IR. Input-Response Models

1. First-order linear ODE's with positive constant coefficient. This is probably the most important first order equation; we use t and y as the variables, and think of the independent variable t as representing time. The IVP in standard form then is

$$(1) \qquad y' + ky = q(t), \quad k > 0; \qquad y(0) = y_0 \ .$$

The integrating factor for the ODE in (1) is e^{kt}; using it, the general solution is

$$(2) \qquad y = e^{-kt}\left(\int q(t)e^{kt}dt + c\right).$$

To get from this an explicit solution to (1), we change (cf. Notes D) the indefinite integral in (2) to a definite integral from 0 to t, which requires us to change the t in the integrand to a different dummy variable, u say; then the explicit solution to (1) is

$$(3) \qquad y = e^{-kt}\int_0^t q(u)e^{ku}du + y_0 e^{-kt}.$$

In this form, note that the first term on the right is the solution to the IVP (1) corresponding to the initial condition $y_0 = 0$.

What we have done so far does not depend on whether k is positive or negative. However, the terminology we will now introduce makes sense only when $k > 0$, which we shall assume from now on.

Looking at (3) and assuming $k > 0$, we observe that as $t \to \infty$, the second term of the solution $y_0 e^{-kt} \to 0$, regardless of the initial value y_0. It is therefore called the **transient** since its effect on the solution dies away as time increases. As it dies away, what is left of the solution is the integral term on the right, which does not involve the initial value y_0; it is called the **steady-state** or **long-term** solution to (1).

$$(4) \qquad y \ = \underbrace{e^{-kt}\int_0^t q(u)e^{ku}du}_{\textit{steady-state}} + \underbrace{y_0 e^{-kt}}_{\textit{transient}}, \qquad k > 0.$$

Despite the use of the definite article, the steady-state solution is not unique: since all the solutions approach the steady-state solution as $t \to \infty$, they all approach each other, and thus any of them can be called the steady-state solution. In practice, it is usually the simplest-looking solution which is given this honor.

2. Input-response; superposition of inputs. When the ODE (1) is used to model a physical situation, the left-hand side usually is concerned with the physical set-up — the "system" — while the right-hand side represents something external which is driving or otherwise affecting the system from the outside. For this reason, the function $q(t)$ is often called the **input**, or in some engineering subjects, the **signal**; the corresponding general solution (2) is called the **response** of the system to this input.

We will indicate the relation of input to response symbolically by

$$q(t) \rightsquigarrow y(t) \qquad (\textit{input} \rightsquigarrow \textit{response}).$$

Superposition principle for inputs.

For the ODE $y' + ky = q(t)$, *let* $q_1(t)$ *and* $q_2(t)$ *be inputs, and* $c_1,\ c_2$ *constants. Then*

$$q_1 \rightsquigarrow y_1, \quad q_2 \rightsquigarrow y_2 \quad \Longrightarrow \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2. \tag{5}$$

Proof. This follows in one line from the sum property of indefinite integrals; note that the proof does not require k to be positive.

$$y = e^{-kt}\int (q_1+q_2)e^{kt}dt = e^{-kt}\int q_1e^{kt}dt + e^{-kt}\int q_2e^{kt}dt = y_1 + y_2\ .$$

The superposition principle allows us to break up a problem into simpler problems and then at the end assemble the answer from its simpler pieces. Here is an easy example.

Example 1. Find the response of $y' + 2y = q(t)$ to $q = 1 + e^{-2t}$.

Solution. The input $q = 1$ generates the response $y = 1/2$, by inspection; the input e^{-2t} generates the response te^{-2t}, by solving; therefore the response to $1 + e^{-2t}$ is $1/2 + te^{-2t}$. □

3. Physical inputs; system responses to linear inputs.

We continue to assume $k > 0$. We want to get some feeling for how the system responds to a variety of inputs. The temperature model for (1) will be a good guide: in two notations – suggestive and neutral, respectively – the ODE is

$$T' + kT = kT_e(t), \qquad\qquad y' + ky = kq_e(t) = q(t). \tag{6}$$

Note that the neutral notation writes the input in two different forms: the $q(t)$ we have been using, and also in the form $kq_e(t)$ with the k factored out. This last corresponds to the way the input appears in certain physical problems (temperature and diffusion problems, for instance) and leads to more natural formulas: for example, q_e and y have the same units, whereas q and y do not.

For this class of problems, the relation of response to input will be clearer if we relate y with q_e, rather than with q. We will use for q_e the generic name **physical input**, or if we have a specific model in mind, the *temperature* input, *concentration* input, and so on.

The expected behavior of the temperature model suggests general questions such as:

Is the response the same type of function as the physical input? What controls its magnitude?

Does the graph of the response lag behind that of the physical input?

What controls the size of the lag?

Our plan will be to get some feeling for the situation by answering these questions for several simple physical inputs. We begin with linear inputs. Throughout, keep the temperature model in mind to guide your intuition.

Example 2. Find the response of the system (6) to the physical inputs 1 and t.

Solution. The ODE is $y' + ky = kq_e$.

If $q_e = 1$, a solution by inspection is $y = 1$, so the response is 1.

If $q_e = t$, the ODE is $y' + ky = kt$; using the integrating factor e^{kt} and subsequent integration by parts leads (cf. (2)) to the simplest steady-state solution

$$\begin{aligned} y &= e^{-kt}\int kte^{kt}dt \\ &= k\,e^{-kt}\left(\frac{te^{kt}}{k} - \frac{e^{kt}}{k^2}\right) \\ &= t - \frac{1}{k}. \end{aligned}$$

Thus the response of (6) is identical to the physical input t, but with a time lag $1/k$. This is reasonable when one thinks of the temperature model: the internal temperature increases linearly at the same rate as the temperature of the external water bath, but with a time lag dependent on the conductivity: the higher the conductivity, the shorter the time lag.

Using the superposition principle for inputs, it follows from Example 2 that for the ODE $y' + ky = kq_e$, its response to a general linear physical input is given by:

(7) **linear input** physical input: $q_e = a + bt$ response: $a + b\left(t - \frac{1}{k}\right)$.

In the previous example, we paid no attention to initial values. If they are important, one cannot just give the steady-state solution as the response, one has to take account of them, either by using a definite integral as in (3), or by giving the value of the arbitrary constant in (2). Examples in the next section will illustrate.

4. Response to discontinuous inputs, $k > 0$.

The most basic discontinuous function is the unit-step function at a point, defined by

(8) $$u_a(t) = \begin{cases} 0, & t < a; \\ 1, & t > a. \end{cases}$$ **unit-step function at a**

(We leave its value at a undefined, though some books give it the value 0, others the value 1 there.)

Example 3. Find the response of the IVP $y' + ky = kq_e$, $y(0) = 0$, for $t \geq 0$, to the unit-step physical input $u_a(t)$, where $a \geq 0$.

Solution. For $t < a$ the input is 0, so the response is 0. For $t \geq a$, the steady-state solution for the physical input $u_a(t)$ is the constant function 1, according to Example 2 or (7).

We still need to fit the value $y(a) = 0$ to the response for $t \geq a$. Using (2) to do this, we get $1 + ce^{-ka} = 0$, so that $c = -e^{ka}$. We now assemble the results for $t < a$ and $t \geq a$ into one expression; for the latter we also put the exponent in a more suggestive form. We get finally

unit-step input

(9) physical input: $u_a(t),\ a \geq 0$ response: $$y(t) = \begin{cases} 0, & 0 \leq t < a; \\ 1 - e^{-k(t-a)}, & t \geq a. \end{cases}$$

Note that the response is just the translation a units to the right of the response to the unit-step input at 0.

Another way of getting the same answer would be to use the definite integral in (3); we leave this as an exercise.

As another example of discontinuous input, we focus on the temperature model, and obtain the response to the temperature input corresponding to the external bath initially ice-water at 0 degrees, then replaced by water held at a fixed temperature for a time interval, then replaced once more by the ice-water bath.

Example 4. Find the response of $y' + ky = kq_e$ to the physical input

$$u_{ab} = \begin{cases} 1, & a \le t \le b; \\ 0, & \text{otherwise,} \end{cases} \qquad 0 \le a < b; \qquad \textbf{unit-box function} \text{ on } [a,b]. \tag{10}$$

Solution. There are at least three ways to do this:

a) Express u_{ab} as a sum of unit step functions and use (9) together with superposition of inputs;

b) Use the function u_{ab} directly in the definite integral expression (3) for the response;

c) Find the response in two steps: first use (9) to get the response $y(t)$ for the physical input $u_a(t)$; this will be valid up to the point $t = b$.

Then, to continue the response for values $t > b$, evaluate $y(b)$ and find the response for $t > b$ to the input 0, with initial condition $y(b)$.

We will follow (c), leaving the first two as exercises.

By (9), the response to the physical input $u_a(t)$ is $y(t) = \begin{cases} 0, & 0 \le t < a; \\ 1 - e^{-k(t-a)}, & t \ge a. \end{cases}$; this is valid up to $t = b$, since $u_{ab}(t) = u_a(t)$ for $t \le b$. Evaluating at b,

$$y(b) = 1 - e^{-k(b-a)}. \tag{11}$$

Using (2) to find the solution for $t \ge b$, we note first that the steady-state solution will be 0, since $u_{ab} = 0$ for $t > b$; thus by (2) the solution for $t > b$ will have the form

$$y(t) = 0 + ce^{-kt} \tag{12}$$

where c is determined from the initial value (11). Equating the initial values $y(b)$ from (11) and (12), we get

$$ce^{-kb} \;=\; 1 - e^{-kb+ka}$$

from which

$$c = e^{kb} - e^{ka};$$

so by (12),

$$y(t) = (e^{kb} - e^{ka})e^{-kt}, \quad t \ge b. \tag{13}$$

After combining exponents in (13) to give an alternative form for the response, we assemble the parts, getting the *response to the physical unit-box input* u_{ab} *:*

$$y(t) = \begin{cases} 0, & 0 \le t \le a; \\ 1 - e^{-k(t-a)}, & a < t < b; \\ e^{-k(t-b)} - e^{-k(t-a)}, & t \ge b. \end{cases} \tag{14}$$

5. Response to sinusoidal inputs.

Of great importance in the applications is the sinusoidal input, i.e., a pure oscillation like $\cos\omega t$ or $\sin\omega t$, or more generally, $A\cos(\omega t - \phi)$. (The last form includes both of the previous two, as you can see by letting $A = 1$ and $\phi = 0$ or $\pi/2$).

In the temperature model, this could represent the diurnal varying of outside temperature; in the conentration model, the diurnal varying of the level of some hormone in the bloodstream, or the varying concentration in a sewer line of some waste product produced periodically by a manufacturing process.

What follows assumes some familiarity with the vocabulary of pure oscillations: *amplitude, frequency, period, phase lag.* Section 6 following this gives a brief review of these terms plus a few other things that we will need: look at it first, or refer to it as needed when reading this section.

Response of $y' + ky = kq_e$ to the physical inputs $\cos\omega t,\ \sin\omega t$.

This calculation is a good example of how the use of complex exponentials can simplify integrations and lead to a more compact and above all more expressive answer. You should study it very carefully, since the ideas in it will frequently recur.

We begin by complexifying the inputs, the response, and the differential equation:

$$\cos\omega t = \mathrm{Re}\,(e^{i\omega t}), \qquad \sin\omega t = \mathrm{Im}\,(e^{i\omega t}); \tag{15}$$

$$\tilde{y}(t) = y_1(t) + iy_2(t); \tag{16}$$

$$\tilde{y}' + k\tilde{y} = ke^{i\omega t}. \tag{17}$$

If (16) is a solution to the complex ODE (17), then substituting it into the ODE and using the rule $(u + iv)' = u' + iv'$ for differentiating complex functions (see Notes C, (19)), gives

$$y_1' + iy_2' + k(y_1 + iy_2) = k(\cos\omega t + i\sin\omega t);$$

equating real and imaginary parts on the two sides gives the two real ODE's

$$y_1' + ky_1 = k\cos\omega t, \qquad y_2' + ky_2 = k\sin\omega t; \tag{18}$$

this shows that the real and imaginary parts of our complex solution $\tilde{y}(t)$ give us respectively the responses to the physical inputs $\cos\omega t$ and $\sin\omega t$.

It seems wise at this point to illustrate the calculations with an example, before repeating them as a derivation. If you prefer, you can skip the example and proceed directly to the derivation, using the example as a solved exercise to test yourself afterwards.

Example 5. Find the response of $y' + y = 0$ to the input $\cos t$; in other words, find a solution to the equation $y' + y = \cos t$; use complex numbers.

Solution. We follow the above plan and complexify the real ODE, getting

$$\tilde{y}' + \tilde{y} = e^{it}\ .$$

We made the right side e^{it}, since $\cos t = \mathrm{Re}\,(e^{it})$. We will find a complex solution $\tilde{y}(t)$ for the complexified ODE; then $\mathrm{Re}\,(\tilde{y})$ will be a real solution to $y' + y = \cos t$, according to (18) and what precedes it.

The complexified ODE is linear, with the integrating factor e^t; multiplying both sides by this factor, and then following the steps for solving the first order linear ODE, we get

$$\tilde{y}' + \tilde{y} = e^{it} \quad \Rightarrow \quad (\tilde{y}\,e^t)' = e^{(1+i)t} \quad \Rightarrow \quad \tilde{y}\,e^t = \frac{1}{1+i}\,e^{(1+i)t} \quad \Rightarrow \quad \tilde{y} = \frac{1}{1+i}\,e^{it}\ .$$

This gives us our complex solution $\tilde{y}$; the rest of the work is calculating $\mathrm{Re}\,(\tilde{y})$. To do this, we can use either the polar or the Cartesian representations.

Using the polar first, conver $1+i$ to polar form and then use the reciprocal rule (Notes C, (12b)):

$$1+i = \sqrt{2}\,e^{i\pi/4} \quad \Rightarrow \quad \frac{1}{1+i} = \frac{1}{\sqrt{2}}\,e^{-i\pi/4};$$

from which it follows from the multiplication rule (12a) that

$$\tilde{y} = \frac{1}{1+i}\,e^{it} = \frac{1}{\sqrt{2}}\,e^{i(t-\pi/4)}$$

and therefore our solution to $y' + y = \cos t$ is

$$\mathrm{Re}\,(\tilde{y}) = \frac{1}{\sqrt{2}}\cos(t - \pi/4)\ ,$$

the pure oscillation with amplitude $1/\sqrt{2}$, circular frequency 1, and phase lag $\pi/4$.

Repeating the calculation, but using the Cartesian representation, we have

$$\tilde{y} = \frac{1}{1+i}\,e^{it} = \left(\frac{1-i}{2}\right)(\cos t + i\sin t)$$

whose real part is

$$\mathrm{Re}\,(\tilde{y}) = \frac{1}{2}(\cos t + \sin t) = \frac{\sqrt{2}}{2}\cos(t - \pi/4),$$

the last equality following from the sinusoidal identity ((25), section 6).

Instead of converting the Cartesian form to the polar, we could have converted the polar form to the Cartesian form by using the trigonometric addition formula. Since $\cos\pi/4 = \sin\pi/4 = \sqrt{2}/2$, it gives

$$\frac{1}{\sqrt{2}}\cos(t-\pi/4) = \frac{1}{\sqrt{2}}(\cos t\cos\pi/4 + \sin t\sin\pi/4) = \frac{1}{2}(\cos t + \sin t)$$

However, in applications the polar form of the answer is generally preferred, since it gives directly important characteristics of the solution — its amplitude and phase lag, whereas these are not immediately apparent in the Cartesian form of the answer.

Resuming our general solving of the ODE (17) using the circular frequency ω and the constant k, but following the same method as in the example, the integrating factor is e^{kt}; multiplying through by it leads to

$$(\tilde{y}e^{kt})' \;=\; ke^{(k+i\omega)t};$$

integrate both sides, multiply through by e^{-kt}, and scale the coefficient to lump constants and make it look better:

$$\tilde{y} \;=\; \frac{k}{k+i\omega}e^{i\omega t} \;=\; \frac{1}{1+i(\omega/k)}\,e^{i\omega t}. \tag{19}$$

This is our complex solution.

Writing $1+i(\omega/k)$ in polar form, then using the reciprocal rule (Notes C, (12b)), we have

$$1+i(\omega/k) = \sqrt{1+(\omega/k)^2}\;e^{i\phi} \quad\Rightarrow\quad \frac{1}{1+i(\omega/k)} \;=\; \frac{1}{\sqrt{1+(\omega/k)^2}}\;e^{-i\phi}, \tag{20}$$
$$\text{where}\quad \phi = \tan^{-1}(\omega/k).$$

Therefore in polar form,

$$\tilde{y} = \frac{1}{\sqrt{1+(\omega/k)^2}}\;e^{i(\omega t-\phi)}. \tag{21}$$

The real and imaginary parts of this complex response give the responses of the system to respectively the real and imaginary parts of the complex input $e^{i\omega t}$; thus we can summarize our work as follows:

First-order Sinusoidal Input Theorem. *For the equation* $y' + ky = kq_e$ *we have*

$$\textit{physical input:}\; \begin{cases} q_e = \cos\omega t \\ q_e = \sin\omega t \end{cases} \qquad \textit{response:}\; \begin{cases} y_1 = A\cos(\omega t-\phi) \\ y_2 = A\sin(\omega t-\phi) \end{cases}, \tag{22}$$

$$A = \frac{1}{\sqrt{1+(\omega/k)^2}}, \qquad \phi = \tan^{-1}(\omega/k). \tag{23}$$

The calculation can also be done in Cartesian form as in Example 5, then converted to polar form using the sinusoidal identity. We leave this as an exercise.

The Sinusoidal Input Theorem is more general than it looks; it actually covers any sinusoidal input $f(t)$ having ω as its circular frequency. This is because any such input can be written as a linear combination of $\cos\omega t$ and $\sin\omega t$:

$$f(t) = a\,\cos\omega t + b\,\sin\omega t, \qquad a,\; b \text{ constants}$$

and then it is an easy exercise to show:

$$\textit{physical input:}\; q_e = \cos\omega t + b\sin\omega t \qquad \textit{response:}\; a\,y_1 + b\,y_2\;. \tag{22'}$$

6. Sinusoidal oscillations: reference.

The work in section 5 uses terms describing a pure, or **sinusoidal** oscillation: analytically, one that can be written in the form

$$A\cos(\omega t-\phi); \tag{24}$$

or geometrically, one whose graph can be obtained from the graph of $\cos t$ by stretching or shrinking the t and y axes by scale factors, then translating the resulting graph along the t-axis.

Terminology. Referring to function (24) whose graph describes the oscillation,

$|A|$ is its **amplitude**: how high its graph rises over the t-axis at its maximum points;

ϕ is its **phase lag**: the smallest non-negative value of ωt for which the graph is at its maximum
(if $\phi=0$, the graph has the position of $\cos\omega t$; if $\phi=\pi/2$, it has the position of $\sin\omega t$);

ϕ/ω is its **time delay** or **time lag**: how far to the right on the t-axis the graph of $\cos\omega t$ has been moved to make the graph of (24);
(to see this, write $A\cos(\omega t-\phi)=A\cos\omega(t-\phi/\omega)$);

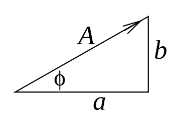
ω is its **circular** or **angular frequency**: the number of complete oscillations it makes in a t-interval of length 2π;

$\omega/2\pi$ (usually written ν) is its **frequency**: the number of complete oscillations the graph makes in a time interval of length 1;

$2\pi/\omega$ or $1/\nu$ is its **period**, the t-interval required for one complete oscillation.

The Sinusoidal Identity. For any real constants a and b,

$$a\cos\omega t+b\sin\omega t \;=\; A\cos(\omega t-\phi), \tag{25}$$



where A, ϕ, a, and b are related as shown in the accompanying right triangle:

$$A=\sqrt{a^2+b^2},\qquad \phi=\tan^{-1}\frac{b}{a},\qquad a=A\cos\phi,\qquad b=A\sin\phi \tag{26}$$

There are at least three ways to verify the sinusoidal identity:

1. Apply the trigonometric addition formula for $\cos(\theta_1-\theta_2)$ to the right side.
(Our good proof, uses only high-school math, but not very satisfying, since it doesn't show where the right side came from.)

2. Observe that the left side is the real part of $(a-bi)(\cos\omega t+i\sin\omega t)$; calculate this product using polar form instead, and take its real part.
(Our better proof, since it starts with the left side, and gives practice with complex numbers to boot.)

3. The left side is the dot product $\langle a,\ b\rangle\cdot\langle\cos\omega t,\ \sin\omega t\rangle$; evaluate the dot product by the geometric formula for it (first day of 18.02).
(Our best proof: starts with the left side, elegant and easy to remember.)

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