Chapter 5 Two Dimensional Kinematics

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Where was the chap I saw in the picture somewhere? Ah yes, in the dead sea floating on his back, reading a book with a parasol open. Couldn't sink if you tried: so thick with salt. Because the weight of the water, no, the weight of the body in the water is equal to the weight of the what? Or is it the volume equal to the weight? It's a law something like that. Vance in High school cracking his fingerjoints, teaching. The college curriculum. Cracking curriculum. What is weight really when you say weight? Thirtytwo feet per second per second. Law of falling bodies: per second per second. They all fall to the ground. The earth. It's the force of gravity of the earth is the weight. [1]

James Joyce

5.1 Introduction to the Vector Description of Motion in Two Dimensions

We have introduced the concepts of position, velocity and acceleration to describe motion in one dimension; however we live in a multidimensional universe. In order to explore and describe motion in more than one dimension, we shall study the motion of a projectile in two-dimension moving under the action of uniform gravitation.

We extend our definitions of position, velocity, and acceleration for an object that moves in two dimensions (in a plane) by treating each direction independently, which we can do with vector quantities by resolving each of these quantities into components. For example, our definition of velocity as the derivative of position holds for each component separately. In Cartesian coordinates, the position vector $\vec{\mathbf{r}}(t)$ with respect to some choice of origin for the object at time t is given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}. \tag{5.1.1}$$

The velocity vector $\vec{\mathbf{v}}(t)$ at time t is the derivative of the position vector,

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}, \tag{5.1.2}$$

where $v_x(t) \equiv dx(t)/dt$ and $v_y(t) \equiv dy(t)/dt$ denote the x- and y-components of the velocity respectively.

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined in a similar fashion as the derivative of the velocity vector,

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[1] James Joyce, *Ulysses*, The Corrected Text edited by Hans Walter Gabler with Wolfhard Steppe and Claus Melchior, Random House, New York.

$$\vec{\mathbf{a}}(t) = \frac{dv_x(t)}{dt}\,\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\,\hat{\mathbf{j}} \equiv a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}},\tag{5.1.3}$$

where $a_x(t) \equiv dv_x(t)/dt$ and $a_y(t) \equiv dv_y(t)/dt$ denote the x- and y-components of the acceleration.

5.2 Projectile Motion

Consider the motion of a body that is released at time $t = 0$ with an initial velocity $\vec{\mathbf{v}}_0$. Two paths are shown in Figure 5.1.
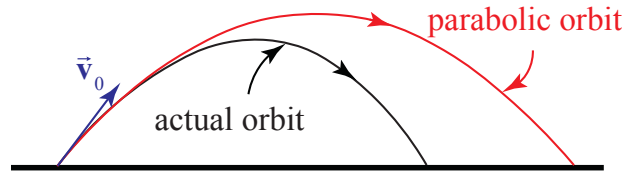

Figure 5.1 Actual orbit accounting for air resistance and parabolic orbit of a projectile

The dotted path represents a *parabolic* trajectory and the solid path represents the actual trajectory. The difference between the two paths is due to air resistance acting on the object, $\vec{\mathbf{F}}^{air} = -bv^2\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}$ is a unit vector in the direction of the velocity. (For the orbits shown in Figure 5.1, $b = 0.01\,\text{N}\cdot\text{s}^2\cdot\text{m}^{-2}$, $|\vec{\mathbf{v}}_0| = 30.0\,\text{m}\cdot\text{s}$, the initial launch angle with respect to the horizontal $\theta_0 = 21°$, and the actual horizontal distance traveled is 71.7% of the projectile orbit.). There are other factors that can influence the path of motion; a rotating body or a special shape can alter the flow of air around the body, which may induce a curved motion or lift like the flight of a baseball or golf ball. We shall begin our analysis by neglecting all interactions except the gravitational interaction.
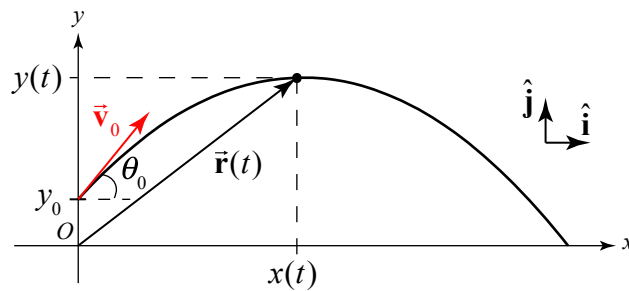

Figure 5.2 A coordinate sketch for parabolic motion.

Choose coordinates with the positive *y*-axis in the upward vertical direction and the positive *x*-axis in the horizontal direction in the direction that the object is moving horizontally. Choose the origin at the ground immediately below the point the object is

released. Figure 5.2 shows our coordinate system with the position of the object $\vec{r}(t)$ at time t, the initial velocity $\vec{v}_0$, and the initial angle θ_0 with respect to the horizontal, and the coordinate functions $x(t)$ and $y(t)$.
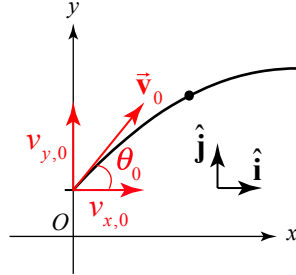
Initial Conditions:

Figure 5.3 A vector decomposition of the initial velocity

Decompose the initial velocity vector into its components:

$$\vec{v}_0 = v_{x,0}\,\hat{\mathbf{i}} + v_{y,0}\,\hat{\mathbf{j}}. \tag{5.1.4}$$

The vector decomposition for the initial velocity is shown in Figure 5.3. Often the description of the flight of a projectile includes the statement, "a body is projected with an initial speed v_0 at an angle θ_0 with respect to the horizontal." The components of the initial velocity can be expressed in terms of the initial speed and angle according to

$$v_{x,0} = v_0 \cos\theta_0, \tag{5.1.5}$$
$$v_{y,0} = v_0 \sin\theta_0. \tag{5.1.6}$$

Because the initial speed is the magnitude of the initial velocity, we have that

$$v_0 = (v_{x,0}^2 + v_{y,0}^2)^{1/2}. \tag{5.1.7}$$

The angle θ_0 is related to the components of the initial velocity by

$$\theta_0 = \tan^{-1}(v_{y,0} / v_{x,0}). \tag{5.1.8}$$

Equation (5.1.8) will give two values for the angle θ_0, so care must be taken to choose the correct physical value. The initial position vector generally is given by

$$\vec{r}_0 = x_0\,\hat{\mathbf{i}} + y_0\,\hat{\mathbf{j}}. \tag{5.1.9}$$

Note that the trajectory in Figure 5.3 has $x_0 = 0$, but this will not always be the case.

3

Force Diagram:

We begin by neglecting all forces other than the gravitational interaction between the object and the earth. This force acts downward with magnitude mg, where m is the mass of the object and $g = 9.8 \text{ m} \cdot \text{s}^{-2}$. Figure 5.4 shows the force diagram on the object.
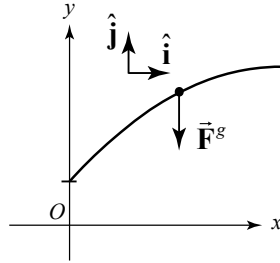

Figure 5.4 Free-body force diagram on the object with the action of gravity

The vector decomposition of the force is

$$\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{j}}. \tag{5.1.10}$$

Equations of Motions:

The force diagram reminds us that the force is acting in the y-direction. Newton's Second Law states that the sum of the force, $\vec{\mathbf{F}}^{\text{total}}$, acting on the object is equal to the product of the mass m and the acceleration vector $\vec{\mathbf{a}}$,

$$\vec{\mathbf{F}}^{\text{total}} = m\vec{\mathbf{a}}. \tag{5.1.11}$$

Because we are modeling the motion with only one force, we have that $\vec{\mathbf{F}}^{\text{total}} = \vec{\mathbf{F}}^g$. This is a vector equation; the components are equated separately:

$$-mg = ma_y, \tag{5.1.12}$$

$$0 = ma_x. \tag{5.1.13}$$

Therefore the y-component of the acceleration is

$$a_y = -g. \tag{5.1.14}$$

We see that the acceleration is a constant and is independent of the mass of the object. Notice that $a_y < 0$. This is because we chose our positive y-direction to point upwards. The sign of the y-component of acceleration is determined by how we choose our

coordinate system. Because there are no horizontal forces acting on the object, we conclude that the acceleration in the horizontal direction is also zero

$$a_x = 0 .\tag{5.1.15}$$

Therefore the x-component of the velocity remains unchanged throughout the flight of the object.

The acceleration in the vertical direction is constant for all bodies near the surface of the Earth, independent of the mass of the object, thus confirming Galileo's Law of Free Falling Bodies. Notice that the equation of motion (Equation (5.1.14)) generalizes the experimental observation that objects fall with constant acceleration. Our statement about the acceleration of objects near the surface of Earth depends on our model force law Eq. (5.1.10), and if subsequent observations show the acceleration is not constant then we either must include additional forces (for example, air resistance), or modify the force law (for objects that are no longer near the surface of Earth, or consider that Earth is a non-symmetric non-uniform body), or take into account the rotational motion of the Earth.

We can now integrate the equation of motions (Eqs. (5.1.14) and (5.1.15)) separately for the x- and y-directions to find expressions for the x- and y-components of velocity and position:

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x(t') \, dt' = 0 \Rightarrow v_x(t) = v_{x,0}$$

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v_x(t') \, dt' = \int_{t'=0}^{t'=t} v_{x,0} \, dt' = v_{x,0}t \Rightarrow x(t) = x_0 + v_{x,0}t$$

$$v_y(t) - v_{y,0} = \int_{t'=0}^{t'=t} a_y(t') \, dt' = -\int_{t'=0}^{t'=t} g \, dt' = -gt \Rightarrow v_y(t) = v_{y,0} - gt$$

$$y(t) - y_0 = \int_{t'=0}^{t'=t} v_y(t') \, dt' = \int_{t'=0}^{t'=t} (v_{y,0} - gt) \, dt' = v_{y,0}t - (1/2)gt^2 \Rightarrow y(t) = y_0 + v_{y,0}t - (1/2)gt^2 .$$

The complete set of vector equations for position and velocity for each independent direction of motion are given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = (x_0 + v_{x,0}t)\,\hat{\mathbf{i}} + (y_0 + v_{y,0}t + (1/2)a_y t^2)\,\hat{\mathbf{j}},\tag{5.1.16}$$

$$\vec{\mathbf{v}}(t) = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = v_{x,0}\,\hat{\mathbf{i}} + (v_{y,0} + a_y t)\,\hat{\mathbf{j}},\tag{5.1.17}$$

$$\vec{\mathbf{a}}(t) = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}} = a_y\,\hat{\mathbf{j}}.\tag{5.1.18}$$

Example 5.1 Time of Flight and Maximum Height of a Projectile

5

A person throws a stone at an initial angle $\theta_0 = 45°$ from the horizontal with an initial speed of $v_0 = 20 \text{ m} \cdot \text{s}^{-1}$. The point of release of the stone is at a height $d = 2 \text{ m}$ above the ground. You may neglect air resistance. a) How long does it take the stone to reach the highest point of its trajectory? b) What was the maximum vertical displacement of the stone? Ignore air resistance.

Solution: Choose the origin on the ground directly underneath the point where the stone is released. We choose the positive y-axis in the upward vertical direction and the positive x-axis in the horizontal direction in the direction that the object is moving horizontally. Set $t = 0$ the instant the stone is released. At $t = 0$ the initial conditions are then $x_0 = 0$ and $y_0 = d$. The initial x- and y-components of the velocity are given by Eqs. (5.1.5) and (5.1.6).

At time t the stone has coordinates $(x(t), y(t))$. These coordinate functions are shown in Figure 5.5.
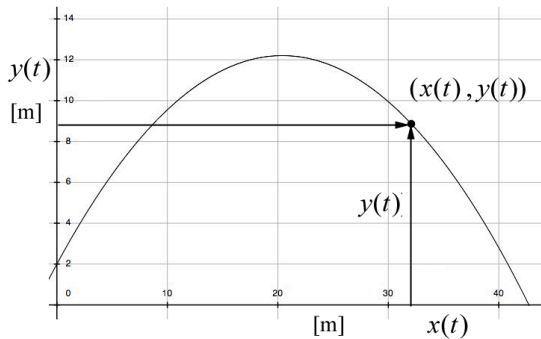

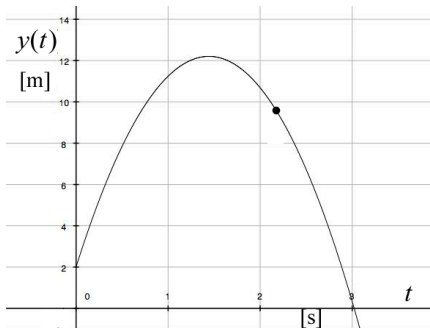**Figure 5.5:** Coordinate functions for stone

Figure 5.6 Plot of the y-component of the position as a function of time

The slope of this graph at any time t yields the instantaneous y-component of the velocity $v_y(t)$ at that time t. Figure 5.5 is a plot of $y(t)$ vs. $x(t)$ and Figure 5.6 is a plot of $y(t)$ vs. t. There are several important things to notice about Figures 5.5 and 5.6. The first point is that the abscissa axes are different in both figures. The second thing to notice is that at $t = 0$, the slope of the graph in Figure 5.5 is equal to

$$\frac{dy}{dx}\bigg|_{t=0} = \left(\frac{dy/dt}{dx/dt}\right)\bigg|_{t=0} = \frac{v_{y,0}}{v_{x,0}} = \tan\theta_0 \ , \tag{5.1.19}$$

while at $t = 0$ the slope of the graph in Figure 5.6 is equal to

$$\frac{dy}{dt}\bigg|_{t=0} = v_{y,0} \ . \tag{5.1.20}$$

The slope of this graph in Figure 5.6 at any time t yields the instantaneous y-component of the velocity $v_y(t)$ at that time t. Let $t = t_1$ correspond to the instant the stone is at its maximal vertical position, the highest point in the flight. The final thing to notice about Figure 5.6 is that at $t = t_1$ the slope is zero or $v_y(t = t_1) = 0$. Therefore

$$v_y(t_1) = v_0 \sin\theta_0 - gt_1 = 0 \ . \tag{5.1.21}$$

Solving Eq. (5.1.21) for t_1 yields,

$$t_1 = \frac{v_0 \sin\theta_0}{g} = \frac{(20 \ \text{m} \cdot \text{s}^{-1})\sin(45°)}{9.8 \ \text{m} \cdot \text{s}^{-2}} = 1.44 \ \text{s}. \tag{5.1.22}$$

The graph in Figure 5.7 shows a plot of $v_y(t)$ as a function of time. Notice that at $t = 0$ the intercept is positive indicting that $v_{y,0}$ is positive which means that the stone was thrown upwards. The y-component of the velocity changes sign at $t = t_1$ indicating that the stone is reversing its direction and starting to move downwards.
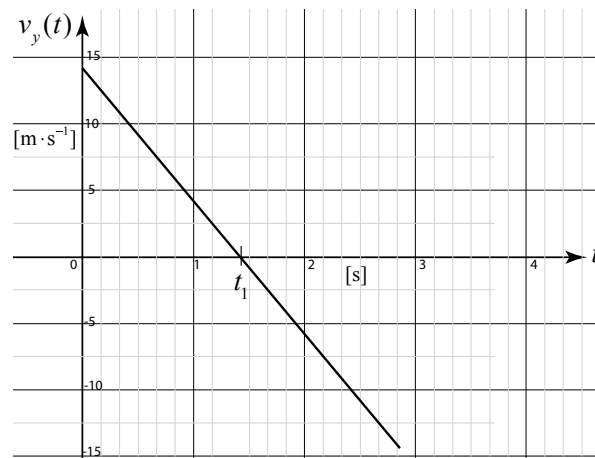

Figure 5.7 y-component of the velocity as a function of time

We now substitute the expression for $t = t_{top}$ (Eq. (5.1.22)) into the y-component of the position in Eq. (5.1.16) to find the maximal height of the stone above the ground

$$y(t = t_{top}) = d + v_0 \sin\theta_0 \frac{v_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g} \right)^2$$

$$= d + \frac{v_0^2 \sin^2\theta_0}{2g} = 2\text{ m} + \frac{(20\text{ m}\cdot\text{s}^{-1})^2 \sin^2(45°)}{2(9.8\text{ m}\cdot\text{s}^{-2})} = 12.2\text{ m}$$

(5.1.23)

5.2.1 Orbit equation

So far our description of the motion has emphasized the independence of the spatial dimensions, treating all of the kinematic quantities as functions of time. We shall now eliminate time from our equation and find the *orbit equation* of the body undergoing projectile motion. We begin with the x-component of the position in Eq. (5.1.16),

$$x(t) = x_0 + v_{x,0} t$$

(5.1.24)

and solve Equation (5.1.24) for time t as a function of $x(t)$,

$$t = \frac{x(t) - x_0}{v_{x,0}} .$$

(5.1.25)

The y-component of the position in Eq. (5.1.16) is given by

$$y(t) = y_0 + v_{y,0} t - \frac{1}{2} g t^2.$$

(5.1.26)

We then substitute Eq. (5.1.25) into Eq. (5.1.26) yielding

$$y(t) = y_0 + v_{y,0} \left(\frac{x(t) - x_0}{v_{x,0}} \right) - \frac{1}{2} g \left(\frac{x(t) - x_0}{v_{x,0}} \right)^2.$$

(5.1.27)

A little algebraic simplification yields the equation for a parabola:

$$y(t) = -\frac{1}{2} \frac{g}{v_{x,0}^2} x(t)^2 + \left(\frac{g\,x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}} \right) x(t) - \frac{v_{y,0}}{v_{x,0}} x_0 - \frac{1}{2} \frac{g}{v_{x,0}^2} x_0^2 + y_0.$$

(5.1.28)

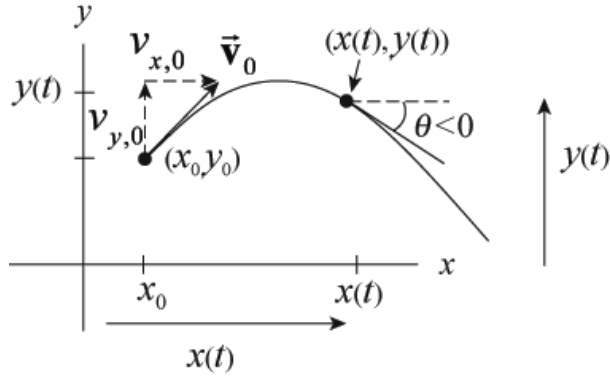The graph of $y(t)$ as a function of $x(t)$ is shown in Figure 5.8.

Figure 5.8 The parabolic orbit

The velocity vector is given by

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}. \tag{5.1.29}$$

The direction of the velocity vector at a point $(x(t), y(t))$ can be determined from the components. Let θ be the angle that the velocity vector forms with respect to the positive x-axis. Then

$$\theta = \tan^{-1}\left(\frac{v_y(t)}{v_x(t)}\right) = \tan^{-1}\left(\frac{dy/dt}{dx/dt}\right) = \tan^{-1}\left(\frac{dy}{dx}\right). \tag{5.1.30}$$

Differentiating Eq. (5.1.28) with respect to x yields

$$\frac{dy}{dx} = -\frac{g}{v_{x,0}^2}x + \left(\frac{g\,x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}}\right). \tag{5.1.31}$$

The direction of the velocity vector at a point $(x(t), y(t))$ is therefore

$$\theta = \tan^{-1}\left(-\frac{g}{v_{x,0}^2}x + \left(\frac{g\,x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}}\right)\right). \tag{5.1.32}$$

Although we can determine the angle of the velocity, we cannot determine how fast the body moves along the parabolic orbit from our graph of $y(x)$; the magnitude of the velocity cannot be determined from information about the tangent line.

If we choose our origin at the initial position of the body at $t = 0$, then $x_0 = 0$ and $y_0 = 0$. Our orbit equation, Equation (5.1.28) can now be simplified to

$$y(t) = -\frac{1}{2}\frac{g}{v_{x,0}^2}x(t)^2 + \frac{v_{y,0}}{v_{x,0}}x(t).$$

(5.1.33)

Example 5.2 Hitting the Bucket

A person is holding a pail while standing on a ladder. The person releases the pail from rest at a height h_1 above the ground. A second person, standing a horizontal distance s from the pail, aims and throws a ball the instant the pail is released in order to hit the pail. The person releases the ball at a height h_2 above the ground, with an initial speed v_0, and at an angle θ_0 with respect to the horizontal. Assume that v_0 is large enough so that the stone will at least travel a horizontal distance s before it hits the ground. You may ignore air resistance.
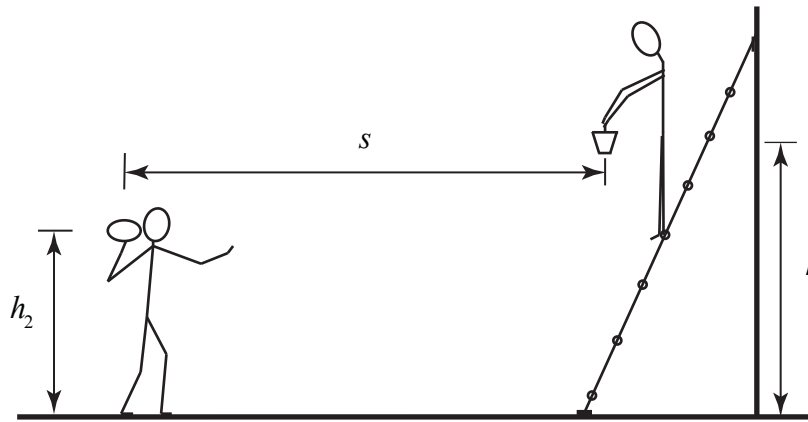

Figure 5.9: Example 5.2

a) Find an expression for the angle θ_0 that the person aims the ball in order to hit the pail. Does the answer depend on the initial velocity?

b) Find an expression for the time of collision as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s.

c) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s.

Solution:

There are two objects involved in this problem. Each object is undergoing free fall, so there is only one stage of motion for each object. The pail is undergoing one-dimensional

motion. The ball is undergoing two-dimensional motion. The parameters h_1, h_2, v_0, and s are unspecified, so our answers will be functions of those quantities. Figure 5.9 shows a sketch of the motion of all the bodies in this problem.

Choose an origin on the ground directly underneath the point where the ball is released, upwards for the positive y-direction and towards the pail for the positive x-direction. Choose position coordinates for the pail as follows. The horizontal coordinate is constant and given by $x_1 = s$. The vertical coordinate represents the height above the ground and is denoted by $y_1(t)$. The ball has coordinates $(x_2(t), y_2(t))$. We show these coordinates in the Figure 5.10.
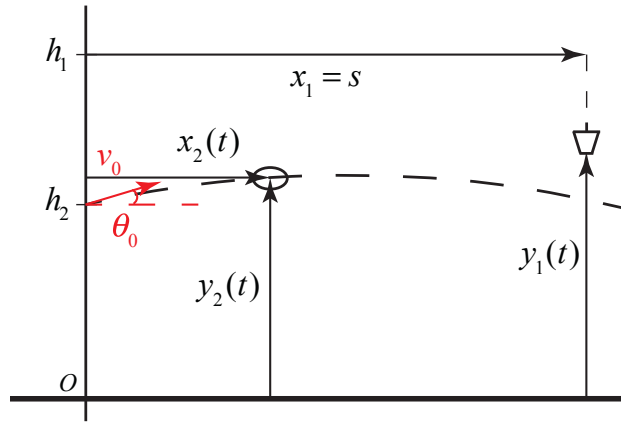

Figure 5.10: Coordinate System

The pail undergoes constant acceleration $a_{1,y} = -g$ in the vertical direction and the ball undergoes uniform motion in the horizontal direction and constant acceleration in the vertical direction, with $a_{2,x} = 0$ and $a_{2,y} = -g$.

The initial conditions for the pail are $(v_{1,0})_y = 0$, $x_{1,0} = s$, $y_{1,0} = h_1$. The equations for position and velocity of the pail simplify to

$$y_1(t) = h_1 - \frac{1}{2}gt^2 \tag{5.1.34}$$

$$v_{y,1}(t) = -gt. \tag{5.1.35}$$

The initial position is given by $x_{2,0} = 0$, $y_{2,0} = h_2$. The components of the initial velocity are given by $(v_{2,0})_y = v_0 \sin(\theta_0)$ and $(v_{2,0})_x = v_0 \cos(\theta_0)$, where v_0 is the magnitude of the initial velocity and θ_0 is the initial angle with respect to the horizontal. The equations for the position and velocity of the ball simplify to

$$x_2(t) = v_0 \cos(\theta_0)t \tag{5.1.36}$$

$$v_{2,x}(t) = v_0 \cos(\theta_0) \tag{5.1.37}$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0)t - \frac{1}{2}gt^2 \tag{5.1.38}$$

$$v_{2,y}(t) = v_0 \sin(\theta_0) - gt. \tag{5.1.39}$$

Note that the quantities h_1, h_2, v_0, and s should be treated as known quantities although no numerical values were given. There are six independent equations with 8 as yet unspecified quantities $y_1(t)$, t, $y_2(t)$, $x_2(t)$, $v_{1,y}(t)$, $v_{2,y}(t)$, $v_{2,x}(t)$, and θ_0.

So we need two more conditions, in order to find expressions for the initial angle, θ_0, the time of collision, t_a, and the spatial location of the collision point specified by $y_1(t_a)$ or $y_2(t_a)$. At the collision time $t = t_a$, the collision occurs when the two balls are located at the same position. Therefore

$$y_1(t_a) = y_2(t_a) \tag{5.1.40}$$

$$x_2(t_a) = x_1 = s. \tag{5.1.41}$$

We shall now apply these conditions that must be satisfied in order for the ball to hit the pail.

$$h_1 - \frac{1}{2}gt_a^2 = h_2 + v_0 \sin(\theta_0)t_a - \frac{1}{2}gt_a^2 \tag{5.1.42}$$

$$s = v_0 \cos(\theta_0)t_a. \tag{5.1.43}$$

Eq. (5.1.42) simplifies to

$$v_0 \sin(\theta_0)t_a = h_1 - h_2 \tag{5.1.44}$$

Dividing Eq. (5.1.44) by Eq. (5.1.43) yields

$$\frac{v_0 \sin(\theta_0)t_a}{v_0 \cos(\theta_0)t_a} = \tan(\theta_0) = \frac{h_1 - h_2}{s_2}. \tag{5.1.45}$$

So the initial angle θ_0 is independent of v_0, and is given by

$$\theta_0 = \tan^{-1}((h_1 - h_2)/s). \tag{5.1.46}$$

From the Figure 5.11 we can see that $\tan(\theta_0) = (h_1 - h_2)/s$ implies that the second person aims the ball at the initial position of the pail.
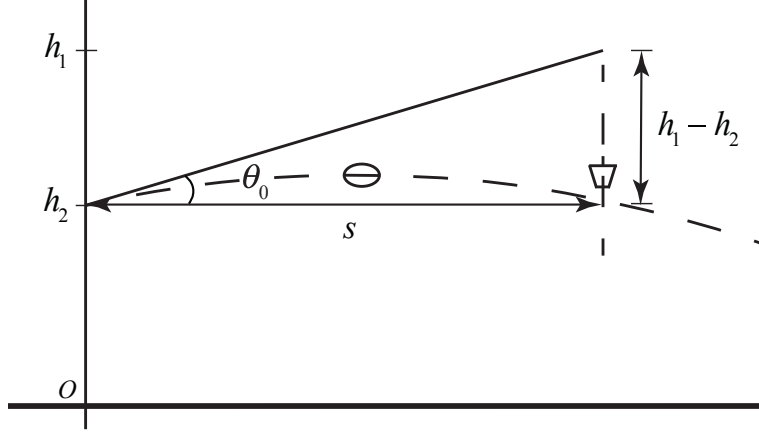
Figure 5.11: Geometry of collision

In order to find the time that the ball collides with the pail, we begin by squaring both Eqs. (5.1.44) and (5.1.43), then utilize the trigonometric identity $\sin^2(\theta_0) + \cos^2(\theta_0) = 1$. Our squared equations become

$$v_0^2 \sin^2(\theta_0) t_a^2 = (h_1 - h_2)^2 \qquad (5.1.47)$$

$$v_0^2 \cos^2(\theta_0) t_a^2 = s^2 . \qquad (5.1.48)$$

Adding these equations together and using the identity $\sin^2(\theta_0) + \cos^2(\theta_0) = 1$ and taking square roots yields

$$v_0 t_a = (s^2 + (h_1 - h_2)^2)^{1/2} . \qquad (5.1.49)$$

We can solve Eq. (5.1.49) for the time of collision

$$t_a = \frac{1}{v_0}(s^2 + (h_1 - h_2)^2)^{1/2} . \qquad (5.1.50)$$

We can now use the y-coordinate function of either the ball or the pail at $t = t_a$ to find the height that the ball collides with the pail. Because the pail had no initial y-component of the velocity, it's easier to use the condition for the pail,

$$y_1(t_a) = h_1 - \frac{g(s^2 + (h_1 - h_2)^2)}{2v_0^2} . \qquad (5.1.51)$$

Comments:

(1) Eqs. (5.1.49) and (5.1.50) can be arrived at in a very direct way. Suppose we analyze the motion in a reference frame that is accelerating downward with $\vec{A} = -g\,\hat{j}$. In that

reference frame both the pail and the stone are not accelerating; the pail is at rest and the stone is travelling with speed v_0, at an angle θ_0. Therefore in order to hit the stationary pail, the stone must be thrown at the angle given by Eq. (5.1.46) and the time that it takes to hit the stone is just given by distance traveled divided by speed, Eq. (5.1.50).