Chapter 8 Applications of Newton's Second Law

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Those who are in love with practice without knowledge are like the sailor who gets into a ship without rudder or compass and who never can be certain whether he is going. Practice must always be founded on sound theory…[1]

Leonardo da Vinci

8.1 Force Laws

There are forces that don't change appreciably from one instant to another, which we refer to as constant in time, and forces that don't change appreciably from one point to another, which we refer to as constant in space. The gravitational force on an object near the surface of the earth is an example of a force that is constant in space.

There are forces that depend on the configuration of a system. When a mass is attached to one end of a spring, the spring force acting on the object increases in strength whether the spring is extended or compressed.

There are forces that spread out in space such that their influence becomes less with distance. Common examples are the gravitational and electrical forces. The gravitational force between two objects falls off as the inverse square of the distance separating the objects provided the objects are of a small dimension compared to the distance between them. More complicated arrangements of attracting and repelling interactions give rise to forces that fall off with other powers of r: constant, $1/r$, $1/r^2$, $1/r^3$, …,.

A force may remain constant in magnitude but change direction; for example the gravitational force acting on a planet undergoing circular motion about a star is directed towards the center of the circle. This type of attractive central force is called a ***centripetal force***.

A ***force law*** describes the relationship between the force and some measurable property of the objects involved. We shall see that some interactions are describable by force laws and other interactions cannot be so simply described.

8.1.1 Hooke's Law

In order to stretch or compress a spring from its equilibrium length, a force must be exerted on the spring. Consider an object of mass m that is lying on a horizontal surface. Attach one end of a spring to the object and fix the other end of the spring to a wall. Let

[1] Notebooks of Leonardo da Vinci Complete, tr. Jean Paul Richter, 1888, Vol.1.

l_0 denote the equilibrium length of the spring (neither stretched nor compressed). Assume that the contact surface is smooth and hence frictionless in order to consider only the effect of the spring force. If the object is pulled to stretch the spring or pushed to compress the spring, then by Newton's Third Law the force of the spring on the object is equal and opposite to the force that the object exerts on the spring. We shall refer to the force of the spring on the object as the *spring force* and experimentally determine a relationship between that force and the amount of stretching or compression of the spring.

Choose a coordinate system with the origin located at the point of contact of the spring and the object when the spring-object system is in the equilibrium configuration. Choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched. Choose the coordinate function x to denote the position of the object with respect to the origin (Figure 8.1).

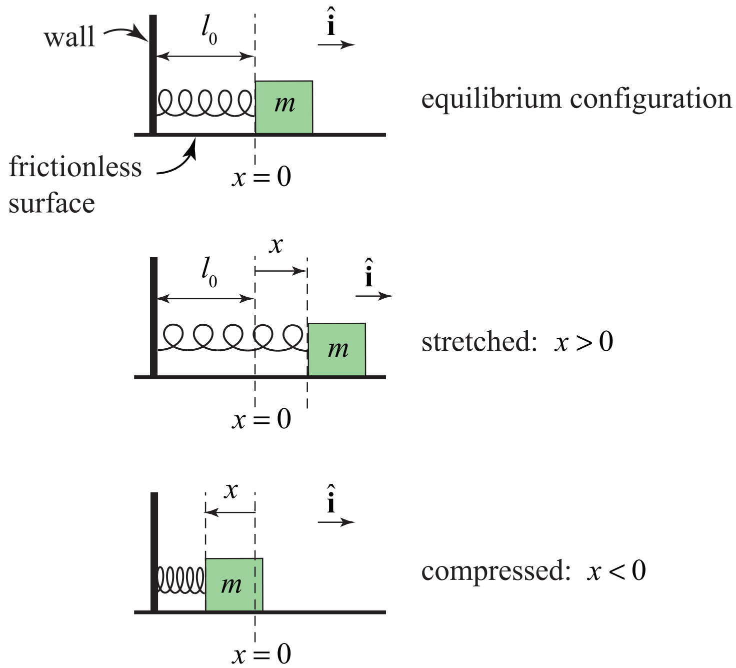


Figure 8.1 Spring attached to a wall and an object

Initially stretch the spring until the object is at position x. Then release the object and measure the acceleration of the object the instant the object is released. The magnitude of the spring force acting on the object is $|\vec{\mathbf{F}}| = m|\vec{\mathbf{a}}|$. Now repeat the experiment for a range of stretches (or compressions). Experiments show that for each spring, there is a range of maximum values $x_{max} > 0$ for stretching and minimum values $x_{min} < 0$ for compressing such that the magnitude of the measured force is proportional to the stretched or compressed length and is given by the formula

$$|\vec{\mathbf{F}}| = k|x|, \tag{8.1.1}$$

where the ***spring constant*** k has units $\text{N}\cdot\text{m}^{-1}$. The free-body force diagram is shown in Figure 8.2.

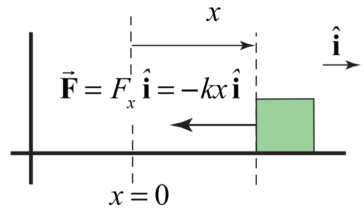


Figure 8.2 Spring force acting on object

The constant k is equal to the negative of the slope of the graph of the force *vs.* the compression or stretch (Figure 8.3).

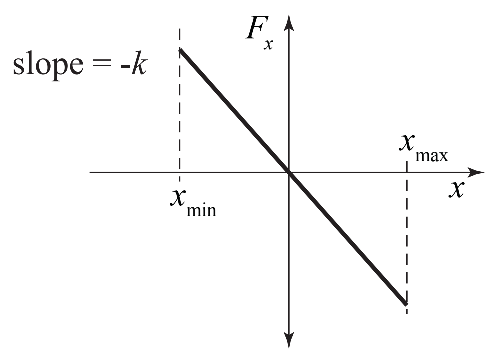


Figure 8.3 Plot of x-component of the spring force F_x vs. x

The direction of the acceleration is always towards the equilibrium position whether the spring is stretched or compressed. This type of force is called a ***restoring force***. Let F_x denote the x-component of the spring force. Then

$$F_x = -kx\,. \tag{8.1.2}$$

Now perform similar experiments on other springs. For a range of stretched lengths, each spring exhibits the same proportionality between force and stretched length, although the spring constant may differ for each spring.

It would be extremely impractical to experimentally determine whether this proportionality holds for all springs, and because a modest sampling of springs has confirmed the relation, we shall *infer* that all *ideal springs* will produce a restoring force, which is linearly proportional to the stretched (or compressed) length. This experimental relation regarding force and stretched (or compressed) lengths for a finite set of springs has now been *inductively* generalized into the above mathematical model for ideal springs, a force law known as a **Hooke's Law**.

This inductive step, referred to as ***Newtonian induction,*** is the critical step that makes physics a predictive science. Suppose a spring, attached to an object of mass m, is stretched by an amount Δx. Use the force law to predict the magnitude of the force between the spring and the object, $|\vec{\mathbf{F}}| = k|\Delta x|$, without having to experimentally measure the acceleration. Now use Newton's Second Law to predict the magnitude of the acceleration of the object

$$|\vec{\mathbf{a}}| = \frac{|\vec{\mathbf{F}}|}{m} = \frac{k|\Delta x|}{m}. \tag{8.1.3}$$

Carry out the experiment, and measure the acceleration within some error bounds. If the magnitude of the predicted acceleration disagrees with the measured result, then the model for the force law needs modification. The ability to adjust, correct or even reject models based on new experimental results enables a description of forces between objects to cover larger and larger experimental domains.

Many real springs have been wound such that a force of magnitude F_0 must be applied before the spring begins to stretch and a force of magnitude F_1 must be applied before the spring begins to compress. The values F_0 and F_1 are referred to as the *pre-tensions* of the spring. Under these circumstances, Hooke's law must be modified to account for these pretensions,

$$\begin{cases} F_x = -F_0 - kx, & x > 0 \\ F_x = +F_1 - kx, & x < 0 \end{cases}. \tag{8.1.4}$$

Note the value of the pre-tensions F_0 and F_1 may differ for compressing or stretching a spring and are determined experimentally.

8.2 Fundamental Laws of Nature

Force laws are mathematical models of physical processes. They arise from observation and experimentation, and they have limited ranges of applicability. Does the linear force law for the spring hold for all springs? Each spring will most likely have a different range of linear behavior. So the model for stretching springs still lacks a universal character. As such, there should be some hesitation to generalize this observation to all springs unless some property of the spring, universal to all springs, is responsible for the force law.

Perhaps springs are made up of very small components, which when pulled apart tend to contract back together. This would suggest that there is some type of force that contracts spring molecules when they are pulled apart. What holds molecules together? Can we find some fundamental property of the interaction between atoms that will suffice to explain the macroscopic force law? This search for ***fundamental forces*** is a central task of physics.

In the case of springs, this could lead into an investigation of the composition and structural properties of the atoms that compose the steel in the spring. We would investigate the geometric properties of the lattice of atoms and determine whether there is some fundamental property of the atoms that create this lattice. Then we ask how stable is this lattice under deformations. This may lead to an investigation into the electron configurations associated with each atom and how they overlap to form bonds between atoms. These particles carry charges, which obey Coulomb's Law, but also the Laws of Quantum Mechanics. So in order to arrive at a satisfactory explanation of the elastic restoring properties of the spring, we need models that describe the fundamental physics that underline Hooke's Law.

8.2.1 Universal Law of Gravitation

At points significantly far away from the surface of Earth, the gravitational force is no longer constant with respect to the distance to the center of Earth. ***Newton's Universal Law of Gravitation*** describes the gravitational force between two objects with masses, m_1 and m_2. This force points along the line connecting the objects, is attractive, and its magnitude is proportional to the inverse square of the distance, $r_{1,2}$, between the two point-like objects (Figure 8.4a). The force on object 2 due to the gravitational interaction between the two objects is given by

$$\vec{\mathbf{F}}^{G}_{1,2} = -G\frac{m_1 m_2}{r^2_{1,2}}\hat{\mathbf{r}}_{1,2}, \qquad (8.2.1)$$

where $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$ is a vector directed from object 1 to object 2, $r_{1,2} = \left|\vec{\mathbf{r}}_{1,2}\right|$, and $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left|\vec{\mathbf{r}}_{1,2}\right|$ is a unit vector directed from object 1 to object 2 (Figure 8.4b). The constant of proportionality in SI units is $G = 6.67\times 10^{-11}\,\mathrm{N\cdot m^2\cdot kg^{-2}}$.

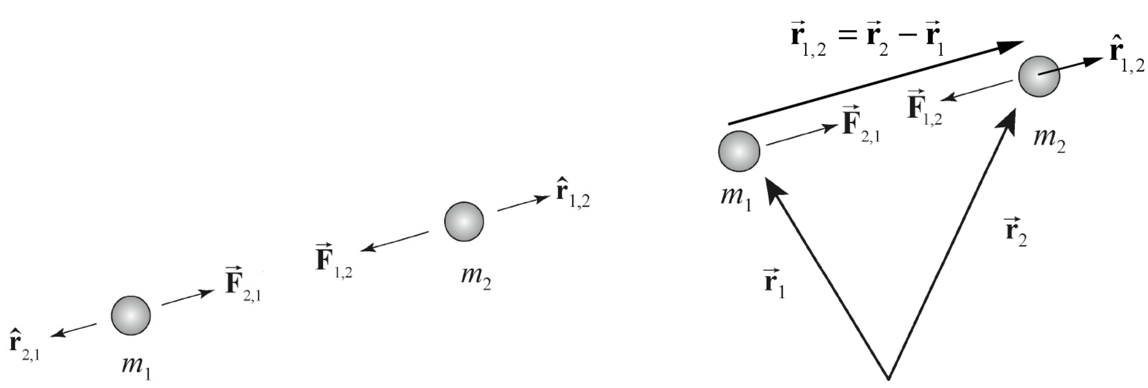


Figure 8.4 (a) Gravitational force between two point-like objects. **Figure 8.4 (b)** Coordinate system for the two-body problem.

8.2.2 Principle of Equivalence:

The Principle of Equivalence states that the mass that appears in the Universal Law of Gravity is identical to the inertial mass that is determined with respect to the standard kilogram. From this point on, the equivalence of inertial and gravitational mass will be assumed and the mass will be denoted by the symbol m.

8.2.3 Gravitational Force near the Surface of the Earth

Near the surface of Earth, the gravitational interaction between an object and Earth is mutually attractive and has a magnitude of

$$\left|\vec{\mathbf{F}}^{G}_{earth,object}\right| = m g \tag{8.2.2}$$

where g is a positive constant.

The International Committee on Weights and Measures has adopted as a standard value for the acceleration of an object freely falling in a vacuum $g = 9.80665\ \text{m}\cdot\text{s}^{-2}$. The actual value of g varies as a function of elevation and latitude. If ϕ is the latitude and h the elevation in meters then the acceleration of gravity in SI units is

$$g = (9.80616 - 0.025928\cos(2\phi) + 0.000069\cos^2(2\phi) - 3.086\times 10^{-4} h)\ \text{m}\cdot\text{s}^{-2}. \tag{8.2.3}$$

This is known as Helmert's equation. The strength of the gravitational force on the standard kilogram at 42° latitude is $9.80345\ \text{N}\cdot\text{kg}^{-1}$, and the acceleration due to gravity at sea level is therefore $g = 9.80345\ \text{m}\cdot\text{s}^{-2}$ for all objects. At the equator, $g = 9.78\ \text{m}\cdot\text{s}^{-2}$ and at the poles $g = 9.83\ \text{m}\cdot\text{s}^{-2}$. This difference is primarily due to the earth's rotation, which introduces an apparent (fictitious) repulsive force that affects the determination of g as given in Equation (8.2.2) and also flattens the spherical shape of Earth (the distance from the center of Earth is larger at the equator than it is at the poles by about $26.5\ \text{km}$). Both the magnitude and the direction of the gravitational force also show variations that depend on local features to an extent that's useful in prospecting for oil, investigating the water table, navigating submerged submarines, and as well as many other practical uses. Such variations in g can be measured with a sensitive spring balance. Local variations have been much studied over the past two decades in attempts to discover a proposed "fifth force" which would fall off faster than the gravitational force that falls off as the inverse square of the distance between the objects.

8.2.4 Electric Charge and Coulomb's Law

Matter has properties other than mass. Matter can also carry one of two types of observed ***electric charge***, positive and negative. Like charges repel, and opposite charges attract each other. The unit of charge in the SI system of units is called the *coulomb* $[\text{C}]$.

The smallest unit of "free" charge known in nature is the charge of an electron or proton, which has a magnitude of

$$e = 1.602 \times 10^{-19}\,\mathrm{C}\,. \tag{8.2.4}$$

It has been shown experimentally that charge carried by ordinary objects is quantized in integral multiples of the magnitude of this free charge. The electron carries one unit of negative charge ($q_e = -e$) and the proton carries one unit of positive charge ($q_p = +e$). In an isolated system, the charge stays constant; in a closed system, an amount of unbalanced charge can neither be created nor destroyed. Charge can only be transferred from one object to another.

Consider two point-like objects with charges q_1 and q_2, separated by a distance $r_{1,2}$ in vacuum. By experimental observation, the two objects repel each other if they are both positively or negatively charged (Figure 8.4a). They attract each other if they are oppositely charged (Figure 8.5b). The force exerted on object 2 due to the interaction between objects 1 and 2 is given by Coulomb's Law,

$$\vec{\mathbf{F}}^{E}_{1,2} = k_e \frac{q_1 q_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2} \tag{8.2.5}$$

where $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left|\vec{\mathbf{r}}_{1,2}\right|$ is a unit vector directed from object 1 to object 2, and in SI units, $k_e = 8.9875 \times 10^{9}\,\mathrm{N \cdot m^2 \cdot C^{-2}}$, as illustrated in the Figure 8.5a. This law was derived empirically by Charles Augustin de Coulomb in the late 18$^{\text{th}}$ century.

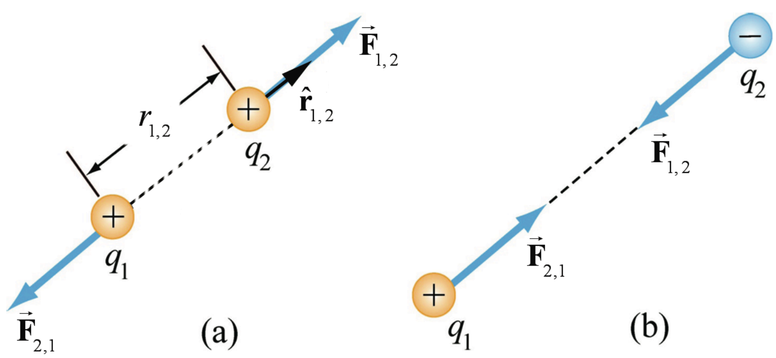


Figure 8.5 (a) and 8.5 (b) Coulomb interaction between two charges

Example 8.1 Coulomb's Law and the Universal Law of Gravitation

Show that both Coulomb's Law and the Universal Law of Gravitation satisfy Newton's Third Law.

Solution: To see this, interchange 1 and 2 in the Universal Law of Gravitation to find the force on object 1 due to the interaction between the objects. The only quantity to change sign is the unit vector

$$\hat{\mathbf{r}}_{2,1} = -\hat{\mathbf{r}}_{1,2} . \tag{8.2.6}$$

Then

$$\vec{\mathbf{F}}^{G}_{2,1} = -G\frac{m_2\, m_1}{r^2_{2,1}}\hat{\mathbf{r}}_{2,1} = G\frac{m_1\, m_2}{r^2_{1,2}}\hat{\mathbf{r}}_{1,2} = -\vec{\mathbf{F}}^{G}_{1,2} . \tag{8.2.7}$$

Coulomb's Law also satisfies Newton's Third Law since the only quantity to change sign is the unit vector, just as in the case of the Universal Law of Gravitation.

8.3 Constraint Forces

Knowledge of all the external and internal forces acting on each of the objects in a system and applying Newton's Second Law to each of the objects determine a set of equations of motion. These equations of motion are not necessarily independent due to the fact that the motion of the objects may be limited by equations of constraint. In addition there are forces of constraint that are determined by their effect on the motion of the objects and are not known beforehand or describable by some force law. For example: an object sliding down an inclined plane is constrained to move along the surface of the inclined plane (Figure 8.6a) and the surface exerts a contact force on the object; an object that slides down the surface of a sphere until it falls off experiences a contact force until it loses contact with the surface (Figure 8.6b); gas particles in a sealed vessel are constrained to remain inside the vessel and therefore the wall must exert force on the gas molecules to keep them inside the vessel (8.6c); and a bead constrained to slide outward along a rotating rod is acted on by time dependent forces of the rod on the bead (Figure 8.6d). We shall develop methods to determine these constraint forces although there are many examples in which the constraint forces cannot be determined.

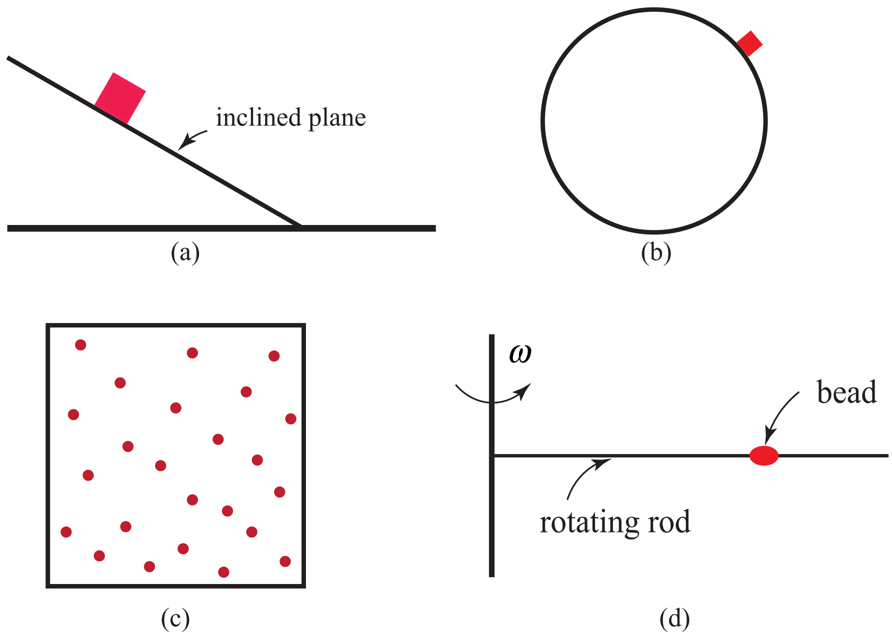


Figure 8.6 Constrained motions: **(a)** particle sliding down inclined plane, **(b)** particles sliding down surface of sphere, **(c)** gas molecules in a sealed vessel, and **(d)** bead sliding on a rotating rod

8.3.1 Contact Forces

Pushing, lifting and pulling are ***contact forces*** that we experience in the everyday world. Rest your hand on a table; the atoms that form the molecules that make up the table and your hand are in contact with each other. If you press harder, the atoms are also pressed closer together. The electrons in the atoms begin to repel each other and your hand is pushed in the opposite direction by the table.

According to Newton's Third Law, the force of your hand on the table is equal in magnitude and opposite in direction to the force of the table on your hand. Clearly, if you push harder the force increases. Try it! If you push your hand straight down on the table, the table pushes back in a direction perpendicular (normal) to the surface. Slide your hand gently forward along the surface of the table. You barely feel the table pushing upward, but you do feel the friction acting as a resistive force to the motion of your hand. This force acts tangential to the surface and opposite to the motion of your hand. Push downward and forward. Try to estimate the magnitude of the force acting on your hand.

The force of the table acting on your hand, $\vec{\mathbf{F}}^C \equiv \vec{\mathbf{C}}$, is called the ***contact force.*** This force has both a normal component to the surface, $\vec{\mathbf{C}}_{\perp} \equiv \vec{\mathbf{N}}$, called the ***normal force,*** and a tangential component to the surface, $\vec{\mathbf{C}}_{\parallel} \equiv \vec{\mathbf{f}}$, called the ***friction force*** (Figure 8.6).

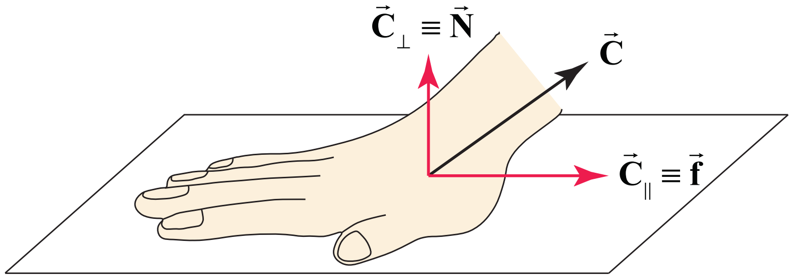


Figure 8.6 Normal and tangential components of the contact force

The contact force, written in terms of its component forces, is therefore

$$\vec{\mathbf{C}} = \vec{\mathbf{C}}_\perp + \vec{\mathbf{C}}_{||} \equiv \vec{\mathbf{N}} + \vec{\mathbf{f}} \,. \tag{8.3.1}$$

Any force can be decomposed into component vectors so the normal component, $\vec{\mathbf{N}}$, and the tangential component, $\vec{\mathbf{f}}$, are not independent forces but the vector components of the contact force, perpendicular and parallel to the surface of contact. The contact force is a *distributed force* acting over all the points of contact between your hand and the surface. For most applications we shall treat the contact force as acting at single point but precaution must be taken when the distributed nature of the contact force plays a key role in constraining the motion of a rigid body.

In Figure 8.7, the forces acting on your hand are shown. These forces include the contact force, $\vec{\mathbf{C}}$, of the table acting on your hand, the force of your forearm, $\vec{\mathbf{F}}_{\text{forearm}}$, acting on your hand (which is drawn at an angle indicating that you are pushing down on your hand as well as forward), and the gravitational interaction, $\vec{\mathbf{F}}^g$, between the earth and your hand.

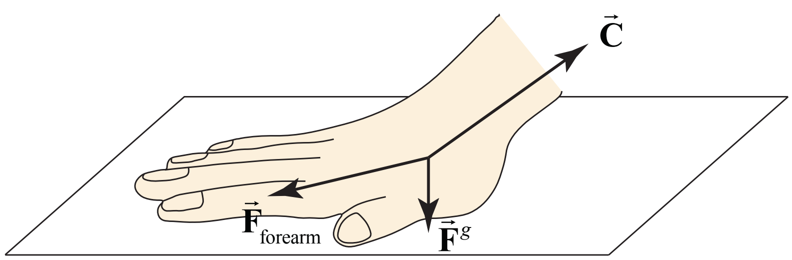


Figure 8.7 Forces on hand when moving towards the left

One point to keep in mind is that the magnitudes of the two components of the contact force depend on how hard you push or pull your hand and in what direction, a characteristic of constraint forces, in which the components are not specified by a force law but dependent on the particular motion of the hand.

Example 8.2 Normal Component of the Contact Force and Weight

Hold a block in your hand such that your hand is at rest (Figure 8.8). You can feel the "weight" of the block against your palm. But what exactly do we mean by "weight"?

Figure 8.8 Block resting in hand

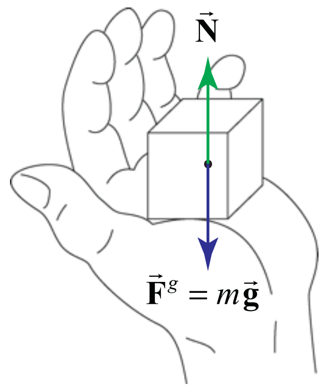


Figure 8.9 Forces on block

There are two forces acting on the block as shown in Figure 8.9. One force is the gravitational force between the earth and the block, and is denoted by $\vec{\mathbf{F}}^g = m\vec{\mathbf{g}}$. The other force acting on the block is the contact force between your hand and the block. Because your hand is at rest, this contact force on the block points perpendicular to the surface, and hence has only a normal component, $\vec{\mathbf{N}}$. Let N denote the magnitude of the normal force. Because the object is at rest in your hand, the vertical acceleration is zero. Therefore Newton's Second Law states that

$$\vec{\mathbf{N}} + \vec{\mathbf{F}}^g = \vec{\mathbf{0}}\,. \tag{8.3.2}$$

Choose the positive direction to be upwards and then in terms of vertical components we have that

$$N - mg = 0\,. \tag{8.3.3}$$

which can be solved for the magnitude of the normal force

$$N = mg\,. \tag{8.3.4}$$

When we talk about the "weight" of the block, we often are referring to the effect the block has on a scale or on the feeling we have when we hold the block. These effects are actually effects of the normal force. We say that a block "feels lighter" if there is an additional force holding the block up. For example, you can rest the block in your hand, but use your other hand to apply a force upwards on the block to make it feel lighter in your supporting hand.

The word "weight," is often used to describe the gravitational force that Earth exerts on an object. We shall always refer to this force as the *gravitational force* instead of "weight." When you jump in the air, you feel "weightless" because there is no normal force acting on you, even though Earth is still exerting a gravitational force on you; clearly, when you jump, you do not turn gravity off!

This example may also give rise to a misconception that the normal force is always equal to the mass of the object times the magnitude of the gravitational acceleration at the surface of the earth. The normal force and the gravitational force are two completely different forces. In this particular example, the normal force is equal in magnitude to the gravitational force and directed in the opposite direction because the object is at rest. The normal force and the gravitational force do not form a Third Law interaction pair of forces. In this example, our system is just the block; the normal force and gravitational force are external forces acting on the block.

Let's redefine our system as the block, your hand, and Earth. Then the normal force and gravitational force are now internal forces in the system and we can now identify the various interaction pairs of forces. We explicitly introduce our interaction pair notation to enable us to identify these interaction pairs: for example, let $\vec{\mathbf{F}}^{g}_{E,B}$ denote the gravitational force on the block due to the interaction with Earth. The gravitational force on Earth due to the interaction with the block is denoted by $\vec{\mathbf{F}}^{g}_{B,E}$, and these two forces form an interaction pair. By Newton's Third Law, $\vec{\mathbf{F}}^{g}_{E,B} = -\vec{\mathbf{F}}^{g}_{B,E}$. Note that these two forces are acting on different objects, the block and Earth. The contact force on the block due to the interaction between the hand and the block is then denoted by $\vec{\mathbf{N}}_{H,B}$. The force of the block on the hand, which we denote by $\vec{\mathbf{N}}_{B,H}$, satisfies $\vec{\mathbf{N}}_{B,H} = -\vec{\mathbf{N}}_{H,B}$. Because we are including your hand as part of the system, there are two additional forces acting on the hand. There is the gravitational force on your hand $\vec{\mathbf{F}}^{g}_{E,H}$, satisfying $\vec{\mathbf{F}}^{g}_{E,H} = -\vec{\mathbf{F}}^{g}_{H,E}$, where $\vec{\mathbf{F}}^{g}_{H,E}$ is the gravitational force on Earth due to your hand. Finally there is the force of your forearm holding your hand up, which we denote $\vec{\mathbf{F}}_{F,H}$. Because we are not including the forearm in our system, this force is an external force to the system. The forces acting on your hand are shown in Figure 8.10, and just the interaction pairing of forces acting on Earth is shown in Figure 8.11 (we are not representing all other external forces acting on the Earth).

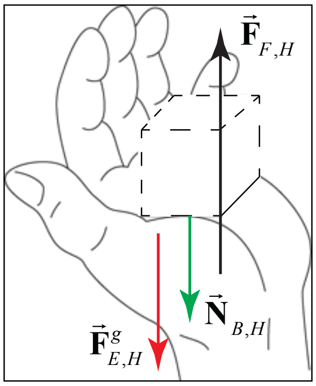


Figure 8.10 Free-body force diagram on hand

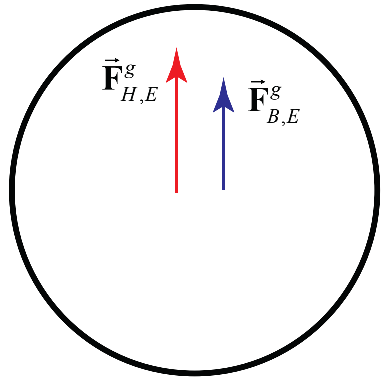


Figure 8.11 Gravitational forces on earth due to object and hand

8.3.2 Kinetic and Static Friction

When a block is pulled along a horizontal surface or sliding down an inclined plane there is a lateral force resisting the motion. If the block is at rest on the inclined plane, there is still a lateral force resisting the motion. This resistive force is known as dry friction, and there are two distinguishing types when surfaces are in contact with each other. The first type occurs when the two objects are moving relative to each other; the friction in that case is called ***kinetic friction*** or ***sliding friction***. When the two surfaces are non-moving but there is still a lateral force as in the example of the block at rest on an inclined plane, the force is called, ***static friction***.

Leonardo da Vinci was the first to record the results of measurements on kinetic friction over a twenty-year period between 1493–4 and about 1515. Based on his measurements, the force of kinetic friction, $\vec{\mathbf{f}}^{k}$, between two surfaces, he identified two key properties of kinetic friction. The magnitude of kinetic friction is proportional to the normal force between the two surfaces,

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} N \,, \tag{8.3.5}$$

where μ_{k} is called the ***coefficient of kinetic friction***. The second result is rather surprising in that the magnitude of the force is independent of the contact surface. Consider two blocks of the same mass, but different surface areas. The force necessary to move the blocks at a constant speed is the same. The block in Figure 8.12a has twice the contact area as the block shown in Figure 8.12b, but when the same external force is applied to either block, the blocks move at constant speed. These results of da Vinci were rediscovered by Guillaume Amontons and published in 1699. The third property that kinetic friction is independent of the speed of moving objects (for ordinary sliding speeds) was discovered by Charles Augustin Coulomb.

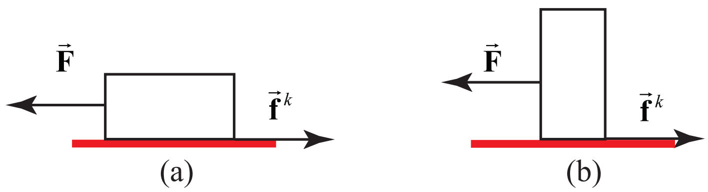


Figure 8.12 (a) and (b): kinetic friction is independent of the contact area

The kinetic friction on surface 2 moving relative to surface 1 is denoted by, $\vec{\mathbf{f}}^{k}_{1,2}$. The direction of the force is always opposed to the relative direction of motion of surface 2 relative to the surface 1. When one surface is at rest relative to our choice of reference frame we will denote the friction force on the moving object by $\vec{\mathbf{f}}^{k}$.

The second type of dry friction, static friction occurs when two surfaces are static relative to each other. Because the static friction force between two surfaces forms a third law interaction pair, we will use the notation $\vec{\mathbf{f}}^{s}_{1,2}$ to denote the static friction force on surface 2 due to the interaction between surfaces 1 and 2. Push your hand forward along a surface; as you increase your pushing force, the frictional force feels stronger and stronger. Try this! Your hand will at first stick until you push hard enough, then your hand slides forward. The magnitude of the static frictional force, f_{s}, depends on how hard you push.

If you rest your hand on a table without pushing horizontally, the static friction is zero. As you increase your push, the static friction increases until you push hard enough that your hand slips and starts to slide along the surface. Thus the magnitude of static friction can vary from zero to some maximum value, $(f_{\mathrm{s}})_{\max}$, when the pushed object begins to slip,

$$0 \le f_{\mathrm{s}} \le (f_{\mathrm{s}})_{\max} . \tag{8.3.6}$$

Is there a mathematical model for the magnitude of the maximum value of static friction between two surfaces? Through experimentation, we find that this magnitude is, like kinetic friction, proportional to the magnitude of the normal force

$$(f_{\mathrm{s}})_{\max} = \mu_{\mathrm{s}} N . \tag{8.3.7}$$

Here the constant of proportionality is μ_{s}, the ***coefficient of static friction.*** This constant is slightly greater than the constant μ_{k} associated with kinetic friction, $\mu_{\mathrm{s}} > \mu_{\mathrm{k}}$. This small difference accounts for the slipping and catching of chalk on a blackboard, fingernails on glass, or a violin bow on a string.

The direction of static friction on an object is always opposed to the direction of the applied force (as long as the two surfaces are not accelerating). In Figure 8.13a, an external force, $\vec{\mathbf{F}}$, is applied the left, and the static friction, $\vec{\mathbf{f}}^s$, is directed to the right opposing the external force. In Figure 8.13b, the external force, $\vec{\mathbf{F}}$, is applied to the right, and the static friction, $\vec{\mathbf{f}}^s$, is now pointing to the left.

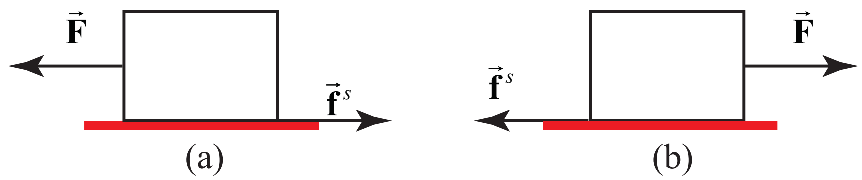


Figure 8.13 (a) and (b): External forces and the direction of static friction.

Although the force law for the maximum magnitude of static friction resembles the force law for sliding friction, there are important differences:

1. The direction and magnitude of static friction on an object always depends on the direction and magnitude of the applied forces acting on the object, where the magnitude of kinetic friction for a sliding object is fixed.

2. The magnitude of static friction has a maximum possible value. If the magnitude of the applied force along the direction of the contact surface exceeds the magnitude of the maximum value of static friction, then the object will start to slip (and be subject to kinetic friction.) We call this the ***just slipping*** condition.

8.4 Free-body Force Diagram

8.4.1 System

When we try to describe forces acting on a collection of objects we must first take care to specifically define the collection of objects that we are interested in, which define our ***system.*** Often the system is a single isolated object but it can consist of multiple objects.

Because force is a vector, the force acting on the system is a vector sum of the individual forces acting on the system

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \qquad (8.4.1)$$

A ***free-body force diagram*** is a representation of the sum of all the forces that act on a single system. We denote the system by a large circular dot, a “point”. (Later on in the course we shall see that the “point” represents the center of mass of the system.) We represent each force that acts on the system by an arrow (indicating the direction of that force). We draw the arrow at the “point” representing the system. For example, the forces that regularly appear in free-body diagram are contact forces, tension, gravitation, friction, pressure forces, spring forces, electric and magnetic forces, which we shall introduce below. Sometimes we will draw the arrow representing the actual point in the system where the force is acting. When we do that, we will not represent the system by a "point" in the free-body diagram.

Suppose we choose a Cartesian coordinate system, then we can resolve the force into its component vectors

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}} \tag{8.4.2}$$

Each one of the component vectors is itself a vector sum of the individual component vectors from each contributing force. We can use the free-body force diagram to make these vector decompositions of the individual forces. For example, the x-component of the force is

$$F_x = F_{1,x} + F_{2,x} + \cdots. \tag{8.4.3}$$

8.4.5 Modeling

One of the most central and yet most difficult tasks in analyzing a physical interaction is developing a physical model. A physical model for the interaction consists of a description of the forces acting on all the objects. The difficulty arises in deciding which forces to include. For example in describing almost all planetary motions, the Universal Law of Gravitation was the only force law that was needed. There were anomalies, for example the small shift in Mercury's orbit. These anomalies are interesting because they may lead to new physics. Einstein corrected Newton's Law of Gravitation by introducing General Relativity and one of the first successful predictions of the new theory was the perihelion precession of Mercury's orbit. On the other hand, the anomalies may simply be due to the complications introduced by forces that are well understood but complicated to model. When objects are in motion there is always some type of friction present. Air friction is often neglected because the mathematical models for air resistance are fairly complicated even though the force of air resistance substantially changes the motion. Static or kinetic friction between surfaces is sometimes ignored but not always. The mathematical description of the friction between surfaces has a simple expression so it can be included without making the description mathematically intractable. A good way to start thinking about the problem is to make a simple model, excluding complications that are small order effects. Then we can check the predictions of the model. Once we are satisfied that we are on the right track, we can include more complicated effects.

8.5 Tension in a Rope

8.5.1 Definition of Tension in a Rope

Let's return to our example of the very light rope (object 2 with $m_2 \simeq 0$) that is attached to a block (object 1) at the point B, and pulled by an applied force at point A, $\vec{\mathbf{F}}_{\text{A},2}$ (Figure 8.18a).

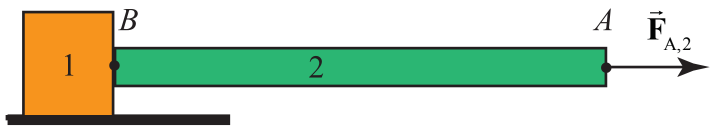


Figure 8.18a Massless rope pulling a block

Choose a coordinate system with the $\hat{\mathbf{j}}$-unit vector pointing upward in the normal direction to the surface, and the $\hat{\mathbf{i}}$-unit vector pointing in the positive x-direction, (Figure 8.18b). The force diagrams for the system consisting of the rope and block is shown in Figure 8.18a, and for the rope and block separately in Figure 8.19, where $\vec{\mathbf{F}}_{2,1}$ is the force on the block (object 1) due to the rope (object 2), and $\vec{\mathbf{F}}_{1,2}$ is the force on the rope due to the block.

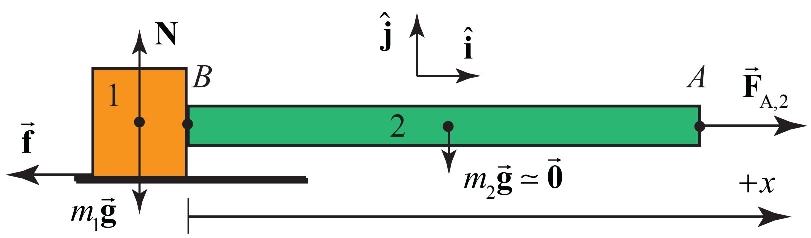


Figure 8.18b Forces acting on system consisting of block and rope

The forces on the rope and the block must each sum to zero. Because the rope is not accelerating, Newton's Second Law applied to the rope requires that $F_{\text{A},2} - F_{1,2} = m_2 a$ (where we are using magnitudes for all the forces).

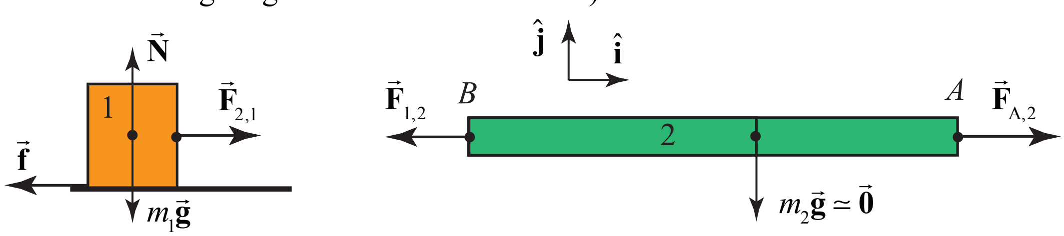


Figure 8.19 Separate force diagrams for rope and block

Because we are assuming the mass of the rope is negligible therefore

$$F_{\text{A},2} - F_{1,2} = 0; \quad \text{(massless rope)} \tag{8.5.1}$$

.

If we consider the case that the rope is very light, then the forces acting at the ends of the rope are nearly horizontal. Then if the rope-block system is moving at constant speed or at rest, Newton's Second Law is now

$$F_{A,2} - F_{1,2} = 0; \quad \text{(constant speed or at rest)} \ . \tag{8.5.2}$$

Newton's Second Law applied to the block in the $+\hat{\mathbf{i}}$-direction requires that $F_{2,1} - f = 0$. Newton's Third Law, applied to the block-rope interaction pair requires that $F_{1,2} = F_{2,1}$. Therefore

$$F_{A,2} = F_{1,2} = F_{2,1} = f \ . \tag{8.5.3}$$

Thus the applied pulling force is transmitted through the rope to the block since it has the same magnitude as the force of the rope on the block. In addition, the applied pulling force is also equal to the friction force on the block.

How do we define "tension" at some point in a rope? Suppose make an imaginary slice of the rope at a point P, a distance x_P from point B, where the rope is attached to the block. The imaginary slice divides the rope into two sections, labeled L (left) and R (right), as shown in Figure 8.20.

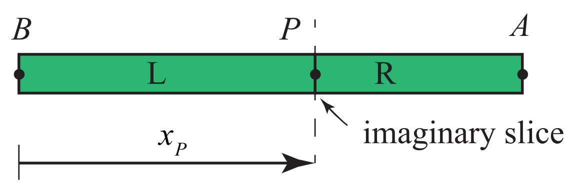


Figure 8.20 Imaginary slice through the rope

There is now a Third Law pair of forces acting between the left and right sections of the rope. Denote the force acting on the left section by $\vec{\mathbf{F}}_{R,L}(x_P)$, and the force acting on the right section by $\vec{\mathbf{F}}_{L,R}(x_P)$. Newton's Third Law requires that the forces in this interaction pair are equal in magnitude and opposite in direction.

$$\vec{\mathbf{F}}_{R,L}(x_P) = -\vec{\mathbf{F}}_{L,R}(x_P) \tag{8.5.4}$$

The force diagram for the left and right sections are shown in Figure 8.21 where $\vec{\mathbf{F}}_{1,L}$ is the force on the left section of the rope due to the block-rope interaction. (We had previously denoted that force by $\vec{\mathbf{F}}_{1,2}$). Now denote the force on the right section of the rope side due to the pulling force at the point A by $\vec{\mathbf{F}}_{A,R}$, (which we had previously denoted by $\vec{\mathbf{F}}_{A,2}$).

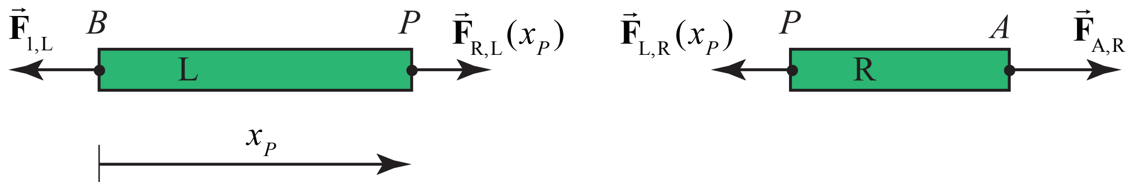


Figure 8.21 Force diagram for the left and right sections of rope

The **tension** $T(x_P)$ at a point P in rope lying a distance x from one the left end of the rope, is the magnitude of the action -reaction pair of forces acting at the point P,

$$T(x_P) = \left|\vec{F}_{R,L}(x_P)\right| = \left|\vec{F}_{L,R}(x_P)\right|. \tag{8.5.5}$$

For a rope of negligible mass, under tension, as in the above case, (even if the rope is accelerating) the sum of the horizontal forces applied to the left section and the right section of the rope are zero, and therefore the tension is uniform and is equal to the applied pulling force,

$$T = F_{A,R}. \tag{8.5.6}$$

Example 8.3 Tension in a Massive Rope

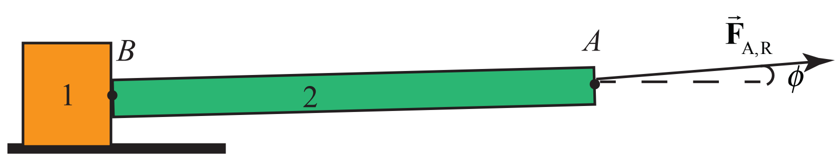


Figure 8.22a Massive rope pulling a block

Consider a block of mass m_1 that is lying on a horizontal surface. The coefficient of kinetic friction between the block and the surface is μ_k. A uniform rope of mass m_2 and length d is attached to the block. The rope is pulled from the side opposite the block with an applied force of magnitude $\left|\vec{F}_{A,2}\right| = F_{A,2}$. Because the rope is now massive, the pulling force makes an angle ϕ with respect to the horizontal in order to balance the gravitational force on the rope, (Figure 8.22a). Determine the tension in the rope as a function of distance x from the block.

Solution: In the following analysis, we shall assume that the angle ϕ is very small and depict the pulling and tension forces as essentially acting in the horizontal direction even though there must be some small vertical component to balance the gravitational forces.

The key point to realize is that the rope is now massive and we must take in to account the inertia of the rope when applying Newton's Second Law. Consider an imaginary slice through the rope at a distance x from the block (Figure 8.22b), dividing the rope into two sections. The right section has length $d-x$ and mass $m_{R}=(m_{2}/d)(d-x)$. The left section has length x and mass $m_{L}=(m_{2}/d)(x)$.

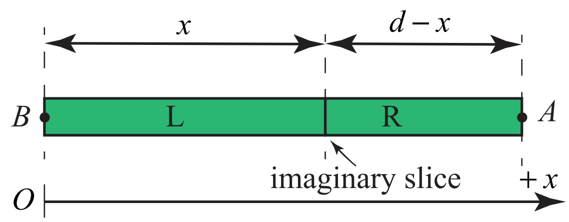


Figure 8.22b Imaginary slice through the rope

The free body force diagrams for the two sections of the rope are shown in Figure 8.22c, where $T(x)$ is the tension in the rope at a distance x from the block, and $F_{1,L}=\left|\vec{\mathbf{F}}_{1,L}\right|\equiv\left|\vec{\mathbf{F}}_{1,2}\right|$ is the magnitude of the force on the left-section of the rope due to the rope-block interaction.

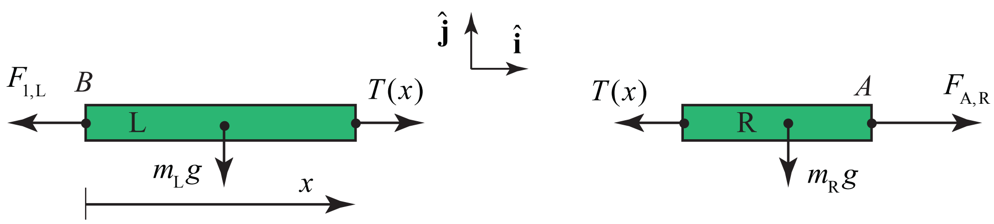


Figure 8.22c Force diagram for the left and right sections of rope

Apply Newton's Second Law to the right section of the rope yielding

$$F_{A,R}-T(x)=m_{R}a_{R}=\frac{m_{2}}{d}(d-x)a_{R}\,, \tag{8.5.7}$$

where a_{R} is the x-component of the acceleration of the right section of the rope. Apply Newton's Second Law to the left slice of the rope yielding

$$T(x)-F_{1,L}=m_{L}a_{L}=(m_{2}/d)x\,a_{L}\,, \tag{8.5.8}$$

where a_{L} is the x-component of the acceleration of the left piece of the rope.

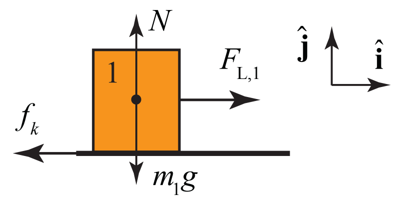


Figure 8.23 Force diagram on sliding block

The force diagram on the block is shown in Figure 8.23. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction is $F_{L,1} - f_k = m_1a_1$ and in the $+\hat{\mathbf{j}}$ -direction is $N - m_1g = 0$. The kinetic friction force acting on the block is $f_k = \mu_k N = \mu_k m_1 g$. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction becomes

$$F_{L,1} - \mu_k m_1 g = m_1 a_1, \tag{8.5.9}$$

Newton's Third Law for the block-rope interaction is given by $F_{L,1} = F_{1,L}$. Eq. (8.5.8) then becomes

$$T(x) - (\mu_k m_1 g + m_1 a_1) = (m_2 / d) x a_L. \tag{8.5.10}$$

Because the rope and block move together, the accelerations are equal which we denote by the symbol $a \equiv a_1 = a_L$. Then Eq. (8.5.10) becomes

$$T(x) = \mu_k m_1 g + (m_1 + (m_2 / d)x)a. \tag{8.5.11}$$

This result is not unexpected because the tension is accelerating both the block and the left section and is opposed by the frictional force.

Alternatively, the force diagram on the system consisting of the rope and block is shown in Figure 8.24.

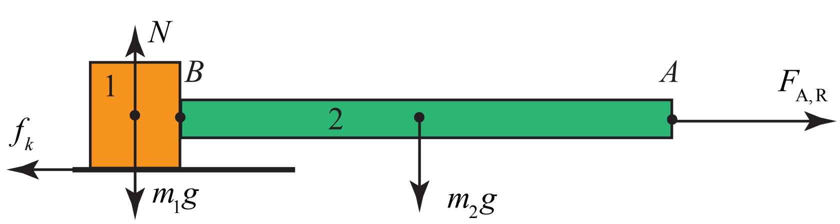


Figure 8.24 Force diagram on block-rope system

Newton's Second Law becomes

$$F_{A,R} - \mu_k m_1 g = (m_2 + m_1)a \tag{8.5.12}$$

Solve Eq. (8.5.12) for $F_{A,R}$ and substitute into Eq. (8.5.7), and solve for the tension yielding Eq. (8.5.11).

Example 8.4 Tension in a Suspended Rope

A uniform rope of mass M and length L is suspended from a ceiling (Figure 8.25). The magnitude of the acceleration due to gravity is g. (a) Find the tension in the rope at the upper end where the rope is fixed to the ceiling. (b) Find the tension in the rope as a function of the distance from the ceiling. (c) Find an equation for the rate of change of the tension with respect to distance from the ceiling in terms of M, L, and g.

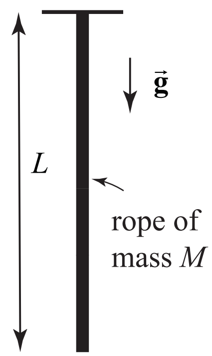


Figure 8.25 Rope suspended from ceiling

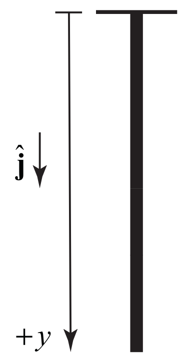


Figure 8.26 Coordinate system for suspended rope

Solution: (a) Begin by choosing a coordinate system with the origin at the ceiling and the positive y-direction pointing downward (Figure 8.26). In order to find the tension at the upper end of the rope, choose as a system the entire rope. The forces acting on the rope are the force at $y = 0$ holding the rope up, $T(y = 0)$, and the gravitational force on the entire rope. The free-body force diagram is shown in Figure 8.27.

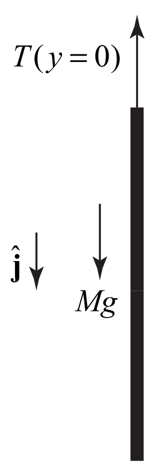


Figure 8.27 Force diagram on rope

Because the acceleration is zero, Newton's Second Law on the rope is $Mg - T(y=0) = 0$. Therefore the tension at the upper end is $T(y=0) = Mg$.

(b) Recall that the tension at a point is the magnitude of the action-reaction pair of forces acting at that point. Make an imaginary slice in the rope a distance y from the ceiling separating the rope into an upper segment 1, and lower segment 2 (Figure 8.28a). Choose the upper segment as a system with mass $m_1 = (M/L)y$. The forces acting on the upper segment are the gravitational force, the force $T(y=0)$ holding the rope up, and the tension $T(y)$ at the point y, that is pulling the upper segment down. The free-body force diagram is shown in Figure 8.28b.

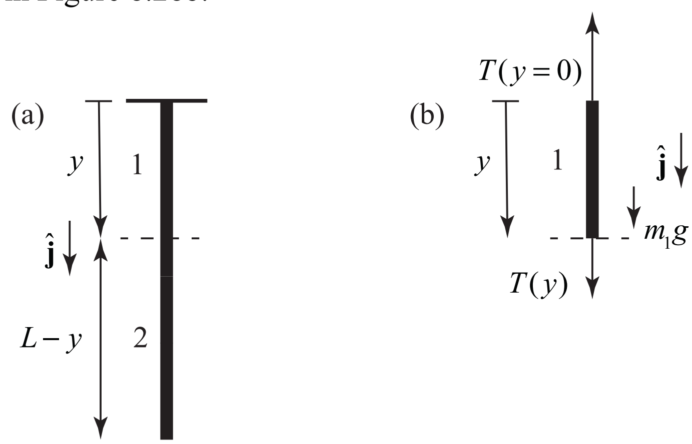


Figure 8.28 (a) Imaginary slice separates rope into two pieces**. (b)** Free-body force diagram on upper piece of rope

Apply Newton's Second Law to the upper segment: $m_1 g + T(y) - T(y=0) = 0$. Therefore the tension at a distance y from the ceiling is $T(y) = T(y=0) - m_1 g$.
Because $m_1 = (M/L)y$ is the mass of the segment piece and Mg is the tension at the upper end, Newton's Second Law becomes

$$T(y) = Mg(1 - y/L) \tag{8.5.13}$$

As a check, we note that when $y = L$, the tension $T(y=L) = 0$, which is what we expect because there is no force acting at the lower end of the rope.

(c) Differentiate Eq. (8.5.13) with respect to y yielding

$$\frac{dT}{dy} = -(M/L)g\,. \tag{8.5.14}$$

The rate that the tension is changing at a constant rate with respect to distance from the top of the rope.

8.5.2 Continuous Systems and Newton's Second Law as a Differential Equations

We can determine the tension at a distance y from the ceiling in Example 8.4, by an alternative method, a technique that will generalize to many types of "continuous systems". Choose a coordinate system with the origin at the ceiling and the positive y-direction pointing downward as in Figure 8.25. Consider as the system a small element of the rope between the points y and $y+\Delta y$. This small element has length Δy, The small element has mass $\Delta m = (M/L)\Delta y$ and is shown in Figure 8.29.

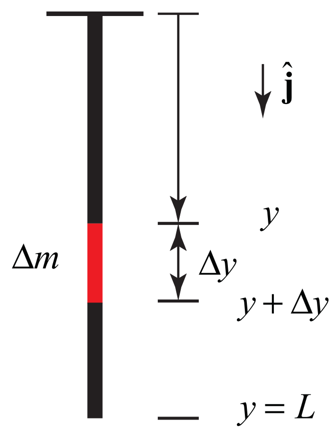


Figure 8.29 Small mass element of the rope

The forces acting on the small element are the tension, $T(y)$ at y directed upward, the tension $T(y+\Delta y)$ at $y+\Delta y$ directed downward, and the gravitational force $\Delta m g$ directed downward. The tension $T(y+\Delta y)$ is equal to the tension $T(y)$ plus a small difference ΔT,

$$T(y+\Delta y) = T(y) + \Delta T \,. \tag{8.5.15}$$

The small difference in general can be positive, zero, or negative. The free body force diagram is shown in Figure 8.30.

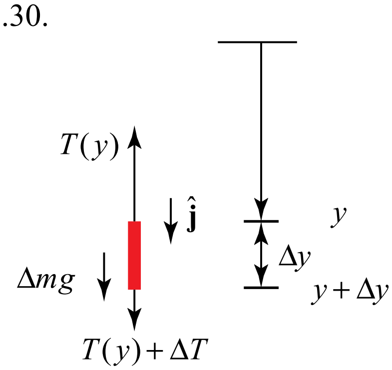


Figure 8.30 Free body force diagram on small mass element

Now apply Newton's Second Law to the small element

$$\Delta mg - T(y) + (T(y) + \Delta T) = 0 \qquad (8.5.16)$$

The difference in the tension is then $\Delta T = -\Delta mg$. We now substitute our result for the mass of the element $\Delta m = (M / L)\Delta y$, and find that that

$$\Delta T = -(M / L)\Delta y g \ . \qquad (8.5.17)$$

Divide through by Δy, yielding $\Delta T / \Delta y = -(M / L)g$. Now take the limit in which the length of the small element goes to zero, $\Delta y \to 0$,

$$\lim_{\Delta y \to 0} \frac{\Delta T}{\Delta y} = -(M / L)g \ . \qquad (8.5.18)$$

Recall that the left hand side of Eq. (8.5.18) is the definition of the derivative of the tension with respect to y, and so we arrive at Eq. (8.5.14),

$$\frac{dT}{dy} = -(M / L)g .$$

We can solve the differential equation, Eq. (8.5.14), by a technique called **separation of variables**. We rewrite the equation as $dT = -(M / L)g dy$ and integrate both sides. Our integral will be a definite integral in which we integrate a 'dummy' integration variable y' from $y' = 0$ to $y' = y$ and the corresponding T' from $T' = T(y = 0)$ to $T' = T(y)$:

$$\int_{T'=T(y=0)}^{T'=T(y)} dT' = -(M / L)g \int_{y'=0}^{y'=y} dy' \ . \qquad (8.5.19)$$

After integration and substitution of the limits, we have that

$$T(y) - T(y = 0) = -(M / L)gy \ . \qquad (8.5.20)$$

Use the fact that tension at the top of the rope is $T(y = 0) = Mg$ and find that

$$T(y) = Mg(1 - y / L)$$

in agreement with our earlier result, Eq. (8.5.13).

8.6 Drag Forces in Fluids

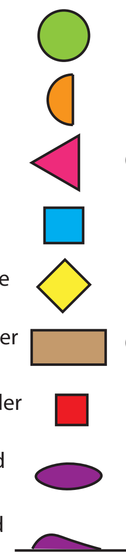
When a solid object moves through a fluid it will experience a resistive force, called the ***drag force***, opposing its motion. The fluid may be a liquid or a gas. This force is a very complicated force that depends on both the properties of the object and the properties of the fluid. The force depends on the speed, size, and shape of the object. It also depends on the density, viscosity and compressibility of the fluid.

For objects moving in air, the air drag is still quite complicated but for rapidly moving objects the resistive force is roughly proportional to the square of the speed v, the cross-sectional area A of the object in a plane perpendicular to the motion, the density ρ of the air, and independent of the viscosity of the air. Traditionally the magnitude of the air drag for rapidly moving objects is written as

$$F_{\text{drag}} = \frac{1}{2} C_D A \rho v^2 \ . \qquad (8.6.1)$$

The coefficient C_D is called the *drag coefficient*, a dimensionless number that is a property of the object. Table 8.1 lists the drag coefficient for some simple shapes, (each of these objects has a Reynolds number of order 10^4).

Table 8.1 Drag Coefficients



Shape	Drag coefficient
Sphere	0.47
Half-sphere	0.42
Cone	0.50
Cube	1.05
Angled cube	0.80
Long cylinder	0.82
Short cylinder	1.15
Streamlined body	0.04
Streamlined half-body	0.09

The above model for air drag does not extend to all fluids. An object dropped in oil, molasses, honey, or water will fall at different rates due to the different viscosities of the fluid. For very low speeds, the drag force depends linearly on the speed and is also proportional to the viscosity η of the fluid. For the special case of a sphere of radius R, the drag force law can be exactly deduced from the principles of fluid mechanics and is given by

$$\vec{\mathbf{F}}_{\text{drag}} = -6\pi\eta R\vec{\mathbf{v}} \quad \text{(sphere)} \ . \qquad (8.6.2)$$

This force law is known as *Stokes' Law*. The coefficient of viscosity η has SI units of $[\text{N}\cdot\text{m}^{-2}\cdot\text{s}] = [\text{Pa}\cdot\text{s}] = [\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1}]$; a cgs unit called the **poise** is often encountered . Some typical coefficients of viscosity are listed in Table 8.2.

Table 8.2: Coefficients of viscosity

fluid	Temperature, ^0C	Coefficient of viscosity η; $[\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1}]$
Acetone	25	3.06×10^{-4}
Air	15	1.81×10^{-5}
Benzene	25	6.04×10^{-4}
Blood	37	$(3-4)\times10^{-3}$
Castor oil	25	0.985
Corn Syrup	25	1.3806
Ethanol	25	1.074×10^{-3}
Glycerol	20	1.2
Methanol	25	5.44×10^{-4}
Motor oil (SAE 10W)	20	6.5×10^{-2}
Olive Oil	25	8.1×10^{-2}
Water	10	1.308×10^{-3}
Water	20	1.002×10^{-3}
Water	60	0.467×10^{-3}
Water	100	0.28×10^{-3}

This law can be applied to the motion of slow moving objects in a fluid, for example: very small water droplets falling in a gravitational field, grains of sand settling in water, or the sedimentation rate of molecules in a fluid. In the later case, If we model a molecule as a sphere of radius R, the mass of the molecule is proportional to R^3 and the drag force is proportion to R, therefore different sized molecules will have different rates of acceleration. This is the basis for the design of measuring devices that separate molecules of different molecular weights.

In many physical situations the force on an object will be modeled as depending on the object's velocity. We have already seen static and kinetic friction between surfaces modeled as being independent of the surfaces' relative velocity. Common experience (swimming, throwing a Frisbee) tells us that the frictional force between an object and a fluid can be a complicated function of velocity. Indeed, these complicated relations are an important part of such topics as aircraft design.

Example 8.5 Drag Force at Low Speeds

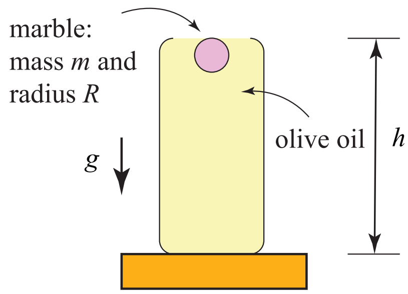


Figure 8.31 Example 8.5

A spherical marble of radius R and mass m is released from rest and falls under the influence of gravity through a jar of olive oil of viscosity η. The marble is released from rest just below the surface of the olive oil, a height h from the bottom of the jar. The gravitational acceleration is g (Figure 8.31). Neglect any force due to the buoyancy of the olive oil. (i) Determine the velocity of the marble as a function of time, (ii) what is the maximum possible velocity $\vec{\mathbf{v}}_{\infty} = \vec{\mathbf{v}}(t = \infty)$ (terminal velocity), that the marble can obtain, (iii) determine an expression for the viscosity of olive oil η in terms of g, m, R, and $v_{\infty} = |\vec{\mathbf{v}}_{\infty}|$, (iv) determine an expression for the position of the marble from just below the surface of the olive oil as a function of time.

Solution: Choose the positive y-direction downwards with the origin at the initial position of the marble as shown in Figure 8.32(a).

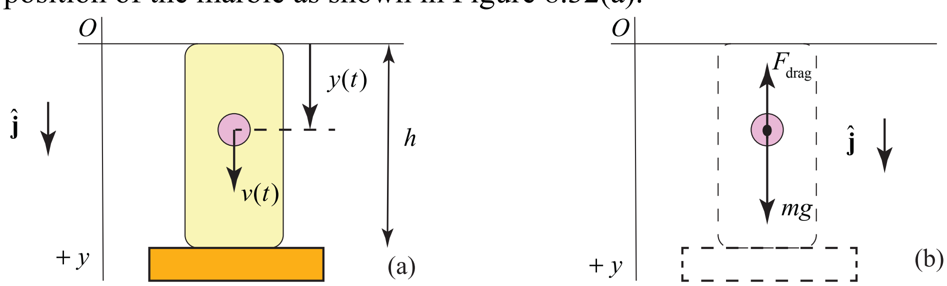


Figure 8.32 (a) Coordinate system for marble; (b) free body force diagram on marble

There are two forces acting on the marble: the gravitational force, and the drag force which is given by Eq. (8.6.2). The free body diagram is shown in the Figure 8.32(b). Newton's Second Law is then

$$mg - 6\pi\eta Rv = m\frac{dv}{dt}, \tag{8.6.3}$$

where v is the y-component of the velocity of the marble. Let $\gamma = 6\pi\eta R / m$; the SI units γ are $[\mathrm{s}^{-1}]$. Then Eq. (8.6.3) becomes

$$g - \gamma v = \frac{dv}{dt}, \qquad (8.6.4)$$

Suppose the object has an initial y-component of velocity $v(t=0)=0$. We shall solve Eq. (8.6.3) using the method of separation of variables. The differential equation may be rewritten as

$$\frac{dv}{(v - g/\gamma)} = -\gamma dt . \qquad (8.6.5)$$

The integral version of Eq. (8.6.5) is then

$$\int_{v'=0}^{v'=v(t)} \frac{dv'}{v' - g/\gamma} = -\gamma \int_{t'=0}^{t'=t} dt' . \qquad (8.6.6)$$

Integrating both sides of Eq. (8.6.6) yields

$$\ln\left(\frac{v(t) - g/\gamma}{-g/\gamma}\right) = -\gamma t . \qquad (8.6.7)$$

Recall that $e^{\ln x} = x$, therefore upon exponentiation of Eq. (8.6.7) yields

$$\frac{v(t) - g/\gamma}{-g/\gamma} = e^{-\gamma t} . \qquad (8.6.8)$$

Thus the y-component of the velocity as a function of time is given by

$$v(t) = \frac{g}{\gamma}(1 - e^{-\gamma t}) = \frac{mg}{6\pi\eta R}(1 - e^{-(6\pi\eta R/m)t}) . \qquad (8.6.9)$$

A plot of $v(t)$ vs. t is shown in Figure 8.31 with parameters $R = 5.00\times10^{-3}\,\mathrm{m}$, $\eta = 8.10\times10^{-2}\ \mathrm{kg\cdot m^{-1}\cdot s^{-1}}$, $m = 4.08\times10^{-3}\ \mathrm{kg}$, and $g/\gamma = 1.87\ \mathrm{m\cdot s^{-1}}$.

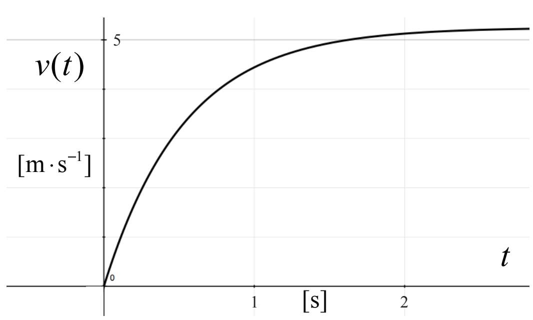


Figure 8.33 Plot of y -component of the velocity $v(t)$ vs. t for marble falling through oil with $g/\gamma = 1.87\ \text{m}\cdot\text{s}^{-1}$.

For large values of t, the term $e^{-(6\pi\eta R/m)t}$ approaches zero, and the marble reaches a terminal velocity

$$v_\infty = v(t=\infty) = \frac{mg}{6\pi\eta R}. \tag{8.6.10}$$

The coefficient of viscosity can then be determined from the terminal velocity by the condition that

$$\eta = \frac{mg}{6\pi R v_{ter}}. \tag{8.6.11}$$

Let ρ_m denote the density of the marble. The mass of the spherical marble is $m = (4/3)\rho_m R^3$. The terminal velocity is then

$$v_\infty = \frac{2\rho_m R^2 g}{9\eta}. \tag{8.6.12}$$

The terminal velocity depends on the square of the radius of the marble, indicating that larger marbles will reach faster terminal speeds.

The position of the marble as a function of time is given by the integral expression

$$y(t) - y(t=0) = \int_{t'=0}^{t'=t} v(t')\,dt', \tag{8.6.13}$$

which after substitution of Eq. (8.6.9) and integration using the initial condition that $y(t=0)=0$, becomes

$$y(t) = \frac{g}{\gamma}t + \frac{g}{\gamma^2}\left(e^{-\gamma t} - 1\right). \tag{8.6.14}$$

Example 8.6 Drag Forces at High Speeds

An object of mass m at time $t = 0$ is moving rapidly with velocity $\vec{\mathbf{v}}_0$ through a fluid of density ρ. Let A denote the cross-sectional area of the object in a plane perpendicular to the motion. The object experiences a retarding drag force whose magnitude is given by Eq. (8.6.1). Determine an expression for the velocity of the object as a function of time.

Solution: Choose a coordinate system such that the object is moving in the positive x-direction, $\vec{\mathbf{v}} = v\hat{\mathbf{i}}$. Set $\beta = (1/2)C_D A\rho$. Newton's Second Law can then be written as

$$-\beta v^2 = \frac{dv}{dt}. \quad (8.6.15)$$

An integral version of Eq. (8.6.15) is then

$$\int_{v'=v_0}^{v'=v(t)} \frac{dv'}{v'^2} = -\beta \int_{t'=0}^{t'=t} dt'. \quad (8.6.16)$$

Integration yields

$$-\left(\frac{1}{v(t)} - \frac{1}{v_0}\right) = -\beta t. \quad (8.6.17)$$

After some algebraic rearrangement the x-component of the velocity as a function of time is given by

$$v(t) = \frac{v_0}{1 + v_0 \beta t} = \frac{1}{1 + t/\tau} v_0, \quad (8.6.18)$$

where $\tau = 1/v_0\beta$. A plot of $v(t)$ vs. t is shown in Figure 8.34 with initial conditions $v_0 = 20\ \text{m}\cdot\text{s}^{-1}$ and $\beta = 0.5\ \text{s}^{-1}$.

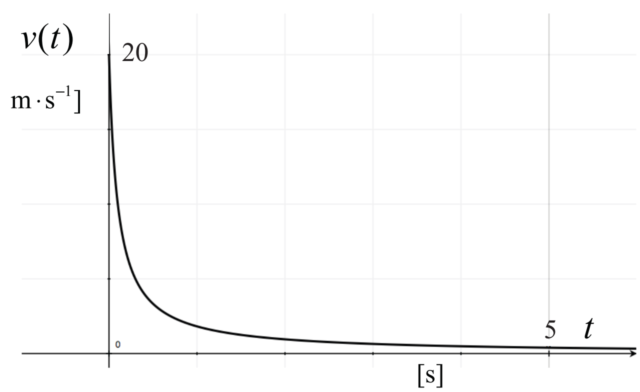


Figure 8.34 Plot of $v(t)$ vs. t for damping force $F_{\text{drag}} = \frac{1}{2} C_D A \rho v^2$

8.7 Worked Examples

Example 8.7 Staircase

An object of mass m at time $t = 0$ has speed v_0. It slides a distance s along a horizontal floor and then off the top of a staircase (Figure 8.35). The coefficient of kinetic friction between the object and the floor is μ_k. The object strikes at the far end of the third stair.

Each stair has a rise of h and a run of d. Neglect air resistance and use g for the gravitational constant. (a) What is the distance s that the object slides along the floor?

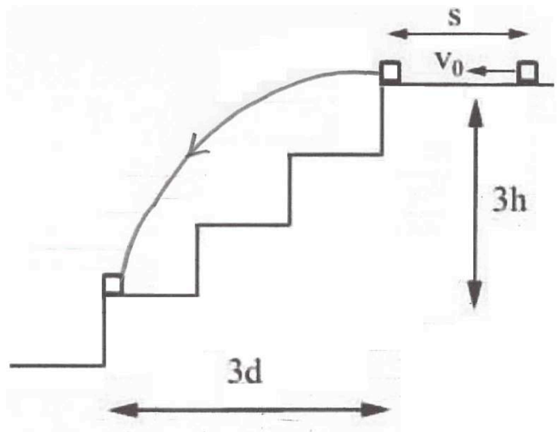


Figure 8.35 Object falling down a staircase

Solution: There are two distinct stages to the object's motion, the initial horizontal motion and then free fall. The given final position of the object, at the far end of the third stair, will determine the horizontal component of the velocity at the instant the object left the top of the stairs. This in turn can be used to determine the time the object decelerated along the floor, and hence the distance traveled on the floor. The given quantities are m, v_0, μ_k, g, h and d.

For the horizontal motion, choose coordinates with the origin at the initial position of the block. Choose the positive $\hat{\mathbf{i}}$-direction to be horizontal, directed to the left in Figure 8.35, and the positive $\hat{\mathbf{j}}$-direction to be vertical (up). The forces on the object are gravity $m\vec{\mathbf{g}} = -mg\,\hat{\mathbf{j}}$, the normal force $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}}$ and the kinetic frictional force $\vec{\mathbf{f}}_k = -f_k\,\hat{\mathbf{i}}$. The components of the vectors in Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, are

$$\begin{aligned} -f_k &= m\,a_x \\ N - m\,g &= m\,a_y. \end{aligned} \tag{8.6.19}$$

The object does not move in the y-direction; $a_y = 0$ and thus from the second expression in (8.6.19), $N = m\,g$. The magnitude of the frictional force is then $f_k = \mu_k N = \mu_k mg$, and the first expression in (8.6.19) gives the x-component of acceleration as $a_x = -\mu_k g$. Becasue the acceleration is constant the x-component of the velocity is given by

$$v_x(t) = v_0 + a_x\,t\,, \tag{8.6.20}$$

where v_0 is the x-component of the velocity of the object when it just started sliding. The displacement is given by

$$x(t) - x_0 = v_0\, t + \frac{1}{2} a_x\, t^2. \tag{8.6.21}$$

Denote the time the block just leaves the landing by t_1, where $x(t_1) = s$, and the speed just when it reaches the landing $v_x(t_1) = v_{x,1}$. The initial speed is v_0 and $x_0 = 0$. Using the initial and final conditions, and the value of the acceleration, Eq. (8.6.21) becomes

$$s = v_0\, t_1 - \frac{1}{2} \mu_k g\, t_1^{\,2}. \tag{8.6.22}$$

Solve Eq. (8.6.20) for the time the block reaches the edge of the landing,

$$t_1 = \frac{v_{x,1} - v_0}{-\mu_\text{k} g} = \frac{v_0 - v_{x,1}}{\mu_\text{k} g}. \tag{8.6.23}$$

Substituting Eq. (8.6.23) into Eq. (8.6.22) yields

$$s = v_0 \left(\frac{v_0 - v_{x,1}}{\mu_\text{k} g} \right) - \frac{1}{2} \mu_k g \left(\frac{v_0 - v_{x,1}}{\mu_\text{k} g} \right)^2 \tag{8.6.24}$$

and after some algebra, we can rewrite Eq. (8.6.24) as

$$s = \frac{v_0^{\,2} - v_{x,1}^{\,2}}{2\mu_\text{k} g}. \tag{8.6.25}$$

From the top of the stair to the far end of the third stair, the object is in free fall. Choose the positive $\hat{\mathbf{i}}$-direction to be horizontal, directed to the left in Figure 8.35, and the positive $\hat{\mathbf{j}}$-direction to be vertical (up) and now choose the origin at the top of the stairs, where the object first goes into free fall. The components of acceleration are $a_x = 0$, $a_y = -g$, the initial x-component of velocity is $v_{x,1}$, the initial y-component of velocity is $v_{y,0} = 0$, the initial x-position is $x_0 = 0$ and the initial y-position is $y_0 = 0$. Reset $t = 0$ when the object just leaves the landing. Let t_2 denote the instant the object hits the stair, where $y(t_2) = -3h$ and $x(t_2) = 3d$. The equations describing the object's position and speed at time $t = t_2$ are

$$x(t_2) = 3d = v_{x,1}\, t_2 \tag{8.6.26}$$

$$y(t_2) = -3h = -\frac{1}{2}gt_2^{\ 2}. \tag{8.6.27}$$

Solve Eq. (8.6.26) for t_2 to yield

$$t_2 = \frac{3d}{v_{x,1}}. \tag{8.6.28}$$

Substitute Eq. (8.6.28) into Eq. (8.6.27) and eliminate the variable t_2,

$$3h = \frac{1}{2}g\frac{9d^2}{v_{x,1}^{\ 2}}. \tag{8.6.29}$$

Eq. (8.6.29) can now be solved for the square of the horizontal component of the velocity,

$$v_{x,1}^{\ 2} = \frac{3gd^2}{2h}. \tag{8.6.30}$$

Now substitute Eq. (8.6.30) into Eq. (8.6.25) to determine the distance the object traveled on the landing,

$$s = \frac{v_0^2 - (3gd^2 / 2h)}{2\mu_k g}. \tag{8.6.31}$$

Example 8.8 Cart Moving on a Track

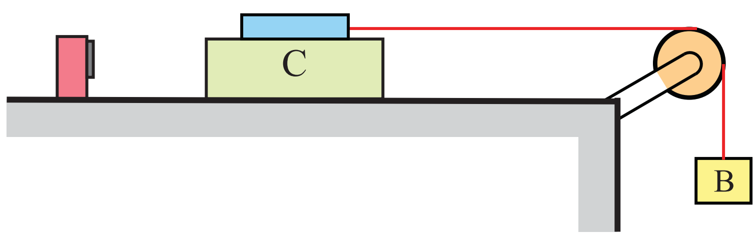


Figure 8.36 A falling block will accelerate a cart on a track via the pulling force of the string. The force sensor measures the tension in the string.

Consider a cart that is free to slide along a horizontal track (Figure 8.36). A force is applied to the cart via a string that is attached to a force sensor mounted on the cart, wrapped around a pulley and attached to a block on the other end. When the block is released the cart will begin to accelerate. The force sensor and cart together have a mass m_C, and the suspended block has mass m_B. Neglect the small mass of the string and pulley, and assume the string is inextensible. The coefficient of kinetic friction between the cart and the track is μ_k. Determine (i) the acceleration of the cart, and (ii) the tension in the string.

Solution: In general, we would like to draw free-body diagrams on all the individual objects (cart, sensor, pulley, rope, and block) but we can also choose a system consisting of two (or more) objects knowing that the forces of interaction between any two objects will cancel in pairs by Newton's Third Law. In this example, we shall choose the sensor/cart as one free-body, and the block as the other free-body. The free-body force diagram for the sensor/cart is shown in Figure 8.37.

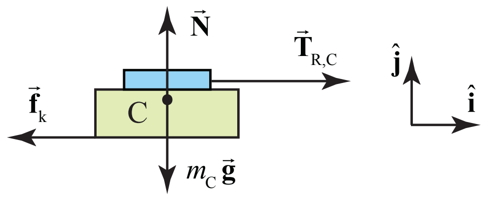


Figure 8.37 Force diagram on sensor/cart with a vector decomposition of the contact force into horizontal and vertical components

There are three forces acting on the sensor/cart: the gravitational force $m_C\vec{\mathbf{g}}$, the pulling force $\vec{\mathbf{T}}_{R,C}$ of the rope on the force sensor, and the contact force between the track and the cart. In Figure 8.34, we decompose the contact force into its two components, the kinetic frictional force $\vec{\mathbf{f}}_k = -f_k\hat{\mathbf{i}}$ and the normal force, $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}}$.

The cart is only accelerating in the horizontal direction with $\vec{\mathbf{a}}_C = a_{C,x}\,\hat{\mathbf{i}}$, so the component of the force in the vertical direction must be zero, $a_{C,y} = 0$. We can now apply Newton's Second Law in the horizontal and vertical directions and find that

$$\hat{\mathbf{i}}:\ T_{R,C} - f_k = m_C a_{C,x} \tag{8.6.32}$$

$$\hat{\mathbf{j}}:\ N - m_C g = 0\,. \tag{8.6.33}$$

From Eq. (8.6.33), we conclude that the normal component is

$$N = m_C g\,. \tag{8.6.34}$$

We use Equation (8.6.34) for the normal force to find that the magnitude of the kinetic frictional force is

$$f_k = \mu_k N = \mu_k m_C g\,. \tag{8.6.35}$$

Then Equation (8.6.32) becomes

$$T_{R,C} - \mu_k m_C g = m_C a_{C,x}\,. \tag{8.6.36}$$

The force diagram for the block is shown in Figure 8.38. The two forces acting on the block are the pulling force $\vec{\mathbf{T}}_{\text{R,B}}$ of the string and the gravitational force $m_{\text{B}}\vec{\mathbf{g}}$. We now apply Newton's Second Law to the block and find that

$$\hat{\mathbf{j}}_{\text{B}} : m_{\text{B}}g - T_{\text{R,B}} = m_{\text{B}}a_{B,y} . \tag{8.6.37}$$

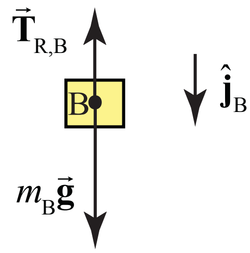


Figure 8.38 Forces acting on the block

In Equation (8.6.37), the symbol $a_{B,y}$ represents the component of the acceleration with sign determined by our choice of downward direction for the unit vector $\hat{\mathbf{j}}_{\text{B}}$. Note that we made a different choice of direction for the unit vector in the vertical direction in the free-body diagram for the block shown in Figure 8.37. Each free-body diagram has an independent set of unit vectors that define a sign convention for vector decomposition of the forces acting on the free-body and the acceleration of the free-body. In our example, with the unit vector pointing downwards in Figure 8.38, if we solve for the component of the acceleration and it is positive, then we know that the direction of the acceleration is downwards.

There is a second subtle way that signs are introduced with respect to the forces acting on a free-body. In our example, the force between the string and the block acting on the block points upwards, so in the vector decomposition of the forces acting on the block that appears on the left-hand side of Equation (8.6.37), this force has a minus sign and the quantity $\vec{\mathbf{T}}_{\text{R,B}} = -T_{\text{R,B}}\hat{\mathbf{j}}_{\text{B}}$ where $T_{\text{R,B}}$ is assumed positive.

Our assumption that the mass of the rope and the mass of the pulley are negligible enables us to assert that the tension in the rope is uniform and equal in magnitude to the forces at each end of the rope,

$$T_{\text{R,B}} = T_{\text{R,C}} \equiv T . \tag{8.6.38}$$

We also assumed that the string is inextensible (does not stretch). This implies that the rope, block, and sensor/cart all have the same magnitude of acceleration,

$$a_{\text{C},x} = a_{\text{B},y} \equiv a . \tag{8.6.39}$$

Using Equations (8.6.38) and (8.6.39), we can now rewrite the equation of motion for the sensor/cart, Equation (8.6.36), as

$$T - \mu_k m_C g = m_C a, \tag{8.6.40}$$

and the equation of motion (8.6.37) for the block as

$$m_B g - T = m_B a. \tag{8.6.41}$$

We have only two unknowns T and a, so we can now solve the two equations (8.6.40) and (8.6.41) simultaneously for the acceleration of the sensor/cart and the tension in the rope. We first solve Equation (8.6.40) for the tension

$$T = \mu_k m_C g + m_C a \tag{8.6.42}$$

and then substitute Equation (8.6.42) into Equation (8.6.41) and find that

$$m_B g - (\mu_k m_C g + m_C a) = m_B a. \tag{8.6.43}$$

We can now solve Equation (8.6.43) for the acceleration,

$$a = \frac{m_B g - \mu_k m_C g}{m_C + m_B}. \tag{8.6.44}$$

Substitution of Equation (8.6.44) into Equation (8.6.42) gives the tension in the string,

$$\begin{aligned} T &= \mu_k m_C g + m_C a \\ &= \mu_k m_C g + m_C \frac{m_B g - \mu_k m_C g}{m_C + m_B} \\ &= (\mu_k + 1)\frac{m_C m_B}{m_C + m_B} g. \end{aligned} \tag{8.6.45}$$

In this example, we applied Newton's Second Law to two objects, one a composite object consisting of the sensor and the cart, and the other the block. We analyzed the forces acting on each object and also any constraints imposed on the acceleration of each object. We used the force laws for kinetic friction and gravitation on each free-body system. The three equations of motion enable us to determine the forces that depend on the parameters in the example: the tension in the rope, the acceleration of the objects, and normal force between the cart and the table.

Example 8.9 Pulleys and Ropes Constraint Conditions

Consider the arrangement of pulleys and blocks shown in Figure 8.39. The pulleys are assumed massless and frictionless and the connecting strings are massless and inextensible. Denote the respective masses of the blocks as m_1, m_2 and m_3. The upper pulley in the figure is free to rotate but its center of mass does not move. Both pulleys have the same radius R. (a) How are the accelerations of the objects related? (b) Draw force diagrams on each moving object. (c) Solve for the accelerations of the objects and the tensions in the ropes.

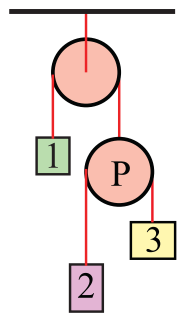


Figure 8.39 Constrained pulley system

Solution: (a) Choose an origin at the center of the upper pulley. Introduce coordinate functions for the three moving blocks, y_1, y_2 and y_3. Introduce a coordinate function y_P for the moving pulley (the pulley on the lower right in Figure 8.40). Choose downward for positive direction; the coordinate system is shown in the figure below then.

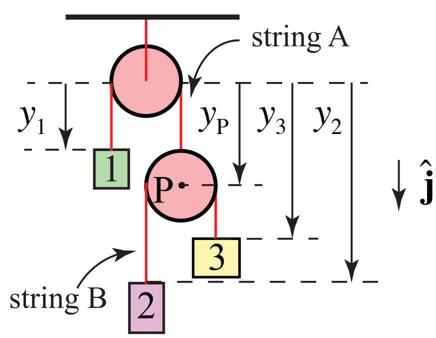


Figure 8.40 Coordinated system for pulley system

The length of string A is given by

$$l_A = y_1 + y_P + \pi R \tag{8.6.46}$$

where πR is the arc length of the rope that is in contact with the pulley. Because the rope is assumed to be inextensible, this length l_A is constant, and so the second derivative with respect to time is zero,

$$0=\frac{d^2l_A}{dt^2}=\frac{d^2y_1}{dt^2}+\frac{d^2y_P}{dt^2}=a_{y,1}+a_{y,P}\,. \tag{8.6.47}$$

Thus block 1 and the moving pulley's components of acceleration are equal in magnitude but opposite in sign,

$$a_{y,P}=-a_{y,1}\,. \tag{8.6.48}$$

The length of string B is given by

$$l_B=(y_3-y_P)+(y_2-y_P)+\pi R=y_3+y_2-2y_P+\pi R \tag{8.6.49}$$

where πR is the arc length of the rope that is in contact with the pulley. The length l_B of the rope is constant and so the second derivative with respect to time is zero,

$$0=\frac{d^2l_B}{dt^2}=\frac{d^2y_2}{dt^2}+\frac{d^2y_3}{dt^2}-2\frac{d^2y_P}{dt^2}=a_{y,2}+a_{y,3}-2a_{y,P}\,. \tag{8.6.50}$$

We can substitute Equation (8.6.48) for the pulley acceleration into Equation (8.6.50) yielding the *constraint relation* between the components of the acceleration of the three blocks,

$$0=a_{y,2}+a_{y,3}+2a_{y,1}\,. \tag{8.6.51}$$

b) Free-body Force diagrams: the forces acting on block 1 are: the gravitational force $m_1\vec{\mathbf{g}}$ and the pulling force $\vec{\mathbf{T}}_{A,1}$ of string A acting on the block 1. Denote the magnitude of this force by T_A. Because the string is assumed to be massless and the pulley is assumed to be massless and frictionless, the tension T_A in the string is uniform and equal in magnitude to the pulling force of the string on the block. The free-body diagram on block 1 is shown in Figure 8.41(a).

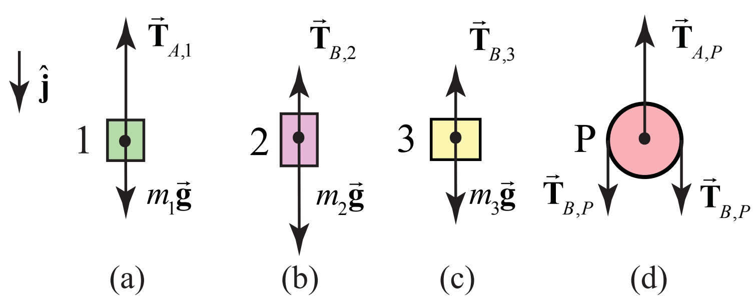


Figure 8.41 Free-body force diagram on (a) block 1; (b) block 2; (c) block 3; (d) pulley

Newton's Second Law applied to block 1 is then

$$\hat{\mathbf{j}}: \; m_1 g - T_A = m_1 \, a_{y,1} \,. \tag{8.6.52}$$

The forces on block 2 are the gravitational force $m_2\vec{\mathbf{g}}$ and string B holding the block, $\vec{\mathbf{T}}_{B,2}$, with magnitude T_B. The free-body diagram for the forces acting on block 2 is shown in Figure 8.41(b). Newton's second Law applied to block 2 is

$$\hat{\mathbf{j}}: \; m_2 g - T_B = m_2 \, a_{y,2} \,. \tag{8.6.53}$$

The forces on block 3 are the gravitational force $m_3\vec{\mathbf{g}}$ and string holding the block, $\vec{\mathbf{T}}_{B,3}$, with magnitude equal to T_B because pulley P has been assumed to be both frictionless and massless. The free-body diagram for the forces acting on block 3 is shown in Figure 8.41(c). Newton's second Law applied to block 3 is

$$\hat{\mathbf{j}}: \; m_3 g - T_B = m_3 \, a_{y,3} \,. \tag{8.6.54}$$

The forces on the moving pulley P are the gravitational force $m_P \, \vec{\mathbf{g}} = \vec{\mathbf{0}}$ (the pulley is assumed massless); string B pulls down on the pulley on each side with a force, $\vec{\mathbf{T}}_{B,P}$, which has magnitude T_B. String A holds the pulley up with a force $\vec{\mathbf{T}}_{A,P}$ with the magnitude T_A equal to the tension in string A. The free-body diagram for the forces acting on the moving pulley is shown in Figure 8.41(d). Newton's second Law applied to the pulley is

$$\hat{\mathbf{j}}: \; 2T_B - T_A = m_P \, a_{y,P} = 0 \,. \tag{8.6.55}$$

Because the pulley is assumed to be massless, we can use this last equation to determine the condition that the tension in the two strings must satisfy,

$$2T_B = T_A \tag{8.6.56}$$

We are now in position to determine the accelerations of the blocks and the tension in the two strings. We record the relevant equations as a summary.

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1} \tag{8.6.57}$$

$$m_1 g - T_A = m_1 \, a_{y,1} \tag{8.6.58}$$

$$m_2 g - T_B = m_2 \, a_{y,2} \tag{8.6.59}$$

$$m_3 g - T_B = m_3 \, a_{y,3} \tag{8.6.60}$$

$$2T_B = T_A \,. \tag{8.6.61}$$

There are five equations with five unknowns, so we can solve this system. We shall first use Equation (8.6.61) to eliminate the tension T_A in Equation (8.6.58), yielding

$$m_1 g - 2T_B = m_1\, a_{y,1}\,. \tag{8.6.62}$$

We now solve Equations (8.6.59), (8.6.60) and (8.6.62) for the accelerations,

$$a_{y,2} = g - \frac{T_B}{m_2} \tag{8.6.63}$$

$$a_{y,3} = g - \frac{T_B}{m_3} \tag{8.6.64}$$

$$a_{y,1} = g - \frac{2T_B}{m_1}\,. \tag{8.6.65}$$

We now substitute these results for the accelerations into the constraint equation, Equation (8.6.57),

$$0 = g - \frac{T_B}{m_2} + g - \frac{T_B}{m_3} + 2g - \frac{4T_B}{m_1} = 4g - T_B\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right). \tag{8.6.66}$$

We can now solve this last equation for the tension in string B,

$$T_B = \frac{4g}{\left(\dfrac{1}{m_2} + \dfrac{1}{m_3} + \dfrac{4}{m_1}\right)} = \frac{4g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4\, m_2\, m_3}\,. \tag{8.6.67}$$

From Equation (8.6.61), the tension in string A is

$$T_A = 2T_B = \frac{8g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}\,. \tag{8.6.68}$$

We find the acceleration of block 1 from Equation (8.6.65), using Equation (8.6.67) for the tension in string B,

$$a_{y,1} = g - \frac{2T_B}{m_1} = g - \frac{8g\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\,\frac{m_1\, m_3 + m_1\, m_2 - 4\, m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4\, m_2\, m_3}\,. \tag{8.6.69}$$

We find the acceleration of block 2 from Equation (8.6.63), using Equation (8.6.67) for the tension in string B,

$$a_{y,2} = g - \frac{T_B}{m_2} = g - \frac{4g\, m_1\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3} = g\frac{-3m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}. \quad (8.6.70)$$

Similarly, we find the acceleration of block 3 from Equation (8.6.64), using Equation (8.6.67) for the tension in string *B*,

$$a_{y,3} = g - \frac{T_B}{m_3} = g - \frac{4g\, m_1\, m_2}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3} = g\frac{m_1\, m_3 - 3m_1 m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}. \quad (8.6.71)$$

As a check on our algebra we note that

$$\begin{aligned} &2a_{1,y} + a_{2,y} + a_{3,y} = \\ &2g\frac{m_1\, m_3 + m_1\, m_2 - 4m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3} + g\frac{-3m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4m_2\, m_3} + g\frac{m_1\, m_3 - 3m_1\, m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4m_2 m_3} \\ &= 0. \end{aligned}$$

Example 8.10 Accelerating Wedge

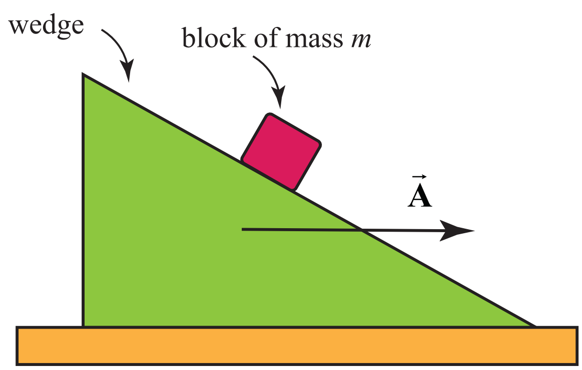


Figure 8.42 Block on accelerating wedge

A 45^o wedge is pushed along a table with constant acceleration $\vec{\mathbf{A}}$ according to an observer at rest with respect to the table. A block of mass m slides without friction down the wedge (Figure 8.42). Find its acceleration with respect to an observer at rest with respect to the table. Write down a plan for finding the magnitude of the acceleration of the block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any free-body force diagrams or sketches that you plan to use.

Solution: Choose a coordinate system for the block and wedge as shown in Figure 8.43. Then $\vec{\mathbf{A}} = A_{x,w}\,\hat{\mathbf{i}}$ where $A_{x,w}$ is the x-component of the acceleration of the wedge.

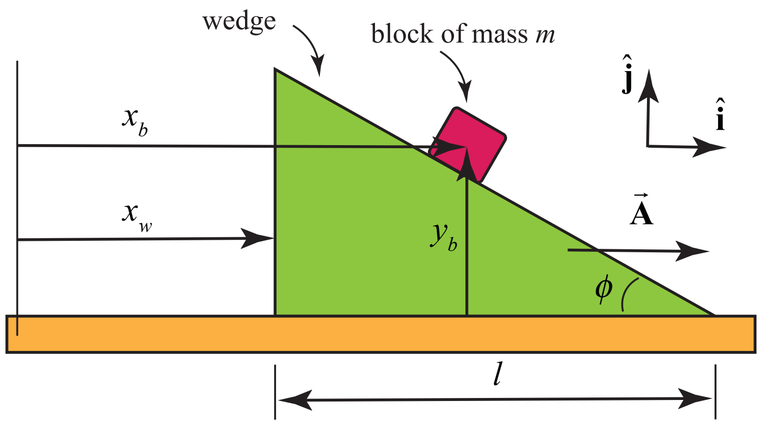


Figure 8.43 Coordinate system for block on accelerating wedge

We shall apply Newton's Second Law to the block sliding down the wedge. Because the wedge is accelerating, there is a constraint relation between the x - and y - components of the acceleration of the block. In order to find that constraint we choose a coordinate system for the wedge and block sliding down the wedge shown in the figure below. We shall find the constraint relationship between the components of the accelerations of the block and wedge by a geometric argument. From the figure above, we see that

$$\tan\phi = \frac{y_b}{l-(x_b - x_w)}. \tag{8.6.72}$$

Therefore

$$y_b = (l-(x_b - x_w))\tan\phi. \tag{8.6.73}$$

If we differentiate Eq. (8.6.73) twice with respect to time noting that

$$\frac{d^2 l}{dt^2} = 0 \tag{8.6.74}$$

we have that

$$\frac{d^2 y_b}{dt^2} = -\left(\frac{d^2 x_b}{dt^2} - \frac{d^2 x_w}{dt^2}\right)\tan\phi. \tag{8.6.75}$$

Therefore

$$a_{b,y} = -(a_{b,x} - A_{x,w})\tan\phi \tag{8.6.76}$$

where

$$A_{x,w} = \frac{d^2 x_w}{dt^2}. \tag{8.6.77}$$

We now draw a free-body force diagram for the block (Figure 8.44). Newton's Second Law in the $\hat{\mathbf{i}}$ - direction becomes

$$N\sin\phi = m a_{b,x}. \tag{8.6.78}$$

and the $\hat{\mathbf{j}}$-direction becomes

$$N\cos\phi - mg = ma_{b,y} \tag{8.6.79}$$

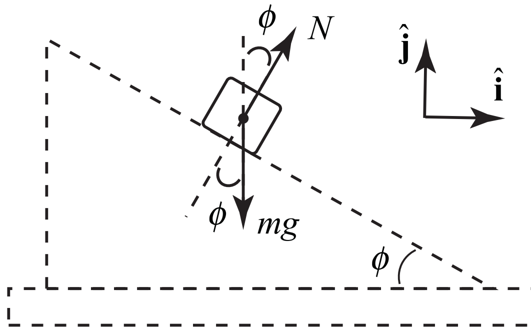


Figure 8.44 Free-body force diagram on block

We can solve for the normal force from Eq. (8.6.78):

$$N = \frac{ma_{b,x}}{\sin\phi} \tag{8.6.80}$$

We now substitute Eq. (8.6.76) and Eq. (8.6.80) into Eq. (8.6.79) yielding

$$\frac{ma_{b,x}}{\sin\phi}\cos\phi - mg = m(-(a_{b,x} - A_{w,x})\tan\phi)\,. \tag{8.6.81}$$

We now clean this up yielding

$$ma_{b,x}(\text{cotan}\,\phi + \tan\phi) = m(g + A_{w,x}\tan\phi) \tag{8.6.82}$$

Thus the x-component of the acceleration is then

$$a_{b,x} = \frac{g + A_{w,x}\tan\phi}{\text{cotan}\,\phi + \tan\phi} \tag{8.6.83}$$

From Eq. (8.6.76), the y-component of the acceleration is then

$$a_{b,y} = -(a_{b,x} - A_{w,x})\tan\phi = -\left(\frac{g + A_{w,x}\tan\phi}{\text{cotan}\,\phi + \tan\phi} - A_{w,x}\right)\tan\phi\,. \tag{8.6.84}$$

This simplifies to

$$a_{b,y} = \frac{A_{w,x} - g\tan\phi}{\text{cotan}\,\phi + \tan\phi} \tag{8.6.85}$$

When $\phi = 45^\circ$, cotan $45^\circ = \tan 45^\circ = 1$, and so Eq. (8.6.83) becomes

$$a_{b,x} = \frac{g + A_{w,x}}{2} \tag{8.6.86}$$

and Eq. (8.6.85) becomes

$$a_{b,y} = \frac{A - g}{2}. \tag{8.6.87}$$

The magnitude of the acceleration is then

$$a = \sqrt{a_{b,x}^{\;2} + a_{b,y}^{\;2}} = \sqrt{\left(\frac{g + A_{w,x}}{2}\right)^2 + \left(\frac{A_{w,x} - g}{2}\right)^2} \tag{8.6.88}$$

$$a = \sqrt{\left(\frac{g^2 + A_{w,x}^2}{2}\right)}.$$

Example 8.11: Capstan

A device called a capstan is used aboard ships in order to control a rope that is under great tension. The rope is wrapped around a fixed drum of radius R, usually for several turns (Figure 8.45 shows about three fourths turn as seen from overhead). The load on the rope pulls it with a force T_A, and the sailor holds the other end of the rope with a much smaller force T_B. The coefficient of static friction between the rope and the drum is μ_s. The sailor is holding the rope so that it is just about to slip. Show that $T_B = T_A e^{-\mu_s \theta_{BA}}$, where θ_{BA} is the angle subtended by the rope on the drum.

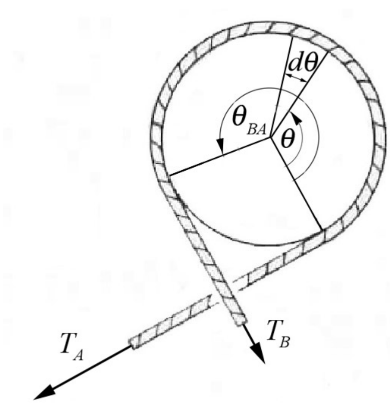


Figure 8.45 Capstan

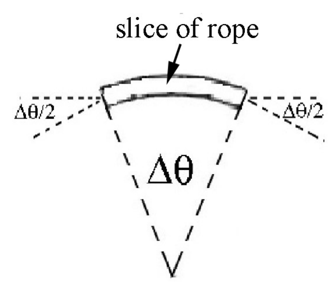


Figure 8.46 Small slice of rope

Solution: We begin by considering a small slice of rope of arc length $R\Delta\theta$, shown in the Figure 8.46. We choose unit vectors for the force diagram on this section of the rope and indicate them on Figure 8.47. The right edge of the slice is at angle θ and the left edge of the slice is at $\theta + \Delta\theta$. The angle edge end of the slice makes with the horizontal is $\Delta\theta / 2$. There are four forces acting on this section of the rope. The forces are the normal force between the capstan and the rope pointing outward, a static frictional force and the tensions at either end of the slice. The rope is held at the just slipping point, so if the load exerts a greater force the rope will slip to the right. Therefore the direction of the static frictional force between the capstan and the rope, acting on the rope, points to the left. The tension on the right side of the slice is denoted by $T(\theta) \equiv T$, while the tension on the left side of the slice is denoted by $T(\theta + \Delta\theta) \equiv T + \Delta T$. Does the tension in this slice from the right side to the left, increase, remain the same, or decrease? The tension decreases because the load on the left side is less than the load on the right side. Note that $\Delta T < 0$.

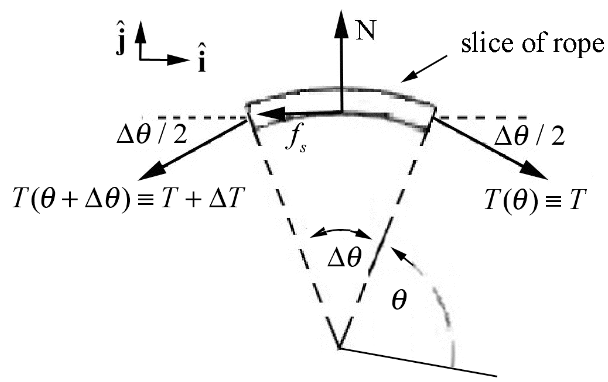


Figure 8.47 Free-body force diagram on small slice of rope

The vector decomposition of the forces is given by

$$\hat{\mathbf{i}} : T\cos(\Delta\theta / 2) - f_s - (T + \Delta T)\cos(\Delta\theta / 2) \tag{8.6.89}$$

$$\hat{\mathbf{j}} : -T\sin(\Delta\theta / 2) + N - (T + \Delta T)\sin(\Delta\theta / 2)\,. \tag{8.6.90}$$

For small angles $\Delta\theta$, $\cos(\Delta\theta / 2) \cong 1$ and $\sin(\Delta\theta / 2) \cong \Delta\theta / 2$. Using the small angle approximations, the vector decomposition of the forces in the x-direction (the $+\hat{\mathbf{i}}$-direction) becomes

$$\begin{aligned} T\cos(\Delta\theta / 2) - f_s - (T + \Delta T)\cos(\Delta\theta / 2) &\simeq T - f_s - (T + \Delta T) \\ &= -f_s - \Delta T \quad . \end{aligned} \tag{8.6.91}$$

By the static equilibrium condition the sum of the x-components of the forces is zero,

$$-f_s - \Delta T = 0\,. \tag{8.6.92}$$

The vector decomposition of the forces in the y-direction (the $+\hat{\mathbf{j}}$-direction) is approximately

$$\begin{aligned}-T\sin(\Delta\theta/2)+N-(T+\Delta T)\sin(\Delta\theta/2) &\simeq -T\Delta\theta/2+N-(T+\Delta T)\Delta\theta/2 \\ &= -T\Delta\theta+N-\Delta T\Delta\theta/2 .\end{aligned} \quad (8.6.93)$$

In the last equation above we can ignore the terms proportional to $\Delta T\,\Delta\theta$ because these are the product of two small quantities and hence are much smaller than the terms proportional to either ΔT or $\Delta\theta$. The vector decomposition in the y-direction becomes

$$-T\Delta\theta+N . \quad (8.6.94)$$

Static equilibrium implies that this sum of the y-components of the forces is zero,

$$-T\Delta\theta+N=0 . \quad (8.6.95)$$

We can solve this equation for the magnitude of the normal force

$$N=T\Delta\theta . \quad (8.6.96)$$

The just slipping condition is that the magnitude of the static friction attains its maximum value

$$f_{\text{s}}=(f_{\text{s}})_{\text{max}}=\mu_{\text{s}}N . \quad (8.6.97)$$

We can now combine the Equations (8.6.92) and (8.6.97) to yield

$$\Delta T=-\mu_s N . \quad (8.6.98)$$

Now substitute the magnitude of the normal force, Equation (8.6.96), into Equation (8.6.98), yielding

$$-\mu_s T\Delta\theta-\Delta T=0 . \quad (8.6.99)$$

Finally, solve this equation for the ratio of the change in tension to the change in angle,

$$\frac{\Delta T}{\Delta\theta}=-\mu_{\text{s}}T . \quad (8.6.100)$$

The derivative of tension with respect to the angle θ is defined to be the limit

$$\frac{dT}{d\theta}\equiv\lim_{\Delta\theta\to 0}\frac{\Delta T}{\Delta\theta}, \quad (8.6.101)$$

and Equation (8.6.100) becomes

$$\frac{dT}{d\theta} = -\mu_s T. \tag{8.6.102}$$

This is an example of a first order linear differential equation that shows that the rate of change of tension with respect to the angle θ is proportional to the negative of the tension at that angle θ. This equation can be solved by integration using the technique of separation of variables. We first rewrite Equation (8.6.102) as

$$\frac{dT}{T} = -\mu_s \, d\theta. \tag{8.6.103}$$

Integrate both sides, noting that when $\theta = 0$, the tension is equal to force of the load T_A, and when angle $\theta = \theta_{A,B}$ the tension is equal to the force T_B the sailor applies to the rope,

$$\int_{T=T_A}^{T=T_B} \frac{dT}{T} = -\int_{\theta=0}^{\theta=\theta_{BA}} \mu_s \, d\theta. \tag{8.6.104}$$

The result of the integration is

$$\ln\left(\frac{T_B}{T_A}\right) = -\mu_s \theta_{BA}. \tag{8.6.105}$$

Note that the exponential of the natural logarithm

$$\exp\left(\ln\left(\frac{T_B}{T_A}\right)\right) = \frac{T_B}{T_A}, \tag{8.6.106}$$

so exponentiating both sides of Equation (8.6.105) yields

$$\frac{T_B}{T_A} = e^{-\mu_s \theta_{BA}}; \tag{8.6.107}$$

the tension decreases exponentially,

$$T_B = T_A \, e^{-\mu_s \theta_{BA}}, \tag{8.6.108}$$

Because the tension decreases exponentially, the sailor need only apply a small force to prevent the rope from slipping.

Example 8.12 Free Fall with Air Drag

Consider an object of mass *m* that is in free fall but experiencing air resistance. The magnitude of the drag force is given by Eq. (8.6.1), where ρ is the density of air, A is the cross-sectional area of the object in a plane perpendicular to the motion, and C_D is the drag coefficient. Assume that the object is released from rest and very quickly attains speeds in which Eq. (8.6.1) applies. Determine (i) the terminal velocity, and (ii) the velocity of the object as a function of time.

Solution: Choose positive y-direction downwards with the origin at the initial position of the object as shown in Figure 8.48(a).

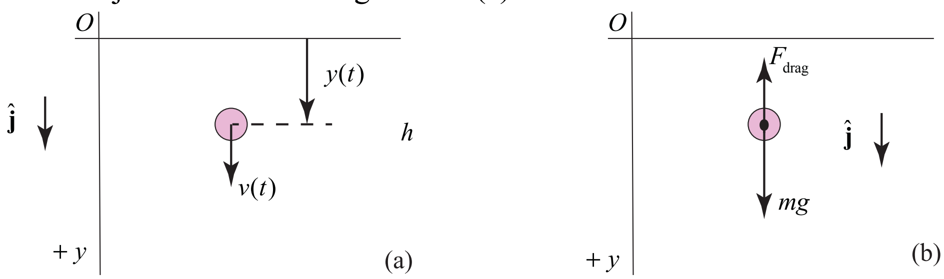


Figure 8.48 (a) Coordinate system for marble; (b) free body force diagram on marble

There are two forces acting on the object: the gravitational force, and the drag force which is given by Eq. (8.6.1). The free body diagram is shown in the Figure 8.48(b). Newton's Second Law is then

$$mg - (1/2)C_D A\rho v^2 = m\frac{dv}{dt}, \tag{8.6.109}$$

Set $\beta = (1/2)C_D A\rho$. Newton's Second Law can then be written as

$$mg - \beta v^2 = m\frac{dv}{dt}. \tag{8.6.110}$$

Initially when the object is just released with $v = 0$, the air drag is zero and the acceleration dv/dt is maximum. As the object increases its velocity, the air drag becomes larger and dv/dt decreases until the object reaches terminal velocity and $dv/dt = 0$. Set $dv/dt = 0$ in Eq. (8.6.15) and solve for the terminal velocity yielding.

$$v_\infty = \sqrt{\frac{mg}{\beta}} = \sqrt{\frac{2mg}{C_D A\rho}}. \tag{8.6.111}$$

Values for the magnitude of the terminal velocity is shown in Table 8.3 for a variety of objects with the same drag coefficient $C_D = 0.5$, and air density $\rho = 1.225\,\text{kg}\cdot\text{m}^{-3}$.

Table 8.3 Terminal Velocities for Different Sized Objects with $C_D = 0.5$, $\rho = 1.225\,\text{kg}\cdot\text{m}^{-3}$

Object	Mass m (kg)	Area A (m^2)	Terminal Velocity v_∞ $(\text{m}\cdot\text{s}^{-1})$
Rain drop	4×10^{-6}	3×10^{-6}	6.5
Hailstone	4×10^{-3}	3×10^{-4}	20
Osprey	1.5	2.5×10^{-1}	14
Human Being	7.5×10^{1}	6×10^{-1}	60

In order to integrate Eq. (8.6.15), we shall apply the technique of separation of variables and integration by partial fractions. First rewrite Eq. (8.6.15) as

$$\frac{-\beta}{m}dt = \frac{dv}{\left(v^2 - \dfrac{mg}{\beta}\right)} = \frac{dv}{\left(v^2 - v_\infty^2\right)} = \left(-\frac{1}{2v_\infty(v+v_\infty)} + \frac{1}{2v_\infty(v-v_\infty)}\right)dv\,. \quad (8.6.112)$$

An integral expression of Eq. (8.6.112) is then

$$-\int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'+v_\infty)} + \int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'-v_\infty)} = -\frac{\beta}{m}\int_{t'=0}^{t'=t} dt'\,. \quad (8.6.113)$$

Integration yields

$$\begin{gathered}-\int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'+v_\infty)} + \int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'-v_\infty)} = -\frac{\beta}{m}\int_{t'=0}^{t'=t} dt' \\ \frac{1}{2v_\infty}\left(-\ln\left(\frac{v(t)+v_\infty}{v_\infty}\right) + \ln\left(\frac{v_\infty - v(t)}{v_\infty}\right)\right) = -\frac{\beta}{m}t\end{gathered}\,. \quad (8.6.114)$$

After some algebraic manipulations, Eq. (8.6.114) can be rewritten as

$$\ln\left(\frac{v_\infty - v(t)}{v(t)+v_\infty}\right) = -\frac{2v_\infty\beta}{m}t \quad (8.6.115)$$

Exponentiate Eq. (8.6.115) yields

$$\left(\frac{v_\infty - v(t)}{v(t)+v_\infty}\right) = e^{-\frac{2v_\infty\beta}{m}t}\,. \quad (8.6.116)$$

After some algebraic rearrangement the y-component of the velocity as a function of time is given by

$$v(t) = v_\infty \left(\frac{1 - e^{-\frac{2v_\infty \beta}{m}t}}{1 + e^{-\frac{2v_\infty \beta}{m}t}} \right) = v_\infty \tan h\left(\frac{v_\infty \beta}{m} t \right). \tag{8.6.117}$$

where $\frac{v_\infty \beta}{m} = \frac{\beta}{m}\sqrt{\frac{mg}{\beta}} = \sqrt{\frac{\beta g}{m}} = \sqrt{\frac{(1/2)C_D A \rho g}{m}}$.