18.03 LA.9: Decoupling

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[1] Multiple springs.

The "Coupled Oscillators" Mathlet shows a system with three springs connecting two masses. The ends of the springs are fixed, and the whole thing is set up so that there is a position in which all three springs are relaxed. Let's see some of the behaviors that are possible.

Wow, this is pretty complicated. Imagine what happens with five springs, or a hundred ....

Let's analyze this. The spring constants are k_1, k_2, k_3; the masses are m_1, m_2. The displacement from relaxed positions are x_1, x_2.

Let's look at the special case when $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$. You can put the subscripts back in on your own.

The forces on the objects are given by

$$\begin{cases} m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\ m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 = kx_1 - 2kx_2 \end{cases}$$

Let's divide through by m, and agree to write $k/m = \omega^2$.

There's a matrix here, $B = \omega^2 \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. (This is the same as the heat conduction matrix!) We have

$$\ddot{\mathbf{x}} = B\mathbf{x}$$

What do do about the second derivative? Let's do the companion trick! Set $\mathbf{y} = \dot{\mathbf{x}}$, so

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{y} \\ \dot{\mathbf{y}} = B\mathbf{x} \end{cases}$$

Breaking this down even further, $y_1 = \dot{x}_1$, $y_2 = \dot{x}_2$; so we have four equations in four unknown functions:

$$\begin{array}{rcllll} \dot{x}_1 & = & & & y_1 & \\ \dot{x}_2 & = & & & & y_2 \\ \dot{y}_1 & = & -2\omega^2 x_1 & + \omega^2 x_2 & & \\ \dot{y}_2 & = & \omega^2 x_1 & - 2\omega^2 x_2 & & \end{array}$$

We might be quite uncomfortable about the prospect of computing eigenvalues of 4×4 matrices without something like Matlab. But we have a block matrix here:

$$A = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$$

and

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

Let's think about the eigenvector equation: It's

$$\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

This breaks down to two simpler equations:

$$\begin{cases} \mathbf{y} = \lambda \mathbf{x} \\ B\mathbf{x} = \lambda \mathbf{y} \end{cases}$$

Plugging the first equation into the second gives

$$B\mathbf{x} = \lambda^2 \mathbf{x}$$

This says that the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector for B associated to the eigenvalue λ^2.

We have learned that the four eigenvalues of A are the square roots of the two eigenvalues of B. And the eigenvectors are gotten by putting $\lambda\mathbf{x}$ below the $\mathbf{x}$.

Well let's see. The characteristic polynomial of B is

$$p_B(\lambda) = \lambda^2 + 4\omega^2\lambda + 3\omega^4 = (\lambda + \omega^2)(\lambda + 3\omega^2)$$

so its eigenvalues are $-\omega^2$ and $-3\omega^2$.

That says the eigenvalues of A are $\pm i\omega$ and $\pm\sqrt{3}i\omega$.

We're almost there. The eigenvectors for B: For $-\omega^2$ we want to find a nonzero vector killed by $B - (-\omega^2 I) = \omega^2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do. The matrix is symmetric so eigenvectors for different eigenvalues are orthogonal; an eigenvector for value $-3\omega^2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So the eigenvectors for A are given by

$$\lambda = \pm i\omega \, : \begin{bmatrix} 1 \\ 1 \\ \pm i\omega \\ \pm i\omega \end{bmatrix} \quad , \quad \lambda = \pm\sqrt{3}i\omega \, : \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{3}i\omega \\ \mp\sqrt{3}i\omega \end{bmatrix}$$

This gives us exponential solutions!

$$e^{i\omega t} \begin{bmatrix} 1 \\ 1 \\ i\omega \\ i\omega \end{bmatrix} \quad , \quad e^{i\sqrt{3}\omega t} \begin{bmatrix} 1 \\ -1 \\ \sqrt{3}i\omega \\ -\sqrt{3}i\omega \end{bmatrix}$$

and their complex conjugates. We can get real solutions by taking real and imaginary parts. Let's just write down the top halves; the bottom halves are just the derivatives.

$$\begin{bmatrix} \cos(\omega t) \\ \cos(\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \sin(\omega t) \\ \sin(\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \cos(\sqrt{3}\omega t) \\ -\cos(\sqrt{3}\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \sin(\sqrt{3}\omega t) \\ -\sin(\sqrt{3}\omega t) \end{bmatrix}$$

The first two combine to give the general sinusoid of angular frequency ω for x_1 *and* $x_2 = x_1$. In this mode the masses are moving together; the spring between them is relaxed.

The second two combine to give a general sinusoid of angular frequency $\sqrt{3}\omega$ for x_1, *and* $x_2 = -x_1$. In this mode the masses are moving back and forth relative to each other.

