18.03 LA.4: Inverses and Determinants

[1] Transposes
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[1] Transposes

The transpose of a matrix A is denoted A^T, or in Matlab, A'.

The transpose of a matrix exchanges the rows and columns. The ith column becomes the ith row. Or the a_{ij} entry becomes the a_{ji} entry.

Example:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 7 \end{bmatrix}$$

Symmetric Matrices are square matrices that satisfy $A = A^T$.

Example:

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 3 & 2 & 5 \\ 9 & 5 & 8 \end{bmatrix}$$

We'll see that the eigenvalues of symmetric matrices are great. The eigenvectors are even better! And symmetric matrices come up all of the time.

Property of transposes:

$$(AB)^T = B^T A^T$$

[2] Inverses

Important questions:

- Is a matrix A invertible?

- How do you compute the inverse?

Let A be a square matrix. Suppose it has an inverse. We denote the inverse by A^{-1}, and it has the property that

$$AA^{-1} = I \qquad A^{-1}A = I.$$

The fact that the inverse is simultaneously a right and left inverse is not immediately obvious. See if you can use the associative property $(AB)C = A(BC)$ to see why this must be the case when A is square.

If the inverse of A and B both exists, and both matrices have the same shape, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Corny Example:

If B represents taking off your jacket, and A represents taking off your sweater, then it makes sense that you first take of your jacket, and then take off your sweater. To find the inverse, which is to reverse this process, it makes sense that we have to reverse the order. First you put the sweater back on, and then you put your jacket on.

So let's start to answer our question: when is a matrix invertible? To answer this question, we'll look at when it is NOT invertible first.

A is NOT invertible when:

- The determinant of A is zero.

- There exists a nonzero vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{0}$.

Examples of matrices that are easy to invert:

2 by 2 matrices are easy to invert:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

Check that the product of this matrix with its inverse is the identity.

The quantity in the denominator $ad - bc$ is the determinant of that matrix. That's why it can't be zero. This is true for 5 by 5 matrices, 10 by 10 matrices, the inverse will always involve dividing by the determinant.

Diagonal matrices are easy to invert:

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & & \\ & 1/d_2 & & \\ & & 1/d_3 & \\ & & & 1/d_4 \end{bmatrix}$$

Example of non-invertible matrix

$$A = \begin{bmatrix} 3 & -1 & -2 \\ -4 & 7 & -3 \\ -3 & -2 & 5 \end{bmatrix}$$

We notices that the sum of each row is zero. So

$$\begin{bmatrix} 3 & -1 & -2 \\ -4 & 7 & -3 \\ -3 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

The vector $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ is in the *nullspace* of A.

Let's see why if there is a nonzero vector in the nullspace, then A can't be invertible.

Proof: Suppose $\mathbf{x} \neq 0$ and $A\mathbf{x} = \mathbf{0}$. If A^{-1} existed, then

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{0}$$
$$I\mathbf{x} = \mathbf{0}$$
$$\mathbf{x} = \mathbf{0}$$

This contradiction forces us to accept that an inverse must not exist!
QED

Conditions for Invertibility:

- $\det A \neq 0$.

- Nullspace of A is $\mathbf{0}$.

- Columns are independent.

- A has full rank (rank of $A = n$ if A is an n by n matrix).

- Rows are independent.

- The row echelon form of A has a full set of nonzero pivots.

- rref A is the identify.

- $A\mathbf{x} = \mathbf{b}$ has a solution for every b.

This last condition is the question we've been interested in, when can we solve this equation. The idea is that when it is always solvable, then $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof:

$$A\mathbf{x} = \mathbf{b}$$
$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$
$$\mathbf{x} = A^{-1}\mathbf{b}$$

QED

Triangular Matrices

If A is upper triangular,

$$A = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

we can tell whether or not A is invertible immediately by looking at the diagonal, or pivot entries. If all the diagonal entries are nonzero, the matrix is invertible.

For example, the matrix

$$\begin{bmatrix} 3 & \pi & e & \rho \\ 0 & 2 & \delta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

is invertible, and its determinant is the product of these pivots, which is 42.

Computing Inverses

We've seen a bunch of connected ideas about when a matrix is invertible. And it is important to understand that you basically never want to compute the inverse. In Matlab, you could solve the equation $A\mathbf{x} = \mathbf{b}$ by typing

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inv(A)*b
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This is correct, but not smart. Instead, use

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A \ b
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which cleverly checks for certain forms of a matrix, and ultimately will row reduce to find the solution.

However, if we want to compute A^{-1} there is a complicated formula involving determinants. But what you want to do is really solve the equation

$AA^{-1} = I$. Let $\mathbf{a}'_1, \mathbf{a}'_2, \ldots, a'_n$ denote the columns of A^{-1}. Then solving for A^{-1} is equivalent to solving the n equations:

$$A\mathbf{a}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\mathbf{a}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad A\mathbf{a}'_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Recall that to solve $A\mathbf{x} = \mathbf{b}$, we augment the matrix A with the column $\mathbf{b}$ and do row operations until we end up with a row reduced matrix where we can read off the solutions:

$$[A|\mathbf{b}] \longrightarrow \text{ Apply row operations } \longrightarrow [R|\mathbf{d}]$$

It turns out we can solve all n of our equations simulaneously by augmenting the matrix A with the matrix I, $[A|I]$ and then performing row operations. Because A is invertible, its reduced row echelon form is I, and what you end up with is I on the left, augmented by the solutions to $A\mathbf{x} = I$ on the right. But that solution is exactly A^{-1}.

Computing the inverse

- Augment A with the identity matrix: $[A|I]$

- Apply row operations until A reaches row reduced echelon form (rref)

- What you are left with on the augmented side is the collection of columns of A^{-1}: $[I|A^{-1}]$

[3] Determinants

Example 2 by 2

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In general, the determinant of a *square* matrix is a single number. This entry depends on all of the entries of the matrix.

Properties of the determinant:

- $\det I = 1$

- If you subtract m times row i and subtract that from row j, the determinants is unchanged! Example, subtract 4 times row 1 from row 2:
$$\det \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 7 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 7 & 2 & 1 \end{bmatrix}$$

- If you exchange rows, the determinant changes sign. Example:
$$\det I = - \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- If you multiply a row by a number c, the determinant is multiplied by c.
$$\det \begin{bmatrix} c & c & c \\ 4 & 3 & 1 \\ 7 & 2 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 7 & 2 & 1 \end{bmatrix}$$

- $\det(AB) = \det A \det B$

Example

Suppose I have a matrix A such that two of its rows are equal. Then if I exchange those rows, the matrix is unchanged. But by the third property, this implies that $\det A = - \det A$, which can only be true if $\det A$ is zero. This tells us that any matrix whose rows and not independent has determinant equal to zero.

Example 3 by 3

The 3 by 3 determinant has $6 = 3!$ terms. Each term is a multiple of a three entries, one from each row and column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = 1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 3 + 3 \cdot 2 \cdot 4 - 2 \cdot 2 \cdot 5 - 1 \cdot 4 \cdot 4 - 3 \cdot 3 \cdot 3$$

This matrix satisfies the equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so the determinant must be zero.

In general, the determinant of an n by n matrix is a sum of $n!$ terms all combined into one number. A 4 by 4 matrix already has 24 terms! That is a lot of terms. The key idea here is that if a matrix is not invertible, its determinant is zero.

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