18.03 LA.6: Diagonalization and Orthogonal Matrices

[1] Diagonal factorization

Recall: if $A\mathbf{x} = \lambda\mathbf{x}$, then the system $\dot{\mathbf{y}} = A\mathbf{y}$ has a general solution of the form

$$y = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2,$$

where the λ_i are eigenvalues with corresponding eigenvectors $\mathbf{x}_i$.

I'm never going to see eigenvectors without putting them into a matrix. And I'm never going to see eigenvalues without putting them into a matrix. Let's look at an example from last class.

$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$. We found that this had eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

I'm going to form a matrix out of these eigenvectors called the *eigenvector matrix* S:

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then lets look at what happens when we multiply AS, and see that we can factor this into S and a diagonal matrix Λ:

$$\underset{A}{\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 7 & 3 \\ 7 & -3 \end{bmatrix} = \underset{S}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}}$$

We call matrix Λ with eigenvalues λ on the diagonal the *eigenvalue matrix.*

So we see that $AS = S\Lambda$, but we can multiply both sides on the right by S^{-1} and we get a factorization $A = S\Lambda S^{-1}$. We've factored A into 3 pieces.

Properties of Diagonalization

- $A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$
- $A^{-1} = (S\Lambda S^{-1})^{-1} = (S^{-1})^{-1}\Lambda^{-1}S^{-1} = S\Lambda^{-1}S^{-1}$

Diagonal matrices are easy to square and invert because you simply square or invert the elements along the diagonal!

[2] Solving systems of first order differential equations

The entire reason we are finding eigenvectors is to solve differential equations. Let's express our solution to the differential equation in terms of S and Λ:

$$y = \underbrace{\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\mathbf{c}}$$

What determines $\mathbf{c}$? Suppose we have an initial condition $\mathbf{y}(0)$. Plugging this into our vector equation above we can solve for $\mathbf{c}$:

$$\begin{aligned} \mathbf{y}(0) &= SI\mathbf{c} \\ S^{-1}\mathbf{y}(0) &= \mathbf{c} \end{aligned}$$

The first line simply expresses our initial condition as a linear combination of the eigenvectors, $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. The second equation just multiplies the first by S^{-1} on both sides to solve for $\mathbf{c}$ in terms of $\mathbf{y}(0)$ and S^{-1}, which we know, or can compute from what we know.

Steps for solving a differential equation

Step 0. Find λ_i and $\mathbf{x}_i$.

Step 1. Use the initial condition to compute the parameters:

$$\mathbf{c} = S^{-1}\mathbf{y}(0)$$

Step 2. Multiply $\mathbf{c}$ by $e^{\Lambda t}$ and S:

$$\mathbf{y} = Se^{\Lambda t}S^{-1}\mathbf{y}(0).$$

[3] Symmetric and Orthonormal Matrices

In our example, we saw that A was symmetric ($A = A^T$) implied that the eigenvectors were perpendicular, or orthogonal. Perpendicular and orthogonal are two words that mean the same thing.

Now, the eigenvectors we chose

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

had length $\sqrt{2}$. If we make them unit length, we can choose eigenvectors that are both orthogonal and unit length. This is called *orthonormal.*

Question: Are the unit length vectors also eigenvectors?

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Yes! If $A\mathbf{x} = \lambda\mathbf{x}$, then

$$A\frac{\mathbf{x}}{||\mathbf{x}||} = \lambda\frac{\mathbf{x}}{||\mathbf{x}||}.$$

It turns out that finding the inverse of a matrix whose columns are orthonormal is extremely easy! All you have to do is take the transpose!

Claim

If S has orthonormal columns, then $S^{-1} = S^T$.

Example

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S \qquad\qquad\qquad S^T = I$$

If the inverse exists, it is unique, so S^T must be the inverse!

If we set $\theta = \pi/4$ we get

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

but what we found was

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Fortunately we can multiply the second column by negative 1, and it is still and eigenvector. So in the 2 by 2 case, we can always choose the eigenvectors of a symmetric matrix so that the eigenvector matrix is not only orthonormal, but also so that it is a rotation matrix!

In general, a set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is said to be orthonormal if the dot product of any vector with itself is 1:

$$\mathbf{x}_i \cdot \mathbf{x}_i = \mathbf{x}_i^T \mathbf{x}_i = 1,$$

and the dot product of any two vectors that are not equal is zero:

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = 0,$$

when $i \neq j$.

This tells us that the matrix product:

$$\begin{bmatrix} - & \mathbf{x}_1^T & - \\ - & \mathbf{x}_2^T & - \\ - & \mathbf{x}_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_2 & \mathbf{x}_1^T\mathbf{x}_3 \\ \mathbf{x}_2^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \mathbf{x}_2^T\mathbf{x}_3 \\ \mathbf{x}_3^T\mathbf{x}_1 & \mathbf{x}_3^T\mathbf{x}_1 & \mathbf{x}_3^T\mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S^T \qquad S \qquad\qquad\qquad = I$$

Example

We've seen that 2 by 2 orthonormal eigenvector matrices can be chosen to be rotation matrices.

Let's look at a 3 by 3 rotation matrix:

$$S = \frac{1}{3} \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

As an exercise, test that all vector dot products are zero if the vectors are not equal, and are one if it is a dot product with itself. This is a particularly nice matrix because there are no square roots! And this is also a rotation matrix! But it is a rotation is 3 dimensions.

Find a symmetric matrix A whose eigenvector matrix is S.

All we have to do is choose any Λ with real entries along the diagonal, and then $A = S\Lambda S^T$ is symmetric!

Recall that $(AB)^T = B^T A^T$. We can use this to check that this A is in fact symmetric:

$$\begin{aligned} A^T &= (S\Lambda S^T)^T \\ &= S^{TT}\Lambda^T S^T \\ &= S\Lambda S^T \end{aligned}$$

This works because transposing a matrix twice returns the original matrix, and transposing a diagonal matrix does nothing!

In physics and engineering this is called the principal axis theorem. In math, this is the spectral theorem.

Why is it called the principal axis theorem?

An ellipsoid whose principal axis are along the standard x, y, and z axes can be written as the equation $ax^2 + by^2 + cz^2 = 1$, which in matrix form is

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

However, what you consider a general ellipsoid, the 3 principal direction can be pointing in any direction. They are orthogonal direction though! And this means that we can get back to the standard basis elements by applying a rotation matrix S whose columns are orthonormal. Thus our equation for a general ellipsoid is:

$$\left(S \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)^T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \left(S \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = 1$$
$$\begin{bmatrix} x & y & z \end{bmatrix} \left(S^T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} S \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises