LS.2   Homogeneous Linear
Systems with Constant Coefficients

1. Using matrices to solve linear systems.

The naive way to solve a linear system of ODE's with constant coefficients is by eliminating variables, so as to change it into a single higher-order equation. For instance, if

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

(1)

we can eliminate x by solving the second equation for x, getting $x = y + y'$, then replacing x everywhere by $y + y'$ in the first equation. This gives

$$y'' - 4y = 0 \ ;$$

the characteristic equation is $(r - 2)(r + 2) = 0$, so the general solution for y is

$$y = c_1 e^{2t} + c_2 e^{-2t} \ .$$

From this we get x from the equation $x = y + y'$ originally used to eliminate x; the whole solution to the system is then

$$\begin{aligned} x &= 3c_1 e^{2t} - c_2 e^{-2t} \\ y &= c_1 e^{2t} + c_2 e^{-2t} \ . \end{aligned}$$

(2)

We now want to introduce linear algebra and matrices into the study of systems like the one above. Our first task is to see how the above equations look when written using matrices and matrix multiplication.

When we do this, the system (1) and its general solution (2) take the forms

(4)
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ ,$$

(5)
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3c_1 e^{2t} - c_2 e^{-2t} \\ c_1 e^{2t} + c_2 e^{-2t} \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \ .$$

Study the above until it is clear to you how the matrices and column vectors are being used to write the system (1) and its solution (2). Note that when we multiply the column vectors by scalars or scalar functions, it does not matter whether we write them behind or in front of the column vector; the way it is written above on the right of (5) is the one usually used, since it is easiest to read and interpret.

We are now going to show a new method of solving the system (1), which makes use of the matrix form (4) for writing it. We begin by noting from (5) that two particular solutions to the system (4) are

(6)
$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} \qquad \text{and} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \ .$$

Based on this, our new method is to look for solutions to (4) of the form

6

$$(7) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \, ,$$

where a_1, a_2 and λ are unknown constants. We substitute (7) into the system (4) to determine what these unknown constants should be. This gives

$$(8) \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \lambda \, e^{\lambda t} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \, .$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$(9) \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \, .$$

We have to solve the matrix equation (9) for the three constants. It is not very clear how to do this. When faced with equations in unfamiliar notation, a reasonable strategy is to rewrite them in more familiar notation. If we try this, (9) becomes the pair of equations

$$(10) \qquad \begin{aligned} \lambda a_1 &= a_1 + 3a_2 \\ \lambda a_2 &= a_1 - a_2 \, . \end{aligned}$$

Technically speaking, these are a pair of non-linear equations in three variables. The trick in solving them is to look at them as a pair of linear equations in the unknowns a_i, with λ viewed as a parameter. If we think of them this way, it immediately suggests writing them in standard form

$$(11) \qquad \begin{aligned} (1-\lambda)a_1 + 3a_2 &= 0 \\ a_1 + (-1-\lambda)a_2 &= 0 \, . \end{aligned}$$

In this form, we recognize them as forming a square system of homogeneous linear equations. According to the theorem on square systems (LS.1, (5)), they have a non-zero solution for the a's if and only if the determinant of coefficients is zero:

$$(12) \qquad \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 \, ,$$

which after calculation of the determinant becomes the equation

$$(13) \qquad \lambda^2 - 4 = 0 \, .$$

The roots of this equation are 2 and -2; what the argument shows is that the equations (10) or (11) (and therefore also (8)) have non-trivial solutions for the a's exactly when $\lambda = 2$ or $\lambda = -2$.

To complete the work, we see that for these values of the parameter λ, the system (11) becomes respectively

$$(14) \qquad \begin{aligned} -a_1 + 3a_2 &= 0 & \qquad 3a_1 + 3a_2 &= 0 \\ a_1 - 3a_2 &= 0 & \qquad a_1 + a_2 &= 0 \\ (\text{for } \lambda = 2) & & (\text{for } \lambda = -2) \end{aligned}$$

It is of course no accident that in each case the two equations of the system become dependent, i.e., one is a constant multiple of the other. If this were not so, the two equations

would have only the trivial solution $(0,0)$. All of our effort has been to locate the two values of λ for which this will *not* be so. The dependency of the two equations is thus a check on the correctness of the value of λ.

To conclude, we solve the two systems in (14). This is best done by assigning the value 1 to one of the unknowns, and solving for the other. We get

(15) $\qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for $\lambda = 2$; $\qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\lambda = -2$,

which thus gives us, in view of (7), essentially the two solutions (6) we had found previously by the method of elimination. Note that the solutions (6) could be multiplied by an arbitrary non-zero constant without changing the validity of the general solution (5); this corresponds in the new method to selecting an arbitrary value of one of the a's, and then solving for the other value.

One final point before we discuss this method in general. Is there some way of passing from (9) (the point at which we were temporarily stuck) to (11) or (12) by using matrices, without writing out the equations separately? The temptation in (9) is to try to combine the two column vectors **a** by subtraction, but this is impossible as the matrix equation stands. If we rewrite it however as

(9′) $\qquad\qquad\qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} ,$

it now makes sense to subtract the left side from the right; using the distributive law for matrix multiplication, the matrix equation (9′) then becomes

(11′) $\qquad\qquad\qquad \begin{pmatrix} 1 - \lambda & 3 \\ 1 & -1 - \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$

which is just the matrix form for (11). Now if we apply the theorem on square homogeneous systems, we see that (11′) has a nontrivial solution for the **a** if and only if its coefficient determinant is zero, and this is precisely (12). The trick therefore was in (9) to replace the scalar λ by the diagonal matrix λI .

2. Eigenvalues and eigenvectors.

With the experience of the preceding example behind us, we are now ready to consider the general case of a homogeneous linear 2×2 system of ODE's with constant coefficients:

(16) $\qquad\qquad\qquad\qquad\qquad \begin{aligned} x' &= ax + by \\ y' &= cx + dy , \end{aligned}$

where the a, b, c, d are constants. We write this system in matrix form as

(17) $\qquad\qquad\qquad\qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$

We look for solutions to (17) having the form

(18) $\qquad\qquad\qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{pmatrix} ,$

where a_1, a_2, and λ are unkinown constants. We substitute (18) into the system (17) to determine these unknown constants. Since $D(ae^{\lambda t}) = \lambda ae^{\lambda t}$, we arrive at

$$\text{(19)} \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} .$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$\text{(20)} \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

As the equation (20) stands, we cannot combine the two sides by subtraction, since the scalar λ cannot be subtracted from the square matrix on the right. As in the previously worked example however (9′), the trick is to replace the scalar λ by the diagonal matrix λI; then (20) becomes

$$\text{(21)} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

If we now proceed as we did in the example, subtracting the left side of (6) from the right side and using the distributive law for matrix addition and multiplication, we get a 2×2 homogeneous linear system of equations:

$$\text{(22)} \qquad \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Written out without using matrices, the equations are

$$\text{(23)} \qquad \begin{aligned} (a - \lambda)a_1 + ba_2 &= 0 \\ ca_1 + (d - \lambda)a_2 &= 0 . \end{aligned}$$

According to the theorem on square homogeneous systems, this system has a non-zero solution for the a's if and only if the determinant of the coefficients is zero:

$$\text{(24)} \qquad \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 .$$

The equation (24) is a quadratic equation in λ, evaluating the determinant, we see that it can be written

$$\text{(25)} \qquad \lambda^2 - (a + d)\lambda + (ad - bc) = 0 .$$

Definition. *The equation (24) or (25) is called the* **characteristic equation** *of the matrix*

$$\text{(26)} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Its roots λ_1 and λ_2 are called the **eigenvalues** *or characteristic values of the matrix A.*

There are now various cases to consider, according to whether the eigenvalues of the matrix A are two distinct real numbers, a single repeated real number, or a pair of conjugate complex numbers. We begin with the first case: *we assume for the rest of this chapter that the eigenvalues are two distinct real numbers λ_1 and λ_2.*

To complete our work, we have to find the solutions to the system (23) corresponding to the eigenvalues λ_1 and λ_2. Formally, the systems become

(27)
$$
\begin{aligned}
(a - \lambda_1)a_1 + ba_2 &= 0 \qquad\qquad (a - \lambda_2)a_1 + ba_2 = 0 \\
ca_1 + (d - \lambda_1)a_2 &= 0 \qquad\qquad ca_1 + (d - \lambda_2)a_2 = 0
\end{aligned}
$$

The solutions to these two systems are column vectors, for which we will use Greek letters rather than boldface.

Definition. *The respective solutions $\mathbf{a} = \vec{\alpha}_1$ and $\mathbf{a} = \vec{\alpha}_2$ to the systems (27) are called the* **eigenvectors** *(or characteristic vectors) corresponding to the eigenvalues λ_1 and λ_2.*

If the work has been done correctly, in each of the two systems in (27), the two equations will be dependent, i.e., one will be a constant multiple of the other. Namely, the two values of λ have been selected so that in each case the coefficient determinant of the system will be zero, which means the equations will be dependent. The solution $\vec{\alpha}$ is determined only up to an arbitrary non-zero constant factor. A convenient way of finding the eigenvector $\vec{\alpha}$ is to assign the value 1 to one of the a_i, then use the equation to solve for the corresponding value of the other a_i.

Once the eigenvalues and their corresponding eigenvectors have been found, we have two independent solutions to the system (16); According to (19), they are

(28)
$$
\mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_2 t}, \qquad \text{where } \mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}.
$$

Then the general solution to the system (16) is

(29)
$$
\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \vec{\alpha}_1 e^{\lambda_1 t} + c_2 \vec{\alpha}_2 e^{\lambda_2 t}.
$$

At this point, you should stop and work another example, like the one we did earlier. Try **5.4** Example 1 in your book; work it out yourself, using the book's solution to check your work. Note that the book uses $\mathbf{v}$ instead of $\vec{\alpha}$ for an eigenvector, and v_i or a, b instead of a_i for its components.

We are still not done with the general case; without changing any of the preceding work, you still need to see how it appears when written out using an even more abridged notation. Once you get used to it (and it is important to do so), the compact notation makes the essential ideas stand out very clearly.

As before, we let A denote the matrix of constants, as in (26). Below, on the left side of each line, we will give the compact matrix notation, and on the right, the expanded version. The equation numbers are the same as the ones above.

We start with the system (16), written in matrix form, with A as in (26):

(17′)
$$
\mathbf{x}' = A\mathbf{x} \qquad\qquad\qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
$$

We use as the trial solution

(18′)
$$
\mathbf{x} = \mathbf{a}\, e^{\lambda t} \qquad\qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}.
$$

We substitute this expression for $\mathbf{x}$ into the system (17'), using $\mathbf{x}' = \lambda \mathbf{a} e^{\lambda t}$:

$$(19') \qquad \lambda\,\mathbf{a}\,e^{\lambda t} \;=\; A\,\mathbf{a}\,e^{\lambda t} \qquad\qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}\;.$$

Cancel the exponential factor from both sides, and replace λ by λI, where I is the identity matrix:

$$(21') \qquad \lambda I \mathbf{a} \;=\; A\mathbf{a} \qquad\qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\;.$$

Subtract the left side from the right and combine terms, getting

$$(22') \qquad (A - \lambda I)\mathbf{a} \;=\; 0 \qquad\qquad \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}\;.$$

This square homogeneous system has a non-trivial solution if and only if the coefficient determinant is zero:

$$(24') \qquad |A - \lambda I| \;=\; 0 \qquad\qquad \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \;=\; 0\;.$$

Definition. *Let A be a square matrix of constants, Then by definition*

(i) $|A - \lambda I| = 0$ *is the* **characteristic equation** *of A ;*

(ii) *its roots λ_i are the* **eigenvalues** *(or characteristic values) of A ;*

(iii) *for each eigenvalue λ_i, the corresonding solution $\vec{\alpha}_i$ to (22') is the* **eigenvector** *(or characteristic vector) associated with λ_i .*

If the eigenvalues are distinct and real, as we are assuming in this chapter, we obtain in this way two independent solutions to the system (17'):

$$(28) \qquad \mathbf{x}_1 \;=\; \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x}_2 \;=\; \vec{\alpha}_2 e^{\lambda_2 t}, \qquad \text{where } \mathbf{x}_i \;=\; \begin{pmatrix} x_i \\ y_i \end{pmatrix}\;.$$

Then the general solution to the system (16) is

$$(29) \qquad \mathbf{x} \;=\; c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \;=\; c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t}\;.$$

The matrix notation on the left above in (17') to (24') is compact to write, makes the derivation look simpler. Moreover, when written in matrix notation, *the derivation applies to square systems of any size*: $n \times n$ just as well as 2×2. This goes for the subsequent definition as well: it defines *characteristic equation, eigenvalue* and *associated eigenvector* for a square matrix of any size.

The chief disadvantage of the matrix notation on the left is that for beginners it is very abridged. Practice writing the sequence of matrix equations so you get some skill in using the notation. Until you acquire some confidence, keep referring to the written-out form on the right above, so you are sure you understand what the abridged form is actually saying.

Since in the compact notation, the definitions and derivations are valid for square systems of any size, you now know for example how to solve a 3×3 system, if its eigenvalues turn out to be real and distinct; **5.4** Example 2 in your book is such a system. First however read the following remarks which are meant to be helpful in doing calculations: remember and *use* them.

Remark 1. Calculating the characteristic equation.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its characteristic equation is given by (cf. (24) and (25)):

(30) $\qquad \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0, \qquad \text{or} \qquad \lambda^2 - (a + d)\lambda + (ad - bc) = 0 .$

Since you will be calculating the characteristic equation frequently, learn to do it using the second form given in (30). The two coefficients have analogs for any square matrix:

$$ad - bc = \det A \qquad\qquad a + d = \operatorname{tr} A \; (\textit{trace } A)$$

where the **trace** of a square matrix A is the sum of the elements on the main diagonal. Using this, the characteristic equation (30) for a 2×2 matrix A can be written

(31) $\qquad\qquad\qquad\qquad \lambda^2 - (\operatorname{tr} A)\, \lambda + \det A = 0 .$

In this form, the characteristic equation of A can be written down by inspection; you don't need the intermediate step of writing down $|A - \lambda I| = 0$. For an $n \times n$ matrix, the characteristic equation reads in part (watch the signs!)

(32) $\qquad\qquad |A - \lambda I| = (-\lambda)^n + \operatorname{tr} A(-\lambda)^{n-1} + \ldots + \det A = 0 .$

In one of the exercises you are asked to derive the two coefficients specified.

Equation (32) shows that the characteristic polynomial $|A - \lambda I|$ of an $n \times n$ matrix A is a polynomial of degree n, so that such a matrix has at most n real eigenvalues. The trace and determinant of A give two of the coefficients of the polynomial. Even for $n = 3$ however this is not enough, and you will have to calculate the characteristic equation by expanding out $|A - \lambda I|$. Nonetheless, (32) is still very valuable, as it enables you to get an independent check on your work. Use it whenever $n > 2$.

Remark 2. Calculating the eigenvectors.

This is a matter of solving a homogeneous system of linear equations (22').

For $n = 2$, there will be just one equation (the other will be a multiple of it); give one of the a_i's the value 1 (or any other convenient non-zero value), and solve for the other a_i.

For $n = 3$, two of the equations will usually be independent (i.e., neither a multiple of the other). Using just these two equations, give one of the a's a convenient value (say 1), and solve for the other two a's. (The case where the three equations are all multiples of a single one occurs less often and will be dealt with later.)

Remark 3. Normal modes.

When the eigenvalues of A are all real and distinct, the corresponding solutions (28)
$$\mathbf{x}_i = \vec{\alpha}_i e^{\lambda_i t}, \qquad i = 1, \ldots, n,$$

are usually called the *normal modes* in science and engineering applications. They often have physical interpretations, which sometimes makes it possible to find them just by inspection of the physical problem. The exercises will illustrate this.

Exercises: Section 4C

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises