18.03 LA.10: The Matrix Exponential

[1] Exponentials

What is e^x?

Very bad definition: e^x is the xth power of the number $e \sim 2.718281828459045\ldots$

Two problems with this: (1) What is e? (2) What does it mean to raise a number to the power of, say, $\sqrt{2}$, or π?

Much better definition: $y(x) = e^x$ is the solution to the differential equation $\frac{dy}{dx} = y$ with initial condition $y(0) = 1$.

Now there's no need to know about e in advance; e is *defined* to be e^1. And e^x is just a function, which can be evaluated at $\sqrt{2}$ or at π just as easily as at an integer.

Note the sublety: you can't use this definition to describe e^x for any *single* x (except $x = 0$); you need to define the entire function at once, and then evaluate that function at the value of x you may want.

As you know, this gives us solutions to other equations: I claim that $y = e^{rt}$ satisfies $\frac{dy}{dt} = ry$. This comes from the chain rule, with $x = rt$:

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = ry$$

A further advantage of this definition is that it can be extended to other contexts in a "brain-free" way.

A first example is Euler's definition

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We *defined* $x(t) = e^{(a+bi)t}$ to be the solution to $\dot{x} = (a+bi)x$, and then *calculated* that

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$$

In all these cases, you get the solution for any initial condition: $e^{rt}x(0)$ is the solution to $\dot{x} = rx$ with initial condition $x(0)$.

[2] Matrix exponential

We're ready for the next step: We have been studying the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where A is a square (constant) matrix.

Definition. e^{At} is the matrix of functions such that the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, in terms of its initial condition, is $e^{At}\mathbf{x}(0)$.

How convenient is that!

If we take $\mathbf{x}(0)$ to be the vector with 1 at the top and 0 below, the product $e^{At}\mathbf{x}(0)$ is the first column of e^{At}. Similarly for the other columns. So:

Each column of e^{At} is a solution of $\dot{\mathbf{x}} = A\mathbf{x}$. We could write this:

$$\frac{d}{dt}e^{At} = Ae^{At}$$

e^{At} is a matrix-valued solution! It satisfies a simple initial condition:

$$e^{A0} = I$$

Not everything about 1×1 matrices extends to the general $n \times n$ matrix. But everything about 1×1 matrices *does* generalize to *diagonal* $n\times n$ matrices.

If $A = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, the given coordinates are already decoupled: the equation $\dot{\mathbf{x}} = A\mathbf{x}$ is just $\dot{x}_1 = \lambda_1 x_1$ and $\dot{x}_2 = \lambda_2 x_2$. Plug in initial condition $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: the first column of $e^{\Lambda t}$ is $\begin{bmatrix} e^{\lambda_1 t} \\ 0 \end{bmatrix}$. Plug in initial condition $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: the second column is $\begin{bmatrix} 0 \\ e^{\lambda_2 t} \end{bmatrix}$. So

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Same works for $n \times n$, of course.

[3] Fundamental matrices

Here's how to compute e^{At}. Suppose we've found the right number (n) independent solutions of $\dot{\mathbf{x}} = A\mathbf{x}$: say $\mathbf{u_1(t)}, \ldots, \mathbf{u_n(t)}$. Line them up in a row: this is a "fundamental matrix" for A:

$$\Phi(t) = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \cdots & \mathbf{u_n} \end{bmatrix}$$

The general solution is

$$\mathbf{x}(t) = \Phi(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\Phi(t)$ may not be quite e^{At}, but it's close. Note that $\mathbf{x}(0) = \Phi(0) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, or $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi(0)^{-1}\mathbf{x}(0)$. Thus

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}(0)$$

So

$$e^{At} = \Phi(t)\Phi(0)^{-1}$$

for any fundamental matrix $\Phi(t)$.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Characteristic polynomial $p_A(\lambda) = \lambda^2 + 1$, so the eigenvalues are $\pm i$. The phase portrait is a "center." Eigenvectors for $\lambda = i$ are killed by $A - iI = \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$; for example $\begin{bmatrix} 1 \\ i \end{bmatrix}$. So the exponential solutions are given by

$$e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and its complex conjugate. To find real solutions, take just the right linear combinations of these to get the real and imaginary parts:

$$\mathbf{u_1}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad , \quad \mathbf{u_2}(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

These both parametrize the unit circle, just starting at different places. The corresponding fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

We luck out, here: $\Phi(0) = I$, so

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

[4] Diagonalization

Suppose that A is diagonalizable: $A = S\Lambda S^{-1}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. You can find the eigenvalues as roots of the characteristic polynomial, but you might as well remember that the eigenvalues of an upper (or lower) triangular matrix are the diagonal entries: here 1 and 3. Also an eigenvalue for 1 is easy: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For the other, subtract 3 from the diagonal entries: $\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ kills $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad , \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Suppose $A = S\Lambda S^{-1}$. Then we have exponential solutions corresponding to the eigenvalues:

$$\mathbf{u_1}(t)e^{\lambda_1 t}\mathbf{v_1}\, , \ldots$$

These give a fine fundamental matrix:

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v_1} & \ldots & e^{\lambda_n t}\mathbf{v_n} \end{bmatrix}$$

$$= Se^{\Lambda t} \quad , \quad S = \begin{bmatrix} \mathbf{u_1} & \ldots & \mathbf{u_n} \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Then $\Phi(0) = S$, so

$$e^{At} = Se^{\Lambda t}S^{-1}$$

In our example,

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

You could multiply this out, but, actually, the exponential matrix is often a pain in the neck to compute, and is often more useful as a symbolic device. Just like e^x, in fact!

[5] The exponential law

I claim that

$$e^{A(t+s)} = e^{At}e^{As}$$

This is a consequence of "time invariance." We have to see that both sides are equal after multiplying by an arbitrary vector $\mathbf{v}$. Let $\mathbf{x}(t)$ be the solution of $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condtion $\mathbf{x}(0) = \mathbf{v}$: so $\mathbf{x}(t) = e^{At}\mathbf{v}$. Now fix s and let

$$\mathbf{y}(t) = \mathbf{x}(t+s) = e^{A(t+s)}\mathbf{v}$$

Calculate using the chain rule:

$$\frac{d}{dt}\mathbf{y}(t) = \frac{d}{dt}\mathbf{x}(t+s) = \dot{\mathbf{x}}(t+s) = A\mathbf{x}(t+s) = A\mathbf{y}(t)$$

So $\mathbf{y}$ is the solution to $\dot{\mathbf{y}} = A\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{x}(s) = e^{As}\mathbf{v}$. That means that $\mathbf{y}(t) = e^{At}e^{As}\mathbf{v}$. QED

This is the proof of the exponential law even in the 1×1 case; and you will recall that as such it contains the trigonometric addition laws. Powerful stuff!

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises