LT. Laplace Transform

1. Translation formula. The usual L.T. formula for translation on the t-axis is

$$(1) \qquad \mathcal{L}\big(u(t-a)f(t-a)\big) \;=\; e^{-as}F(s), \qquad \text{where } \; F(s)=\mathcal{L}\big(f(t)\big)\;, \quad a>0.$$

This formula is useful for computing the inverse Laplace transform of $e^{-as}F(s)$, for example. On the other hand, as written above it is not immediately applicable to computing the L.T. of functions having the form $u(t-a)f(t)$. For this you should use instead this form of (1):

$$(2) \qquad \mathcal{L}\big(u(t-a)f(t)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big), \quad a>0.$$

Example 1. Calculate the Laplace transform of $u(t-1)(t^2+2t)$.

Solution. Here $f(t)=t^2+2t$, so (check this!) $f(t+1)=t^2+4t+3$. So by (2),

$$\mathcal{L}\big(u(t-1)(t^2+2t)\big) \;=\; e^{-s}\mathcal{L}(t^2+4t+3) \;=\; e^{-s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{3}{s}\right) \;.$$

Example 2. Find $\mathcal{L}\big(u(t-\frac{\pi}{2})\sin t\big)$.

Solution.

$$\begin{aligned}\mathcal{L}\big(u(t-\tfrac{\pi}{2})\sin t\big) \;&=\; e^{-\pi s/2}\mathcal{L}\big(\sin(t+\tfrac{\pi}{2}\big)\\ &=\; e^{-\pi s/2}\mathcal{L}(\cos t) \;=\; e^{-\pi s/2}\frac{s}{s^2+1}\;.\end{aligned}$$

Proof of formula (2). According to (1), for any $g(t)$ we have

$$\mathcal{L}\big(u(t-a)g(t-a)\big) \;=\; e^{-as}\mathcal{L}\big(g(t)\big);$$

this says that to get the factor on the right side involving g, we should replace $t-a$ by t in the function $g(t-a)$ on the left, and then take its Laplace transform.

Apply this procedure to the function $f(t)$, written in the form $f(t)=f((t-a)+a)$; we get ("replacing $t-a$ by t and then taking the Laplace Transform")

$$\mathcal{L}\big(u(t-a)f((t-a)+a)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big),$$

exactly the formula (2) that we wanted to prove. $\square$

Exercises. Find: a) $\mathcal{L}\big(u(t-a)e^t\big)$ b) $\mathcal{L}\big(u(t-\pi)\cos t\big)$ c) $\mathcal{L}\big(u(t-2)te^{-t}\big)$

Solutions. a) $e^{-as}\dfrac{e^a}{s-1}$ b) $-e^{-\pi s}\dfrac{s}{s^2+1}$ c) $e^{-2s}\dfrac{e^{-2}(2s+3)}{(s+1)^2}$

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