1 \pm 2i, -2$ . The largest real part among these is -1, so the exponential growth rate is -1.

Note that in item (4) in this set of examples the growth rate is negative because  $f(t)$  actually *decays* to 0 as  $t \rightarrow \infty$ . We have the following

**Rule:**

1. If  $f(t)$  has a negative exponential growth rate then  $f(t) \rightarrow 0$  as  $t \rightarrow \infty$ .
2. If  $f(t)$  has a positive exponential growth rate then  $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

### 2.5. An Example of What the Poles Don't Tell Us

Consider an arbitrary function  $f(t)$  with Laplace transform  $F(s)$  and  $a > 0$ . Shift  $f(t)$  to produce  $g(t) = u(t - a)f(t - a)$ , which has Laplace transform  $G(s) = e^{-as}F(s)$ . Since  $e^{-as}$  does not have any poles,  $G(s)$  and  $F(s)$  have exactly the same poles. That is, the poles can't detect this type of shift in time.

## 3. Pole Diagrams

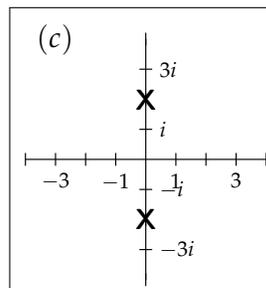
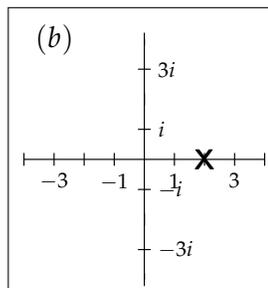
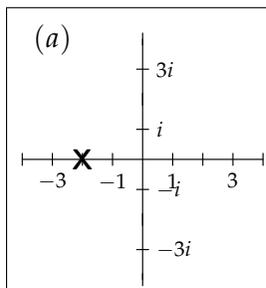
### 3.1. Definition of the Pole Diagram

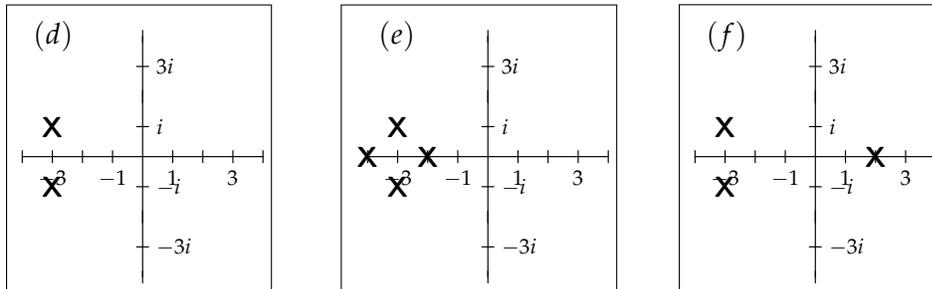
The **pole diagram** of a function  $F(s)$  is simply the complex  $s$ -plane with an **X** marking the location of each pole of  $F(s)$ .

**Example 1.** Draw the pole diagrams for each of the following functions.

- a)  $F_1(s) = \frac{1}{s+2}$       b)  $F_2(s) = \frac{1}{s-2}$       c)  $F_3(s) = \frac{1}{s^2+4}$   
 d)  $F_4(s) = \frac{s}{s^2+6s+10}$       e)  $F_5(s) = \frac{1}{((s^2+3)^2+1)(s+2)(s+4)}$       f)  $F_6(s) = \frac{1}{((s+3)^2+1)(s-2)}$

**Solution.**





For (d) we found the poles by first completing the square:  $s^2 + 6s + 10 = (s + 3)^2 + 1$ , so the poles are at  $s = -3 \pm i$ .

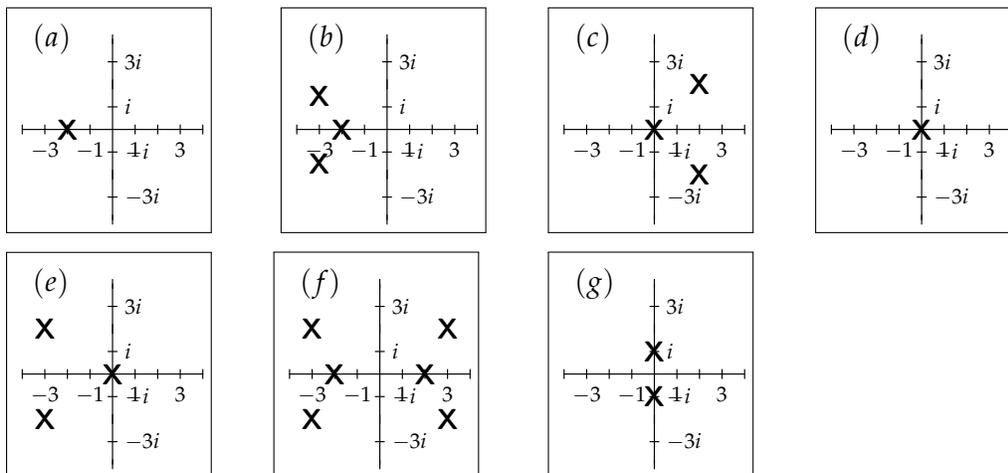
**Example 2.** Use the pole diagram to determine the exponential growth rate of the inverse Laplace transform of each of the functions in example 1.

**Solution.**

- The largest pole is at -2, so the exponential growth rate is -2.
- The largest pole is at 2, so the exponential growth rate is 2.
- The poles are  $\pm 2i$ , so the largest real part of a pole is 0. The exponential growth rate is 0.
- The largest real part of a pole is -3. The exponential growth rate is -3.
- The largest real part of a pole is -2. The exponential growth rate is -2.
- The largest real part of a pole is 2. The exponential growth rate is 2.

**Example 3.** Each of the pole diagrams below is for a function  $F(s)$  which is the Laplace transform of a function  $f(t)$ . Say whether

- $f(t) \rightarrow 0$  as  $t \rightarrow \infty$
- $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$
- You don't know the behavior of  $f(t)$  as  $t \rightarrow 0$ ,



**Solution.** a) Exponential growth rate is -2, so  $f(t) \rightarrow 0$ .

b) Exponential growth rate is -2, so  $f(t) \rightarrow 0$ .

c) Exponential growth rate is 2, so  $f(t) \rightarrow \infty$ .

d) Exponential growth rate is 0, so we can't tell how  $f(t)$  behaves.

Two examples of this: (i) if  $F(s) = 1/s$  then  $f(t) = 1$ , which stays bounded; (ii) if  $F(s) = 1/s^2$  then  $f(t) = t$ , which does go to infinity, but more slowly than any positive exponential.

- e) Exponential growth rate is 0, so don't know the behavior of  $f(t)$ .  
 f) Exponential growth rate is 3, so  $f(t) \rightarrow \infty$ .  
 g) Exponential growth rate is 0, so don't know the behavior of  $f(t)$ . (e.g. both  $\cos t$  and  $t \cos t$  have poles at  $\pm i$ ).

### 3.2. The Pole Diagram for an LTI System

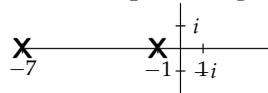
**Definition:** The pole diagram for an LTI system is defined to be the pole diagram of its transfer function.

**Example 4.** Give the pole diagram for the system

$$x'' + 8x' + 7x = f(t),$$

where we take  $f(t)$  to be the input and  $x(t)$  the output.

**Solution.** The transfer function for this system is  $W(s) = \frac{1}{s^2 + 8s + 7} = \frac{1}{(s+1)(s+7)}$ . Therefore, the poles are  $s = -1, -7$  and the pole diagram is



**Example 5.** Give the pole diagram for the system

$$x'' + 4x' + 6x = y',$$

where we consider  $y(t)$  to be the input and  $x(t)$  to be the output.

**Solution.** Assuming rest IC's, Laplace transforming this equation gives us  $(s^2 + 4s + 6)X = sY$ . This implies  $X(s) = \frac{s}{s^2 + 4s + 6}Y(s)$  and the transfer function is  $W(s) = \frac{s}{s^2 + 4s + 6}$ . This has poles at  $s = -2 \pm \sqrt{2}i$ .

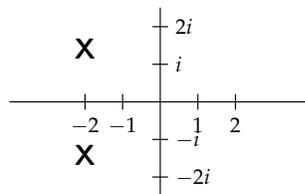


Figure: Pole diagram for the system in example 5.

## 4. Poles and Stability

Recall that the LTI system

$$p(D)x = f \tag{3}$$

has an associated homogeneous equation

$$p(D)x = 0 \tag{4}$$

In Notes S we saw the following stability criteria.

1. The system is stable if every solution to (4) goes to 0 as  $t \rightarrow \infty$ . In words, the unforced system always returns to equilibrium.

2. Equivalently, the system is stable if all the roots of the characteristic equation have negative real part.

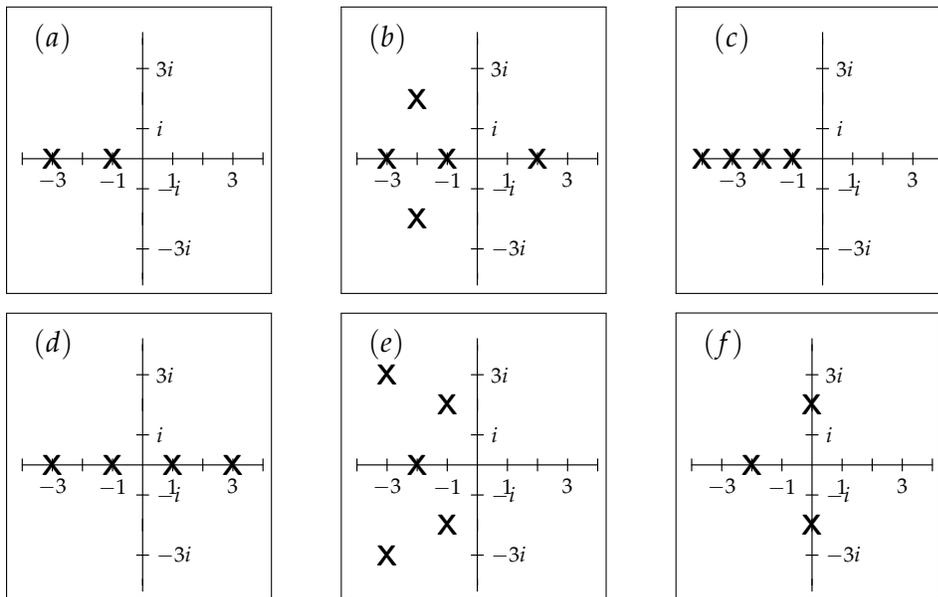
Now, since the transfer function for the system in (3) is  $\frac{1}{p(s)}$  the poles of the system are just the characteristic roots. Comparing this with the stability criterion 2, gives us another way of expressing the stability criteria.

3. The system is stable if all its poles have negative real part.

4. Equivalently, the system is stable if all its poles lie strictly in the left half of the complex plane  $\text{Re}(s) < 0$ .

Criterion 4 tells us how to see at a glance if the system is stable, as illustrated in the following example.

**Example.** Each of the following six graphs is the pole diagram of an LTI system. Say which of the systems are stable.



**Solution.** (a), (c) and (e) have all their poles in the left half-plane, so they are stable. The others do not, so they are not stable.

## 5. Poles and Amplitude Response

We started by considering the poles of functions  $F(s)$ , and saw that, by definition, the graph of  $|F(s)|$  went off to infinity at the poles. Since it tells us where  $|F(s)|$  is infinite, the pole diagram provides a crude graph of  $|F(s)|$ : roughly speaking,  $|F(s)|$  will be large for values of  $s$  near the poles. In this note we show how this basic fact provides a useful graphical tool for spotting resonant or near-resonant frequencies for LTI systems.

**Example 1.** Figure 1 shows the pole diagram of a function  $F(s)$ . At which of the points A, B, C on the diagram would you guess  $|F(s)|$  is largest?

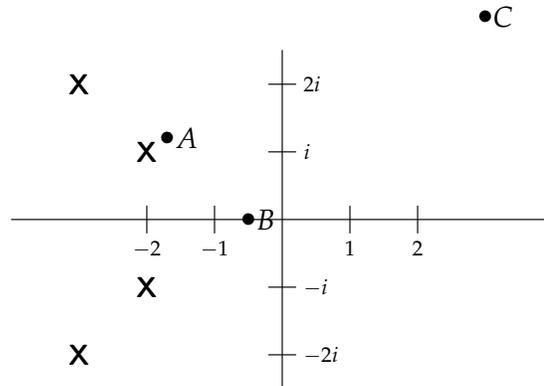


Figure 2: Pole diagram for example 1.

**Solution.** Point  $A$  is close to a pole and  $B$  and  $C$  are both far from poles so we would guess point  $|F(s)|$  is largest at point  $A$ .

**Example 2.** The pole diagram of a function  $F(s)$  is shown in Figure 2. At what point  $s$  on the positive imaginary axis would you guess that  $|F(s)|$  is largest?

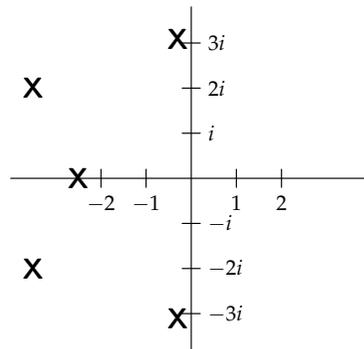


Figure 2: Pole diagram for example 2.

**Solution.** We would guess that  $s$  should be close to  $3i$ , which is near a pole. There is not enough information in the pole diagram to determine the exact location of the maximum, but it is most likely to be near the pole.

### 5.1. Amplitude Response and the System Function

Consider the system

$$p(D)x = f(t). \quad (5)$$

where we take  $f(t)$  to be the input and  $x(t)$  to be the output. The transfer function of this system is

$$W(s) = \frac{1}{p(s)}. \quad (6)$$

If  $f(t) = B \cos(\omega t)$  then equation FR (2) gives the following periodic solution to (5)

$$x_p(t) = \frac{B \cos(\omega t - \phi)}{|p(i\omega)|}, \quad \text{where } \phi = \text{Arg}(p(i\omega)). \quad (7)$$

If the system is stable, then all solutions are asymptotic to the periodic solution in (7). In

this case, we saw (FR (3)) that the amplitude response of the system as a function of  $\omega$  is

$$g(\omega) = \frac{1}{|p(i\omega)|}. \quad (8)$$

Comparing (6) and (8), we see that for a stable system the amplitude response is related to the transfer function by

$$g(\omega) = |W(i\omega)|. \quad (9)$$

**Note:** The relation (9) holds for all stable LTI systems.

Using equation (9) and the language of amplitude response we will now re-do example 2 to illustrate how to use the pole diagram to estimate the practical resonant frequencies of a stable system.

**Example 3.** Figure 3 shows the pole diagram of a stable LTI system. At approximately what frequency will the system have the biggest response?

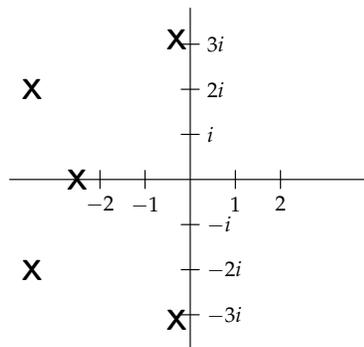


Figure 3: Pole diagram for example 3 (same as Figure 2).

**Solution.** Let the transfer function be  $W(s)$ . Equation (9) says the amplitude response  $g(\omega) = |W(i\omega)|$ . Since  $i\omega$  is on the positive imaginary axis, the amplitude response  $g(\omega)$  will be largest at the point  $i\omega$  on the imaginary axis where  $|W(i\omega)|$  is largest. This is exactly the point found in example 2. Thus, we choose  $i\omega \approx 3i$ , i.e. the practical resonant frequency is approximately  $\omega = 3$ .

**Note:** Rephrasing this in graphical terms: we can graph the magnitude of the system function  $|W(s)|$  as a surface over the  $s$ -plane. The amplitude response of the system  $g(\omega) = |W(i\omega)|$  is given by the part of the system function graph that lies above the imaginary axis. This is all illustrated beautifully by the applet [Amplitude: Pole Diagram](#).

**M.I.T. 18.03 Ordinary Differential  
Equations  
18.03 Notes and Exercises**

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