18.03 LA.5: Eigenvalues and Eigenvectors

[1] Eigenvectors and Eigenvalues

Example from Differential Equations

Consider the system of first order, linear ODEs.

$$\frac{dy_1}{dt} = 5y_1 + 2y_2$$
$$\frac{dy_2}{dt} = 2y_1 + 5y_2$$

We can write this using the companion matrix form:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Note that this matrix is symmetric. Using notation from linear algebra, we can write this even more succinctly as

$$\mathbf{y}' = A\mathbf{y}.$$

This is a coupled equation, and we want to uncouple it.

Method of Optimism

We've seen that solutions to linear ODEs have the form e^{rt}. So we will look for solutions

$$
\begin{aligned}
y_1 &= e^{\lambda t} a \\
y_2 &= e^{\lambda t} b
\end{aligned}
$$

Writing in vector notation:

$$
\mathbf{y} = e^{\lambda t} \begin{bmatrix} a \\ b \end{bmatrix} = e^{\lambda t} \mathbf{x}
$$

Here λ is the eigenvalue and $\mathbf{x}$ is the eigenvector.

To find a solution of this form, we simply plug in this solution into the equation $\mathbf{y}' = A\mathbf{y}$:

$$
\begin{aligned}
\frac{d}{dt} e^{\lambda t} \mathbf{x} &= \lambda e^{\lambda t} \mathbf{x} \\
A e^{\lambda t} \mathbf{x} &= e^{\lambda t} A \mathbf{x}
\end{aligned}
$$

If there is a solution of this form, it satisfies this equation

$$
\lambda e^{\lambda t} \mathbf{x} = e^{\lambda t} A \mathbf{x}.
$$

Note that because $e^{\lambda t}$ is never zero, we can cancel it from both sides of this equation, and we end up with the central equation for eigenvalues and eigenvectors:

$$
\boxed{\lambda \mathbf{x} = A \mathbf{x}}
$$

Definitions

- A nonzero vector $\mathbf{x}$ is an *eigenvector* if there is a number λ such that $A\mathbf{x} = \lambda \mathbf{x}$.

- The scalar value λ is called the *eigenvalue*.

Note that it is always true that $A\mathbf{0} = \lambda \cdot \mathbf{0}$ for any λ. This is why we make the distinction than an eigenvector must be a nonzero vector, and an eigenvalue must correspond to a nonzero vector. However, the scalar value λ can be any real or complex number, including 0.

This is a subtle equation. Both λ and $\mathbf{x}$ are unknown. This isn't exactly a linear problem. There are more unknowns.

What is this equation saying? It says that we are looking for a vector $\mathbf{x}$ such that $\mathbf{x}$ and $A\mathbf{x}$ point in the same direction. But the length can change, the length is scaled by λ.

Note that this isn't true for most vectors. Typically $A\mathbf{x}$ does not point in the same direction as $\mathbf{x}$.

Example

If $\lambda = 0$, our central equation becomes $A\mathbf{x} = 0\mathbf{x} = 0$. The eigenvector $\mathbf{x}$ corresponding to the eigenvalue 0 is a vector in the nullspace!

Example

Let's find the eigenvalues and eigenvectors of our matrix from our system of ODEs. That is, we want to find $\mathbf{x}$ and λ such that

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \lambda \begin{bmatrix} ? \\ ? \end{bmatrix}$$

By inspection, we can see that

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We have found the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_1 = 7$.

So a solution to a differential equation looks like

$$\mathbf{y} = e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Check that this is a solution by pluging

$$\begin{aligned} y_1 &= e^{7t} \qquad \text{and} \\ y_2 &= e^{7t} \end{aligned}$$

into the system of differential equations.

We can find another eigenvalue and eigenvector by noticing that

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We've found the nonzero eigenvector $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with corresponding eigenvalue $\lambda_2 = 3$.

Check that this also gives a solution by plugging

$$\begin{aligned} y_1 &= e^{3t} \qquad \text{and} \\ y_2 &= -e^{3t} \end{aligned}$$

back into the differential equations.

Notice that we've found two independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. More is true, you can see that $\mathbf{x}_1$ is actually perpendicular to $\mathbf{x}_2$. This is because the matrix was symmetric. Symmetric matrices always have perpendicular eigenvectors.

[2] Observations about Eigenvalues

We can't expect to be able to eyeball eigenvalues and eigenvectors everytime. Let's make some useful observations.

We have

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

and eigenvalues

$$\begin{aligned} \lambda_1 &= 7 \\ \lambda_2 &= 3 \end{aligned}$$

- The sum of the eigenvalues $\lambda_1 + \lambda_2 = 7 + 3 = 10$ is equal to the sum of the diagonal entries of the matrix A is $5 + 5 = 10$.

The sum of the diagonal entries of a matrix A is called the *trace* and is denoted $\mathrm{tr}\,(A)$.

It is always true that

$$\lambda_1 + \lambda_2 = \mathrm{tr}\,(A).$$

If A is an n by n matrix with n eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\boxed{\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{tr}\,(A)}$$

- The product of the eigenvalues $\lambda_1 \lambda_2 = 7 \cdot 3 = 21$ is equal to $\det A = 25 - 4 = 21$.

In fact, it is always true that

$$\boxed{\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det A}.$$

For a 2 by 2 matrix, these two pieces of information are enough to compute the eigenvalues. For a 3 by 3 matrix, we need a 3rd fact which is a bit more complicated, and we won't be using it.

[3] Complete Solution to system of ODEs

Returning to our system of ODEs:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

We see that we've found 2 solutions to this homogeneous system.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \textit{and} \qquad e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution is obtained by taking linear combinations of these two solutions, and we obtain the general solution of the form:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The complete solution for any system of two first order ODEs has the form:

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2,$$

where c_1 and c_2 are constant parameters that can be determined from the initial conditions $y_1(0)$ and $y_2(0)$. It makes sense to multiply by this parameter because when we have an eigenvector, we actually have an entire line of eigenvectors. And this line of eigenvectors gives us a line of solutions. This is what we're looking for.

Note that this is the general solution to the homogeneous equation $\mathbf{y}' = A\mathbf{y}$. We will also be interested in finding particular solutions $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. But this isn't where we start. We'll get there eventually.

Keep in mind that we know that all linear ODEs have solutions of the form e^{rt} where r can be complex, so this method has actually allowed us to find *all* solutions. There can be no more and no less than 2 independent solutions of this form to this system of ODEs.

In this example, our matrix was *symmetric*.

- Symmetric matrices have real eigenvalues.
- Symmetric matrices have perpendicular eigenvectors.

[4] Computing Eigenvectors

Let's return to the equation $A\mathbf{x} = \lambda\mathbf{x}$.

Let's look at another example.

Example

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

This is a 2 by 2 matrix, so we know that

$$\lambda_1 + \lambda_2 = \mathrm{tr}\,(A) = 5$$
$$\lambda_1 \cdot \lambda_2 = \det(A) = 6$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. In fact, because this matrix was upper triangular, the eigenvalues are on the diagonal!

But we need a method to compute eigenvectors. So lets' solve

$$A\mathbf{x} = 2\mathbf{x}.$$

This is back to last week, solving a system of linear equations. The key idea here is to rewrite this equation in the following way:

$$(A - 2I)\mathbf{x} = \mathbf{0}$$

How do I find $\mathbf{x}$? I am looking for $\mathbf{x}$ in the nullspace of $A - 2I$! And we already know how to do this.

We've reduced the problem of finding eigenvectors to a problem that we already know how to solve. Assuming that we can find the eigenvalues λ_i, finding $\mathbf{x}_i$ has been reduced to finding the nullspace $N(A - \lambda_i I)$.

And we know that $A - \lambda I$ is singular. So let's compute the eigenvector $\mathbf{x}_1$ corresponding to eigenvalue 2.

$$A - 2I = \begin{bmatrix} 0 & 4 \\ 0 & 1 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By looking at the first row, we see that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a solution. We check that this works by looking at the second row.

Thus we've found the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to eigenvalue $\lambda_1 = 2$.

Let's find the eigenvector $\mathbf{x}_2$ corresponding to eigenvalue $\lambda_2 = 3$. We do this by finding the nullspace $N(A - 3I)$, we wee see is

$$A - 3I = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second eigenvector is $\mathbf{x}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ corresponding to eigenvalue $\lambda_2 = 3$.

Important observation: this matrix is NOT symmetric, and the eigenvectors are NOT perpendicular!

[5] Method for finding Eigenvalues

Now we need a general method to find eigenvalues. The problem is to find λ in the equation $A\mathbf{x} = \lambda\mathbf{x}$.

The approach is the same:

$$(A - \lambda I)\mathbf{x} = 0.$$

Now I know that $(A-\lambda I)$ is singular, and singular matrices have determinant 0! This is a key point in LA.4. To find λ, I want to solve $\det(A-\lambda I) = 0$. The beauty of this equation is that $\mathbf{x}$ is completely out of the picture!

Consider a general 2 by 2 matrix A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}.$$

The determinant is a polynomial in λ:

$$\det(A-\lambda I) = \lambda^2 - \underset{\substack{\uparrow \\ \operatorname{tr}(A)}}{(a+d)}\lambda + \underset{\substack{\uparrow \\ \det(A)}}{(ad-bc)} = 0$$

This polynomial is called the *characteristic polynomial*. This polynomial is important because it encodes a lot of important information.

The determinant is a polynomial in λ of degree 2. If A was a 3 by 3 matrix, we would see a polynomial of degree 3 in λ. In general, an n by n matrix would have a corresponding nth degree polynomial.

Definition

The *characteristic polynomial* of an n by n matrix A is the nth degree polynomial $\det(A - \lambda I)$.

- The roots of this polynomial are the eigenvalues of A.
- The constant term (the coefficient of λ^0) is the determinant of A.
- The coefficient of λ^{n-1} term is the trace of A.
- The other coefficients of this polynomial are more complicated invariants of the matrix A.

Note that it is not fun to try to solve polynomial equations by hand if the degree is larger than 2! I suggest enlisting some computer help.

But the fundamental theorem of arithmetic tells us that this polynomial always has n roots. These roots can be real or complex.

Example of imaginary eigenvalues and eigenvectors

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Take $\theta = \pi/2$ and we get the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

What does this matrix do to vectors?

To get a sense for how this matrix acts on vectors, check out the Matrix Vector Mathlet `http://mathlets.org/daimp/MatrixVector.html`

Set $a = d = 0$, $b = -1$ and $c = 1$. You see the input vector $\mathbf{v}$ in yellow, and the output vector $A\mathbf{v}$ in blue.

What happens when you change the radius? How is the magnitude of the output vector related to the magnitude of the input vector?

Leave the radius fixed, and look at what happens when you vary the angle of the input vector. What is the relationship between the direction of the input vector and the direction of the output vector?

This matrix rotates vectors by 90 degrees! For this reason, there can be no *real* nonzero vector that points in the same direction after being multiplied by the matrix A. Let's look at the characteristic polynomial and find the eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.

Let's do a quick check:

- $\lambda_1 + \lambda_2 = i - i = \operatorname{tr}(A)$
- $\lambda_1 \cdot \lambda_2 = (i)(-i) = -1 = \det(A)$

Let's find the eigenvector corresponding to eigenvalue i:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}$$

Solving for the nullspace we must find the solution to the equation:

$$\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To solve this equation, I look at the first row, and checking against the second row we find that the solution is

$$\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

What ODE does this correspond to?

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This is the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 \end{aligned}$$

Using the method of elimination we get that:

$$y_1'' = -y_2' = -y_1$$

We are very familiar with this differential equation, it is the harmonic oscillator $y'' + y = 0$. This linear, 2nd order equation parameterized motion around a circle! It is a big example and physics, and we know that the solution space has a basis spanned by e^{it} and e^{-it}. Notice that the i and $-i$ are the eigenvalues!

Properties of Eigenvalues

Suppose A has eigenvalue λ and nonzero eigenvector $\mathbf{x}$.

- The the eigenvalues of A^2 are λ^2.

Why?

$$A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

We see that the vector $\mathbf{x}$ will also be an eigenvector corresponding to λ. However, be careful!!! In the example above, $\lambda_1 = i$ and $\lambda_2 = -1$, we get repeated eigenvalues $\lambda_1 = \lambda_2 = -1$. And in fact

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Since $-I\mathbf{x} = -\mathbf{x}$ for all nonzero vectors $\mathbf{x}$, in fact every vector in the plane is an eigenvector with eigenvalue -1!

We know that the exponential function is important.

- The eigenvalues of e^A are e^λ, with eigenvector $\mathbf{x}$.

If $e^A\mathbf{x}$ had meaning,

$$e^A\mathbf{x} = e^\lambda\mathbf{x}$$

where $\mathbf{x}$ is an eigenvector of A, and λ is the corresponding eigenvalue.

- The eigenvalues of e^{-1} are λ^{-1}, with eigenvector $\mathbf{x}$.

Let's look at the example $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$, which had eigenvalues 7 and 3. Check that A^{-1} has eigenvalues $1/7$ and $1/3$. We know that $\det(A) * \det(A^{-1}) = 1$, and $\det(A) = 21$ and $\det(A^{-1}) = 1/21$, which is good.

- The eigenvalues of $A + 12I$ are $\lambda + 12$, with eigenvector $\mathbf{x}$.

Check this with our favorite symmetric matrix A above.

Nonexamples

Let A and B be n by n matrices.

- The eigenvalues of $A + B$ are generally NOT the eigenvalues of A plus eigenvalues of B.
- The eigenvalues of AB are generally NOT the eigenvalues of A times the eigenvalues of B.

Question: What would be necessary for the eigenvalues of $A + B$ to be the sum of the eigenvalues of A and B? Similarly for AB.

Keep in mind that $A\mathbf{x} = \lambda\mathbf{x}$ is NOT an easy equation.

In matlab, the command is

```
eig(A)
```

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises