18.03 PDE.2: Decoupling; Insulated ends

1. Normal Modes: $e^{\lambda_k t} v_k$

2. Superposition

3. Decoupling; dot product

4. Insulated ends

In this note we will review the method of separation of variables and relate it to linear algebra. There is a direct relationship between Fourier's method and the one we used to solve systems of equations.

We compare a system of ODE $\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)$ where A is a matrix and $\boldsymbol{u}(t)$ is a vector-valued function of t to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u, \; 0 < x < \pi, \; t > 0; \quad u(0,t) = u(\pi,t) = 0$$

with zero temperature ends. To establish the parallel, we write

$$\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t) \qquad - - - - - \qquad \dot{u} = \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u \quad (A = (\partial/\partial x)^2)$$

To solve the equations we look for normal modes:

$$\text{Try } \boldsymbol{u}(t) = w(t)\boldsymbol{v}. \qquad - - - - - \qquad \text{Try } u(x,t) = w(t)v(x).$$

This leads to equations for eigenvalues and eigenvectors:

$$\begin{cases} A\boldsymbol{v} = \lambda \boldsymbol{v} \\ \dot{w} = \lambda w \end{cases} \qquad - - - - - - \qquad \begin{cases} Av = v''(x) = \lambda v(x) \text{ [and } v(0) = v(\pi) = 0] \\ \dot{w}(t) = \lambda w(t) \end{cases}$$

There is one new feature: in addition to the differential equation for $v(x)$, there are endpoint conditions. The response to the system $\dot{u} = A\boldsymbol{u}$ is determined by the initial condition $\boldsymbol{u}(0)$, but the heat equation response is only uniquely identified if we know the endpoint conditions as well as $u(x,0)$.

Eigenfunction Equation. The solutions to

$$v''(x) = \lambda v(x) \text{ and } v(0) = v(\pi) = 0,$$

are known as *eigenfunctions.* They are

$$v_k(x) = \sin kx, \quad k = 1, 2, \ldots$$

and the eigenvalues $\lambda_k = -k^2$ lead to $w_k(t) = e^{-k^2 t}$.

$$\text{normal modes} : e^{\lambda_k t}\boldsymbol{v}_k \qquad - - - - - - \qquad e^{-k^2 t}\sin(kx)$$

The principle of superposition, then says that

$$\boldsymbol{u}(0) = \sum c_k \boldsymbol{v}_k \implies \boldsymbol{u}(t) = \sum c_k e^{\lambda_k t}\boldsymbol{v}_k$$

and, similarly,

$$u(x,0) = \sum b_k \sin kx \implies u(x,t) = \sum b_k e^{-k^2 t}\sin kx$$

More generally, we will get formats for solutions of the form

$$u(x,t) = \sum b_k e^{-\beta k^2 t} \sin(\alpha k x) \quad \text{or cosines}$$

The scaling will change if the units are different (inches versus meters in x; seconds versus hours in t) and depending on physical constants like the conductivity factor in front of the $(\partial/\partial x)^2$ term, or if the interval is $0 < x < L$ instead of $0 < x < \pi$. Also, we'll see an example with cosines below.

The final issue is how to find the coefficients c_k or b_k. If we have a practical way to find the coefficients c_k in

$$\boldsymbol{u}(0) = \sum c_k \boldsymbol{v}_k,$$

then we say we have decoupled the system. The modes $e^{\lambda_k t}\boldsymbol{v}_k$ evolve according to separate equations $\dot{w}_k = \lambda_k w_k$.

Recall that the dot product of vectors is given, for example, by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot 2 + (2)(-1) + 3 \cdot 0 = 0$$

When the dot product is zero the vectors are perpendicular. We can also express the length squared of a vector in terms of the dot product:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad \boldsymbol{v} \cdot \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1^2 + 2^2 + 3^2 = (\text{length})^2 = \|\boldsymbol{v}\|^2$$

There is one favorable situation in which it's easy to calculate the coefficients c_k, namely if the eigenvectors $\boldsymbol{v}_k$ are perpendicular to each other

$$\boldsymbol{v}_k \perp \boldsymbol{v}_\ell \iff v_k \cdot \boldsymbol{v}_\ell = 0$$

This happens, in particular, if the matrix A is symmetric. In this case we also normalize the vectors so that their length is one:

$$\|\boldsymbol{v}_k\|^2 = \boldsymbol{v}_k \cdot \boldsymbol{v}_k = 1$$

Then

$$\boxed{c_k = \boldsymbol{v}_k \cdot \boldsymbol{u}(0)}$$

The proof is

$$\boldsymbol{v}_k(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n) = 0 + \cdots + 0 + c_k\boldsymbol{v}_k \cdot \boldsymbol{v}_k + 0 + \cdots = c_k\,.$$

The same mechanism is what makes it possible to compute Fourier coefficients. We have

$$v_k \perp v_\ell \iff \int_0^\pi v_k(x) v_\ell(x)\,dx = 0$$

and

$$\int_0^\pi v_k(x)^2\,dx = \int_0^\pi \sin^2(kx)\,dx = \frac{\pi}{2}$$

To compensate for the length not being 1 we divide by the factor $\pi/2$. It follows that

$$b_k = \frac{2}{\pi}\int_0^\pi u(x,0)\sin(kx)\,dx$$

The analogy between these integrals and the corresponding dot products is very direct. When evaluating integrals, it makes sense to think of functions as a vectors

$$\vec{f} = [f(x_1), f(x_2), \ldots, f(x_N)]; \quad \vec{g} = [g(x_1), g(x_2), \ldots, g(x_N)].$$

The Riemann sum approximation to an integral is written

$$\int_0^\pi f(x)g(x)\,dx \approx \sum_j f(x_j)g(x_j)\Delta x = \vec{f}\cdot\vec{g}\Delta x$$

We have not explained the factor Δx, but this is a normalizing factor that works out after taking into account proper units and dimensional analysis. *To repeat, functions are vectors: we can take linear combinations of them and even use dot products to find their "lengths" and the angle between two of them, as well as distances between them.*

Example 1. Zero temperature ends. We return to the problem from PDE.1, in which the initial conditions and end point conditions were

$$u(x,0) = 1 \quad 0 < x < \pi; \quad u(0,t) = u(\pi,t) = 0 \quad t > 0.$$

Our goal is to express

$$1 = \sum_1^\infty b_k \sin(kx), \quad 0 < x < \pi$$

The physical problem does not dictate any value for the function $u(x,0)$ outside $0 < x < \pi$. But if we want it to be represented by this sine series, it's natural to consider the odd function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Moreover, because the sine functions are periodic of period 2π, it's natural to extend f to have period 2π. In other words, $f(x) = Sq(x)$, the square wave. We computed this series in L26 (same formula as above for b_k) and found

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Therefore the solution is

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin x + e^{-3^2 t}\frac{\sin 3x}{3} + \cdots\right)$$

Example 2. Insulated Ends.

When the ends of the bar are insulated, we have the usual heat equation (taken here for simplicity with conductivity 1 and on the interval $0 < x < \pi$) given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0,$$

with the new feature that the heat flux across 0 and π is zero. This is expressed by the equations

$$\text{insulated ends}: \quad \boxed{\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad t > 0}$$

Separation of variables $u(x,t) = v(x)w(t)$ yields a new eigenfunction equation:

$$v''(x) = \lambda v(x), \qquad \boxed{v'(0) = v'(\pi) = 0}$$

whose solution are

$$v_k(x) = \cos(kx), \quad k = 0,\, 1,\, 2 \ldots$$

Note that the index starts at $k = 0$ because $\cos 0 = 1$ is a nonzero function. The eigenvalues are $\lambda_k = -k^2$, but now the first eigenvalue is

$$\lambda_0 = 0.$$

This will make a difference when we get to the physical interpretation. Since $\dot{w}_k(t) = -k^2 w_k(t)$, we have

$$w_k(t) = e^{-k^2 t}$$

and the normal modes are

$$e^{-k^2 t}\cos(kx), \quad k = 0,\, 1,\, \ldots$$

The general solution has the form (or format)

$$\boxed{u(x,t) = \frac{a_0}{2} e^{0t} + \sum_1^\infty a_k e^{-k^2 t} \cos(kx)}$$

(Here we have anticipated the standard Fourier series format by treating the constant term differently.)

Let us look at one specific case, namely, initial conditions

$$u(x,0) = x, \quad 0 < x < \pi$$

We can imagine an experiment in which the temperature of the bar is 0 on one end and 1 on the other. After a fairly short period, it will have stabilized to the equilibrium distribution x. Then we insulate both ends (cease to provide heat or cooling that would maintain the ends at 0 and 1 respectively). What happens next?

To find out we need to express x as a cosine series. So we extend it evenly to

$$g(x) = |x|, \quad |x| < \pi, \quad \text{with period } 2\pi$$

This is a triangular wave and we calculated its series using $g'(x) = Sq(x)$ as

$$g(x) = \frac{a_0}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

The constant term is not determined by $g'(x) = Sq(x)$ and must be calculated separately. Recall that

$$a_0 = \frac{2}{\pi}\int_0^\pi g(x)\cos 0\, dx = \frac{2}{\pi}\int_0^\pi x\, dx = \left.\frac{x^2}{\pi}\right|_0^\pi = \pi$$

Put another way,

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^\pi g(x)\,dx = \text{average}(g) = \frac{\pi}{2}$$

Thus, putting it all together,

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\left(e^{-t}\cos x + e^{-3^2 t}\frac{\cos 3x}{3^2} + \cdots\right)$$

Lastly, to check whether this makes sense physically, consider what happens as $t \to \infty$. In that case,

$$u(x,t) \to \frac{\pi}{2}$$

In other words, when the bar is insulated, the temperature tends to a constant equal to the average of the initial temperature.

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18.03 Extra Notes and Exercises