These are "normal modes." I have used this term as a synonym for "exponential solutions" earlier in the course, but now we have a better definition. From Wikipedia:

> A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation.

We can see them here on the Mathlet, if I adjust the initial conditions.

Behind the chaotic movement there are two very regular, sinuoidal motions. They happen at different frequencies, and that makes the linear combinations look chaotic. In fact the two frequencies never match up, because $\sqrt{3}$ is an irrational number. Except for the normal mode solutions, no solutions are perioidic.

The physics department has graciously provided some video footage of this in real action': `http://www.youtube.com/watch?v=zlzns5PjmJ4`Coupled Air Carts You can't see the springs here, and at the outset the masses have brakes on. When they turn the switch, the carts become elevated and move.

Check out the action at 5:51 as well: five carts!

I didn't really finish what could be gleaned from the "Coupled Oscillators" applet. All three springs have the same strength k, and the masses all have the same value m. What do you think? Are you seeing periodic motion here?

[The class was split on this question.]

Let's see. We calculated that there at least two families of periodic, even sinusoidal, solutions. They come from the eigenvalues $\pm\omega i$ and $\pm\sqrt{3}\omega i$, where $\omega = \sqrt{k/m}$. I have the applet set to $m = 1$ and $k = 4$. These special "normal mode" solutions are:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} \cos(\omega t - \phi) \\ \cos(\omega t - \phi) \end{bmatrix} \quad , \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} \cos(\sqrt{3}\omega t - \phi) \\ -\cos(\sqrt{3}\omega t - \phi) \end{bmatrix}$$

The general solution is a linear combination of these two. I claim that if both normal modes occur in the linear combination, then you will definitely NOT have a periodic solutions. This comes from the number-theoretic fact that $\sqrt{3}$ is irrational!

[2] Heat Phase Portrait.

I want to discuss how initial conditions can be used to specify which linear combination you get, in a situation like this—we are talking about $\dot{\mathbf{x}} = A\mathbf{x}$. For an example I'd like to go back to the insulated rod from LA.1.

For 3 thermometers, placed at 1, 2, and 3 feet, the details are a little msesy. The same ideas are already visible with two thermometers, so let's focus on that. The temperatures at 0, 1, 2, and 3 feet are x_0, x_1, x_2, x_3. The equation controlling this is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_0 \\ x_3 \end{bmatrix}$$

We're interested in the homogeneous case, so we'll take $x_0 = x3 = 0$ (in degrees centegrade, not Kelvin!), and I'm taking the conduction constant to be $k = 1$.

You can easily find eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. This tells us right off that some solutions decay like e^{-t}, some others decay like e^{-3t}, and in general you get a mix of the two. In paricular: no oscillation, and stable. In fact we can answer:

Question 18.1 This is a

1. Stable Spiral
2. Stable Saddle
3. Stable Node
4. Unstable Spiral
5. Unstable Saddle
6. Unstable Node
7. Don't know.

[There was uniform agreement that **3** is correct. Of course there is no such thing as a "stable saddle."

To get more detail we need to know the eigenvectors. Get them by subtracting the eigenvalue from the diagonal entry and finding a vector killed by the result. For $\lambda = -1$ we get $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, which kills $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and all multiples. We could do the same for the other eigenvalue, but as pointed out in LA.6 symmetric matrices have orthogonal eigenvectors (for distict eigenvalues), so a nonzero eigenvector for $\lambda = -3$ is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So we get two basic exponential solutions: $e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We can now draw the two eigenlines. Ray solutions converge exponentially to zero along them. These solutions are characterized by the fact that the ratio between x_1 and x_2 is constant in time. These are the **normal modes** for the heat model. The Wikipedia definition was applicable only to oscillating motion. You know, the word "normal mode" is like the word "happiness." It's hard to define, you know it when you see it, and it's really important, the basis of everything.

Other solutions are a mix of these two normal modes. IMPORTANT: The smaller the (real part of) the eigenvalue, the quicker the decay. $e^{-3t} = (e^{-t})^3$, so when $e^{-t} = 0.1$, $e^{-3t} = 0.001$. So the component of $\mathbf{v_2}$ decays much more rapidly than the component of $\mathbf{v_1}$, so these other trajectories approach tangency with the eigenline for value -1.

[3] Initial conditions.

Suppose I have a different initial condition, maybe $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$: so the temperatures are $0, 1, 2, 0$. What happens? What are c_1 and c_2 in

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

How can we find c_1 and c_2?

More generally, when we are studying $\dot{\mathbf{x}} = A\mathbf{x}$, and we've found eigenvalues $\lambda_1, \ldots$ and nonzero eigenvectors $\mathbf{v_1}, \ldots$, the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v_1} + c_2 e^{\lambda_2 t} \mathbf{v_2} + \cdots$$

We're interested in the cofficients c_i giving specified initial condition $\mathbf{x}(0) = \mathbf{v}$. Since $e^{\lambda \cdot 0} = 1$ alwasy, our equation is

$$\mathbf{v} = c_1 \mathbf{v_1} + \cdots + c_n \mathbf{v_n}$$

As with many computations, the first step is to rewrite this as a matrix equation:

$$\mathbf{v} = \begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We will write S for the matrix whose columns are the eigenvectors, so $\mathbf{v} =$

$S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Now it's clear: we find $c_1, \ldots$ by inverting S:

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}\mathbf{v}$$

This is an important principle!

The entries in $S^{-1}\mathbf{v}$ are the coefficients in

$$\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$$

In our example, $S^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

sou our particular solution is

$$\mathbf{x}(t) = \frac{3}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The second term decays much faster than the first.

[4] Coordinates.

The big idea: a system can seem complicated just because we are using the wrong coordinates to describe it.

I want to let $\mathbf{x}$ be any vector; maybe it varies with time, as a solution of $\dot{\mathbf{x}} = A\mathbf{x}$. We can write it as a linear combination of $\mathbf{v_1}, \ldots$, but now the coefficients can vary too. I'll write $y_1, y_2, \ldots$ for them to make them look more variable:

$$\mathbf{x} = y_1\mathbf{v_1} + y_2\mathbf{v_2} + \cdots$$

or

$$\mathbf{x} = S\mathbf{y}$$

For example,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$$

This is a change of cooridinates. The y_1 axis (where $y_2 = 0$) is the λ_1 eigenline; the y_2 axis (where $y_1 = 0$) is the λ_2 eigenline.

We can change back using S^{-1}:

$$\mathbf{y} = S^{-1}\mathbf{x}$$

so for us

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i.e.

$$y_1 = \frac{x_1 + x_2}{2} \quad , \quad y_2 = \frac{x_1 - x_2}{2}$$

[5] Diagonalization.

OK, but these vectors $\mathbf{v_i}$ have something to do with the matrix A: they are eigenvectors for it. What does this mean about the relationship between S and A? Well, the eigenvector equation is

$$A\mathbf{v} = \lambda \mathbf{v}$$

that is,

$$A\mathbf{v_1} = \lambda_1 \mathbf{v_1} \quad , \quad \cdots \quad , \quad A\mathbf{v_n} = \lambda_n \mathbf{v_n}$$

Line these up as the columns of a matrix product:

$$AS = A\begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v_1} & \cdots & \lambda_n\mathbf{v_n} \end{bmatrix}$$

Now comes a clever trick: the right hand side is a matrix product as well:

$$\cdots = \begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = S\Lambda$$

where Λ (that's a big "lambda") is the "eigenvalue matrix," with little λ's down the diagonal. That is:

$$AS = S\Lambda$$

or

$$A = S\Lambda S^{-1}$$

This is a *diagonalization* of A. It exhibits the simlicity hidden inside of A. There are only n eigenvalues, but n^2 entries in A. They don't completely determine A, of course, but they say a lot about it.

[6] Decoupling.

Now let's apply this to the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. I'm going to plug

$$\mathbf{x} = S\mathbf{y} \quad \text{and} \quad A = S\Lambda S^{-1}$$

into this equation.

$$\dot{\mathbf{x}} = \frac{d}{dt} S\mathbf{y} = S\dot{\mathbf{y}}$$

and

$$A\mathbf{x} = S\Lambda S^{-1}\mathbf{x} = S\Lambda\mathbf{y}$$

Put it together and cancel the S:

$$\dot{\mathbf{y}} = \Lambda\mathbf{y}$$

Spelling this out:

$$\begin{array}{c} \dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2 \\ \cdots \\ \dot{y}_n = \lambda_n y_n \end{array}$$

Each variable keeps to itself; its derivatives don't depend on the other variables. They are **decoupled**.

In our example,

$$\begin{array}{lll} y_1 = \frac{x_1+x_2}{2} & \text{satisfies} & \dot{y}_1 = -y_1 \\ y_2 = \frac{x_1-x_2}{2} & \text{satisfies} & \dot{y}_3 = -3y_2 \end{array}$$

So the smart variables to use to record the state of our insulated bar are not x_1 and x_2, but rather the *average* and the *difference* (or half the difference). The average decays exponentially like e^{-t}. The difference decays much faster. So quite quickly, the two temperatures become very close, and then the both of them die off exponentially to zero.

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises