18.03 LA.2: Matrix multiplication, rank, solving linear systems

[1]Matrix Vector Multiplication $A\mathbf{x}$

Example 1:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

I like to think of this multiplication as a linear combination of the columns:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}.$$

Many people think about taking the dot product of the rows. That is also a perfectly valid way to multiply. But this column picture is very nice because it gets right to the heart of the two fundamental operation that we can do with vectors.

We can multiply them by scalar numbers, such as x_1 and x_2, and we can add vectors together. This is *linearity.*

[2] When is $A\mathbf{x} = \mathbf{b}$ Solvable?

There are two main components to linear algebra:

1. Solving an equation $A\mathbf{x} = \mathbf{b}$
2. Eigenvalues and Eigenvectors

This week we will focus on this first part. Next week we will focus on the second part.

Given an equation $A\mathbf{x} = \mathbf{b}$, the first questions we ask are :

Question Is there a solution?

Question If there is a solution, how many solutions are there?

Question What is the solution?

Example 2: Let's start with the first question. Is there a solution to this equation?

$$\underset{3\times\mathbf{2}}{\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix}} \quad \underset{\mathbf{2}\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \quad = \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Notice the dimensions or shapes. The number of columns of A must be equal to the number of rows of $\mathbf{x}$ to do the multiplication, and the vector we get has the dimension with the same number of rows as A and the same number of columns as $\mathbf{x}$.

Solving this equation is equivalent to finding x_1 and x_1 such that the linear combination of columns of A gives the vector $\mathbf{b}$.

Example 3:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

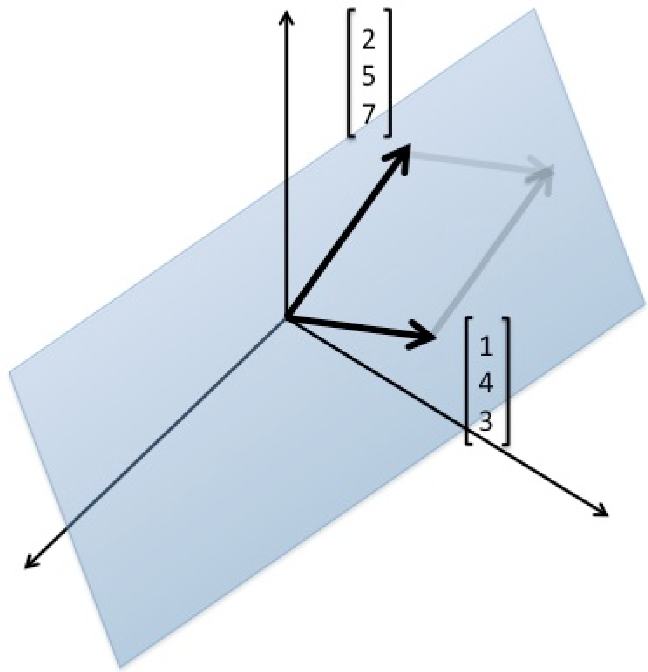
The vector $\mathbf{b}$ is the same as the 2nd column of A, so we can find this solution by inspection, the answer is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

But notice that this more general linear equation from Example 2:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is a system with 3 equations and 2 unknowns. This most likely doesn't have a solution! This would be like trying to fit 3 points of data with a line. Maybe you can, but most likely you can't!

To understand when this system has a solution, let's draw a picture.



All linear combinations of columns of A lie on a plane.

What are *all* linear combinations of these two columns vectors? How do you describe it?

It's a plane! And it is a plane that goes through the origin, because

$$0 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \mathbf{0}.$$

It is helpful to see the geometric picture in these small cases, because the goal of linear algebra is to deal with very large matrices, like 1000 by 100. (That's actually a pretty small matrix.)

What would the picture of a 1000 by 100 matrix be?

It lives in 1000-dimensional space. What would the 100 column vectors span? Or, what is the space of all possible solutions?

Just a hyperplane, a flat, thin, at most 100-dimensional space inside of 1000-dimensions. But linear algebra gets it right! That is the power of linear algebra, and we can use our intuition in lower dimensions to do math on much larger data sets.

The picture exactly tells us what are the possible right hand sides to the equation. $A\mathbf{x} = \mathbf{b}$/ If $\mathbf{b}$ is in the plane, then there is a solution to the equation!

$$\text{All possible } \mathbf{b} \Longleftrightarrow \text{all possible combinations } A\mathbf{x}.$$

In our case, this was a plane. We will call this plane, this subspace, the *column space* of the matrix A.

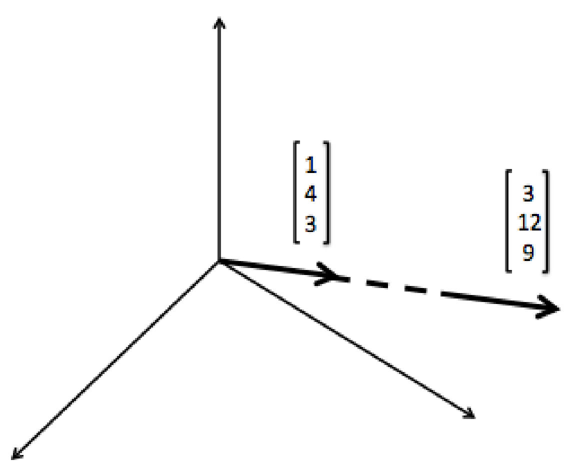
If $\mathbf{b}$ is not on that plane, not in that column space, then there is no solution.

Example 4: What do you notice about the columns of the matrix in this equation?

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The 2nd column is a multiple of the 1st column. We might say these columns are *dependent.*

The picture of the column space in this case is a *line*.



All linear combinations of columns of A lie along a line.

[3] Reduced Row Echelon Form and Rank

You learned about reduced row echelon form in recitation. Matlab and other learn algebra systems do these operation.

Example 5: Find the reduced row echelon form of our matrix:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix}.$$

It is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is very obvious that the reduced row echelon form of the matrix has 2 columns that are independent.

Let's introduce a new term the **rank** of a matrix.

$$\textit{Rank} \text{ of } A \quad = \text{the number of independent columns of } A.$$

Example 6: Find the row echelon form of

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix}.$$

But what do you notice about the rows of this matrix? We made this matrix by making the columns dependent. So the rank is 1. We didn't touch the rows, this just happened.

How many independent rows are there?

1!

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}?$$

This suggests that we can define rank equally well as the number of independent rows of A.

$$\begin{aligned} \textit{Rank} \text{ of } A \quad &= \text{the number of independent columns of } A \\ &\quad \text{the number of independent rows of } A \end{aligned}$$

The process of row reduction provides the algebra, the mechanical steps that make it obvious that the matrix in example 5 has rank 2! The steps of row reduction don't change the rank, because they don't change the number of independent rows!

Example 7: What if I create a 7x7 matrix with random entries. How many linearly independent columns does it have?

7 if it is random!

What is the row echelon form? It's the identity.

To create a random 7x7 matrix with entries random numbers between 0 and 1, we write

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rand(7).
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The command for reduced row echelon form is rref. So what you will see is that

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rref(rand(7))=eye(7).
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The command for the identity matrix is

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eye(n).
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That's a little Matlab joke.

[4] Matrix Multiplication

We thought of $A\mathbf{x} = \mathbf{b}$ as a combination of columns. We can do the same thing with matrix multiplication.

$$AB = A \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \\ | & | & & | \end{bmatrix}$$

This perspective leads to a rule $A(BC) = (AB)C$. This seemingly simple rule takes some messing around to see. But this observation is key to many ideas in linear algebra. Note that in general $AB \neq BA$.

[5] Key Connection to Differential Equations

We want to find solutions to equations $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ is an unknown.

You've seen how to solve differential equations like

$$\frac{dx}{dt} - 3t^5 x = b(t).$$

The key property of this differential equation is that it is *linear* in x. To find a complete solution we need to find one particular solution and add to it all the homogeneous solution. This complete solution describes all solutions to the differential equation.

The same is true in linear algebra! Solutions behave the same way. This is a consequence of **linearity**.

Example: 8 Let's solve

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 6 \end{bmatrix}.$$

This matrix has rank 1, it has dependent columns. We chose $\mathbf{b}$ so that this equation is solvable. What is one particular solution?

$$\mathbf{x}_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

is one solution. There are many solutions. For example

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} -22 \\ 8 \end{bmatrix}$$

But we only need to choose one particular solution. And we'll choose [2;0].

What's the homogeneous solution? We need to solve the equation

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One solution is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and all solutions are given by scalar multiples of this one solution, so all homogenous solutions are described by

$$\mathbf{x}_h = c \begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad c \text{ a real number.}$$

The complete solution is

$$\mathbf{x}_{complete} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Here is how linearity comes into play!

$$A\mathbf{x}_{complete} = A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises