Chapter 31 Non-Inertial Rotating Reference Frames

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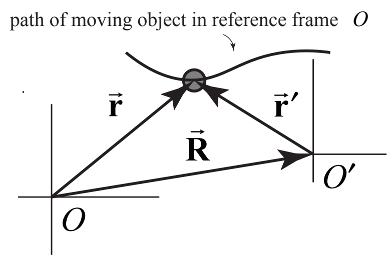
31.1 Introduction

An object is called an *isolated* object if there are no physical interactions between the object and the surroundings. According to Newton's First Law an isolated object will undergo uniform motion. Choose a coordinate system such that the isolated body is at rest or is moving with a constant velocity. That coordinate system is called an ***inertial reference frame***. Do such coordinate systems exist? Newton's First Law states that it is always possible to find such a coordinate system. Newton's Second Law $\vec{\mathbf{F}}_{\text{physical}} = m\vec{\mathbf{a}}$ only holds in inertial reference frames, where $\vec{\mathbf{F}}_{\text{physical}}$ are the forces that arise from the interactions of objects.

Summary: Non-inertial Reference Frames

This is a short summary of results for analyzing motion in non-inertial reference frames.



The position, velocity and acceleration vectors of a moving object of an object in an inertial references O and a non-inertial reference frame O' are related by

$$\begin{aligned}\vec{\mathbf{r}}(t) &= \vec{\mathbf{r}}'(t) = \vec{\mathbf{r}}(t) - \vec{\mathbf{R}}(t)\\ \vec{\mathbf{v}}(t) &= \vec{\mathbf{v}}'(t) = \vec{\mathbf{v}}(t) - \vec{\mathbf{V}}(t)\\ \vec{\mathbf{a}}'(t) &= \vec{\mathbf{a}}(t) - \vec{\mathbf{A}}(t)\end{aligned} \quad . \tag{31.1.1}$$

Newton's Second Law in O is

$$\vec{\mathbf{F}}_{\text{physical}} = m\vec{\mathbf{a}} \quad . \tag{31.1.2}$$

Define the total *fictitious force* by

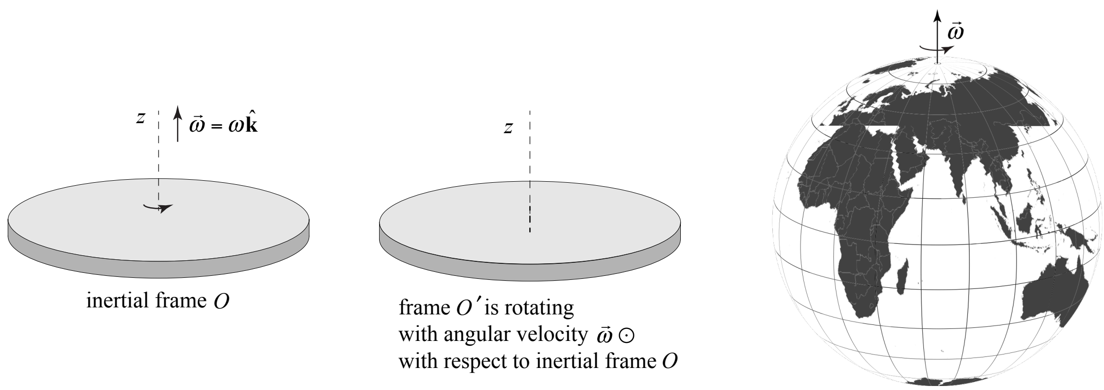
$$\vec{\mathbf{F}}_{\text{fictitious}} = -m\vec{\mathbf{A}} \quad . \tag{31.1.3}$$

Then the modified Newton's Second Law in the non-inertial reference frame O' becomes

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}}' \quad . \tag{31.1.4}$$

Rotating Frames

Let O designate an inertial reference frame and O' a rotating reference frame that is rotating with an angular velocity $\vec{\boldsymbol{\omega}}$ with respect to O. We shall consider two types of rotating reference frames, (i) a reference frame fixed to a platform that is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ with respect to an inertial frame O and (ii) the earth rotating with an angular velocity $\vec{\boldsymbol{\omega}}$ with respect to an inertial frame at rest with respect to the distant stars.



Velocity Transformation Law for Rotaitng Frames

The transformation law for the velocity of an object in the two reference frames O and O' is given by

$$\vec{\mathbf{v}}'(t) = \vec{\mathbf{v}}(t) - \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) \ , \tag{31.1.5}$$

where $\vec{\mathbf{v}} = (d\vec{\mathbf{r}} / dt)_{in}$ is the derivative of the position vector $\vec{\mathbf{r}}(t)$ in the inertial frame and $\vec{\mathbf{v}}' = (d\vec{\mathbf{r}}' / dt)_{rot}$ is the derivative of the position vector $\vec{\mathbf{r}}'(t)$.

Accelertion Transformation Law for Rotaitng Frames

The transformation law for the acceleration of an object in the two reference frames O and O' is given by

$$\vec{\mathbf{a}}' = \vec{\mathbf{a}} - 2\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) \ . \tag{31.1.6}$$

Fictitious Forces and Newton's Second Law in Rotating Frames:

The ***centrifugal fictitious force*** is given by

$$\vec{\mathbf{F}}_{centrifugal} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})), \tag{31.1.7}$$

and the ***Coriolis fictitious force***:

$$\vec{\mathbf{F}}_{coriolis} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' \ . \tag{31.1.8}$$

Then the modified Newton's Second law in the rotating frame becomes

$$\vec{\mathbf{F}}_{physical} + \vec{\mathbf{F}}_{coriolis} + \vec{\mathbf{F}}_{centrifugal} = m\vec{\mathbf{a}}' \,. \tag{31.1.9}$$

31.2 Linearly Accelerating Reference Frames

Let O designate an inertial reference frame and O' designate a second reference frame that is accelerating with a **linear acceleration** $\vec{\mathbf{A}}$ with respect to the inertial frame O (Figure 1).

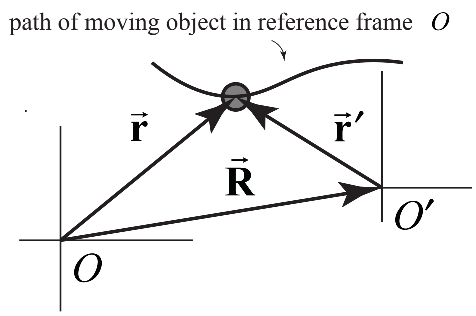


Figure 31.1 Two reference frames

At $t = 0$, the origins of the two reference frames coincide. Let $\vec{\mathbf{R}}(t)$ denote the position vector of the origin in O' as seen by an observer located at O. Then $\vec{\mathbf{V}}(t) = d\vec{\mathbf{R}}(t)/dt$ and $\vec{\mathbf{A}}(t) = d\vec{\mathbf{V}}(t)/dt$ are the velocity and acceleration of reference frame O' with respect to O.

Suppose a particle undergoes an acceleration $\vec{\mathbf{a}}(t)$ in O. The path of the moving particle in reference frame O is shown in Figure 1. The position vector $\vec{\mathbf{r}}(t)$ of the object in O is related to the position vector $\vec{\mathbf{r}}'(t)$ of the object in O' by

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}'(t) + \vec{\mathbf{R}}(t) \;. \tag{31.2.1}$$

.
Differentiating Eq. (31.2.1) yields the relationship between the velocities of the object in the two frames:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\mathbf{V}}(t) \;. \tag{31.2.2}$$

Eq. (31.2.2) is called the *Law of Addition of Velocities.* Differentiating Eq. (31.2.2) yields the relationship between the accelerations of the object in the two frames:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{a}}'(t) + \vec{\mathbf{A}} \;. \tag{31.2.3}$$

Recall that in the inertial reference frame O, $m\vec{\mathbf{a}} = \vec{\mathbf{F}}_{\text{physical}}$. In the non-inertial frame O', Newton's Second Law needs to be modified, because

$$m\vec{\mathbf{a}}' = m\vec{\mathbf{a}} - m\vec{\mathbf{A}} = \vec{\mathbf{F}}_{\text{physical}} - m\vec{\mathbf{A}} \ . \tag{31.2.4}$$

Define a *fictitious force* by

$$\vec{\mathbf{F}}_{\text{fictitious}} = -m\vec{\mathbf{A}} \ . \tag{31.2.5}$$

Then the modified Newton's Second Law in the non-inertial reference frame O' becomes

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}}' \ . \tag{31.2.6}$$

Concept Question 1: Inertial or Non-inertial Reference Frame

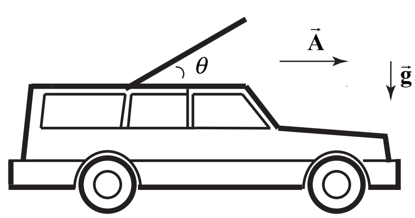
You are in a spaceship with the engines turned off in a zero gravitational field. You are standing on a frictionless floor at rest. Suppose you start to slide backwards. Which of the following statements is true immediately after you start to slide backwards.

1. The spaceship is still an inertial reference frame and has not changed its speed.
2. The spaceship is accelerating backwards.
3. The spaceship is accelerating forwards.

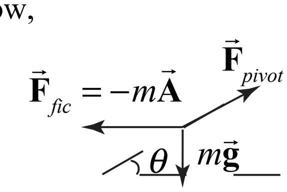
Answer 3. Initially the spaceship defined an inertial reference frame because you, as an isolated, body, remained at rest. Once you start to slide backwards, you conclude that a fictitious force is acting on you in the direction you are moving and hence the spaceship is accelerating in the opposite (forward) direction.

Example 1: Accelerating Car with Hinged Roof

A uniform thin rod of length L and mass m is pivoted at one end. The pivot is attached to the top of a car accelerating at rate $\vec{\mathbf{A}}$. What is the equilibrium value of the angle θ between the rod and the top of the car?



Solution: The free body force diagram on the hinged roof in the accelerating reference frame is shown in the figure below,



where we have added a fictitious force $\vec{\mathbf{F}}_{fic} = -m\vec{\mathbf{A}}$. Because the rod is at rest in the accelerating reference frame, Newton's Second Law becomes

$$m\vec{\mathbf{g}} - m\vec{\mathbf{A}} + \vec{\mathbf{F}}_{pivot} = \vec{\mathbf{0}} .$$

Therefore the pivot force must satisfy $m\vec{\mathbf{g}} - m\vec{\mathbf{A}} + \vec{\mathbf{F}}_{pivot} = -m(\vec{\mathbf{g}} - \vec{\mathbf{A}})$. Note that $\vec{\mathbf{g}}' = \vec{\mathbf{g}} - \vec{\mathbf{A}}$ acts like an effective gravitational field point in the direction given by

$$\theta = \tan^{-1}(g / A)$$

which is the direction that the hinged roof is angled.

31.3 Angular Velocity of a Rigid Body

In Chapter 6 we defined the angular velocity vector $\vec{\boldsymbol{\omega}}$ of a point object undergoing circular motion about the z-axis by

$$\vec{\boldsymbol{\omega}} = \frac{d\theta_z}{dt}\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}} . \qquad (31.3.1)$$

where θ_z is the angle that the position vector of the object makes with the positive x-axis as shown in Figure 31.2.

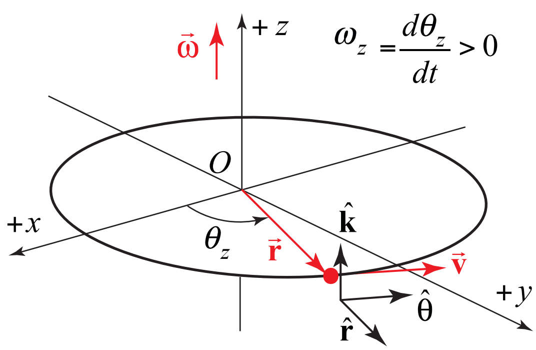


Figure 31.2 Angular velocity for circular motion about z-axis

Now consider a rigid body at time t that is instantaneously rotating about an axis, with unit normal $\hat{\mathbf{n}}$, angle θ, and angular velocity as shown in Figure 31.3.

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\hat{\mathbf{n}} . \qquad (31.3.2)$$

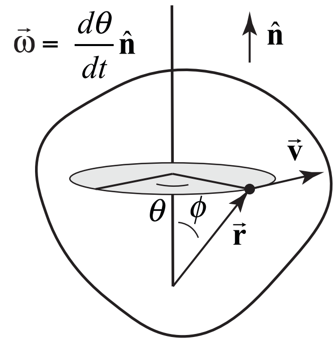


Figure 31.3: Rigid body undergoing rotation about the instantaneous axis of rotation

Introduce angular coordinates θ_x, θ_y, and θ_z, corresponding to angles about the x, y, and z axes. The angular velocity vector in this coordinate system is then

$$\vec{\boldsymbol{\omega}} = \frac{d\theta_x}{dt}\hat{\mathbf{i}} + \frac{d\theta_y}{dt}\hat{\mathbf{j}} + \frac{d\theta_z}{dt}\hat{\mathbf{k}} \equiv \omega_x\,\hat{\mathbf{i}} + \omega_y\,\hat{\mathbf{j}} + \omega_z\,\hat{\mathbf{k}}\,. \tag{31.3.3}$$

The velocity of the rigid body and the angular velocity are related as follows. Every particle in the rigid body is instantaneously undergoing circular motion about the instantaneous axis of rotation (Figure 31.3), $\vec{\boldsymbol{\omega}} = (d\theta / dt)\,\hat{\mathbf{n}}$. Recall that the position vector $\vec{\mathbf{r}}$ of the particle is constant in length and hence the velocity is given by the derivative

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \left|\vec{\mathbf{r}}\right|\sin\phi\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\ , \tag{31.3.4}$$

where $\hat{\theta}$ is a unit vector tangent to the circular path. Note also that $\hat{\mathbf{n}}\times\vec{\mathbf{r}} = \left|\vec{\mathbf{r}}\right|\sin\phi\,\hat{\boldsymbol{\theta}}$. Therefore the velocity is given by the vector product

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}\ . \tag{31.3.5}$$

(Note that $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}} = \frac{d\theta}{dt}\,\hat{\mathbf{n}}\times\vec{\mathbf{r}} = \left|\vec{\mathbf{r}}\right|\sin\phi\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$.)

31.4 Non-inertial Rotating Reference Frame

Let O designate an inertial reference frame and O' a *rotating reference frame* that is rotating with an angular velocity $\vec{\boldsymbol{\omega}}$ with respect to O. We shall consider two types of rotating reference frames. The first example is a reference frame fixed to a platform that is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ with respect to an inertial frame O (Figure 31.4).

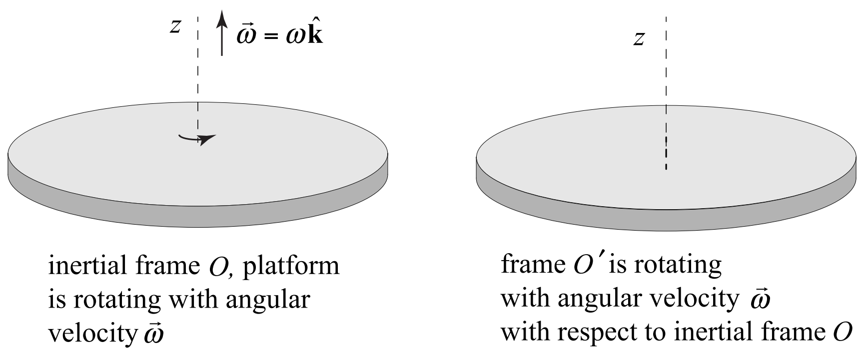


Figure 31.4: Non-inertial reference frame fixed to a rotating platform

The second example is the earth rotating with an angular velocity $\vec{\boldsymbol{\omega}}$ with respect to an inertial frame at rest with respect to the distant stars (Figure 31.5).

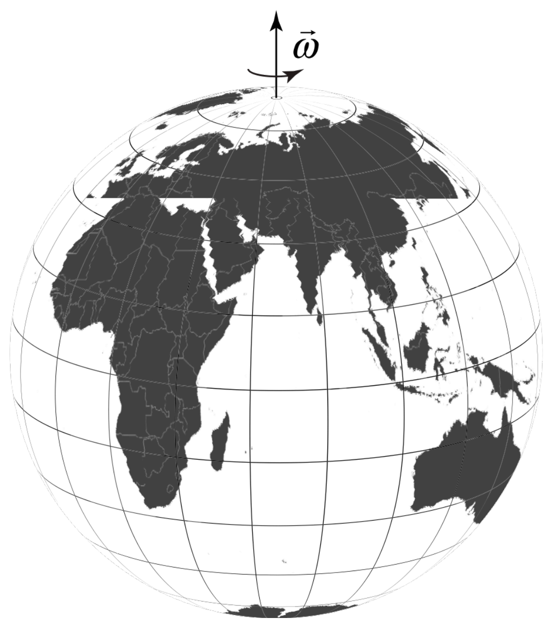


Figure 31.5: Non-inertial reference frame fixed to the earth

31.4.1 Kinematics in Rotating Reference Frames

Let O denote an inertial reference frame. Let O' denote a reference frame that is rotating with an angular velocity $\vec{\boldsymbol{\omega}}$ respect to O. Choose a Cartesian coordinate systems for O, with coordinates (x,y,z), and O', with coordinates (x',y',z'), such that the origins of O and O' coincide at time t, and the axis of rotation of O' passes through the origin in the positive $\hat{\mathbf{k}}$-direction, with $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$. During the time interval $[t, t+\Delta t]$, the x'- and y'-axes have rotated by the angle $\Delta\theta = \omega\Delta t$ as shown in the Figure 31.6.

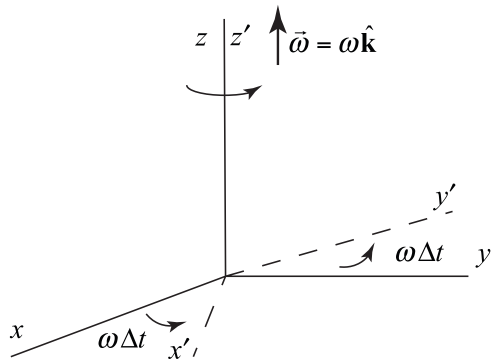


Figure 31.6: Instantaneous rotation about z and z'-axes

Consider the motion of a particle as seen by an observer in reference frame O. Suppose at time t, the position of the particle is located in the (x,z) plane. Denote the position vector by $\vec{\mathbf{r}}(t)$ (Figure 31.7a). During the time interval Δt, the particle has moved to the position $\vec{\mathbf{r}}(t+\Delta t)$, with displacement (Figure 31.7b). In the reference frame O', the position of the particle at time t is given by $\vec{\mathbf{r}}'(t)$. At time $t+\Delta t$, the position of the particle in O' is given in $\vec{\mathbf{r}}'(t+\Delta t)$. The displacement of the particle in O' is given by $\Delta\vec{\mathbf{r}}' = \vec{\mathbf{r}}'(t+\Delta t) - \vec{\mathbf{r}}'(t)$, (Figure 31.8).

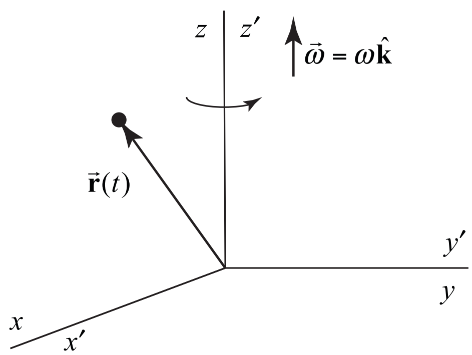


Figure 31.7a: position at time t

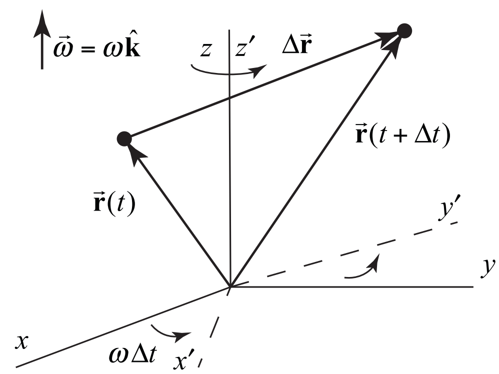


Figure 31.7b: position at time $t+\Delta t$

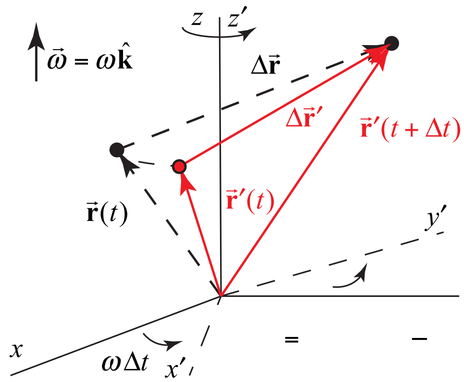


Figure 31.8: Displacement vectors in O and O' at time $t+\Delta t$

This displacement $\Delta\vec{\mathbf{r}}'$ is not equal to the displacement $\Delta\vec{\mathbf{r}}$ in O because the x' and y' axes have rotated by an angle $\Delta\theta = \omega\Delta t$. The initial position vector $\vec{\mathbf{r}}'(t)$ still lies in the (x', z') plane in O' but at time $t+\Delta t$, this vector has rotated with respect to the position $\vec{\mathbf{r}}(t)$ as seen by an observer in O (Figure 31.8). The lengths of the two vectors $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}'(t)$ are equal, $\left|\vec{\mathbf{r}}(t)\right| = \left|\vec{\mathbf{r}}'(t)\right|$. The difference vector between the two displacement vectors, $\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'$, is perpendicular to the axes of rotation (Figure 31.9a).

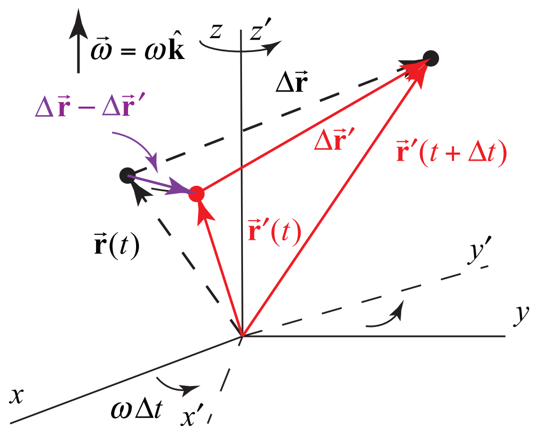


Figure 31.9a: $\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'$ at time $t+\Delta t$

The magnitude of $\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'$ is given by (Figure 31.9b)

$$\left|\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'\right| = 2\left|\vec{\mathbf{r}}(t)\right|\sin(\phi)\sin(\Delta\theta/2) \tag{31.4.1}$$

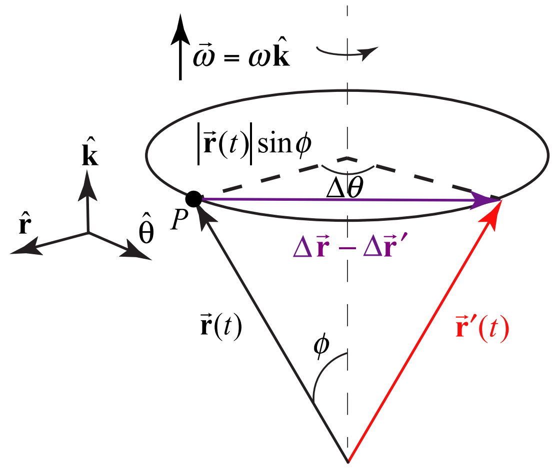


Figure 31.9b: $\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'$ at time $t + \Delta t$

In the limit as $\Delta\theta \to 0$, $\sin(\Delta\theta / 2) \to \Delta\theta / 2$, and thus in the limit as $\Delta\theta \to 0$

$$\left|\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'\right| = \left|\vec{\mathbf{r}}(t)\right|\sin(\phi)\Delta\theta \,. \tag{31.4.2}$$

Introduce a set of unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{k}})$ at the point P as shown in Figure 31.9c. The vector $\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}'$ is in the $\hat{\boldsymbol{\theta}}$-direction, hence the difference in the displacement vectors is given by

$$\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}' = \left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\Delta\boldsymbol{\theta}\,\hat{\boldsymbol{\theta}} \tag{31.4.3}$$

Dividing both sides by Δt and taking the limit as $\Delta t \to 0$ yields

$$\vec{\mathbf{v}}(t) - \vec{\mathbf{v}}'(t) = \lim_{\Delta t \to 0}\left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\frac{\Delta\boldsymbol{\theta}}{\Delta t}\hat{\boldsymbol{\theta}} \tag{31.4.4}$$

Thus

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\boldsymbol{\omega}\,\hat{\boldsymbol{\theta}} \,, \tag{31.4.5}$$

where $\omega = d\theta / dt$. In cylindrical coordinates, the position vector is

$$\vec{\mathbf{r}}(t) = \left|\vec{\mathbf{r}}(t)\right|\sin\phi\,\hat{\mathbf{r}} + \left|\vec{\mathbf{r}}(t)\right|\cos\phi\,\hat{\mathbf{k}} \,. \tag{31.4.6}$$

The vector cross product $\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t)$ is then

$$\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = \boldsymbol{\omega}\,\hat{\mathbf{k}} \times (\left|\vec{\mathbf{r}}(t)\right|\sin\boldsymbol{\phi}\,\hat{\mathbf{r}} + \left|\vec{\mathbf{r}}(t)\right|\cos\boldsymbol{\phi}\,\hat{\mathbf{k}}) = \boldsymbol{\omega}\left|\vec{\mathbf{r}}(t)\right|\sin\boldsymbol{\phi}\,\hat{\boldsymbol{\theta}} \,. \tag{31.4.7}$$

Substituting Eq. (31.4.7) into Eq. (31.4.5) yields

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) \,, \tag{31.4.8}$$

where $\vec{\mathbf{v}} = (d\vec{\mathbf{r}}/dt)_{in}$ is the derivative of the position vector $\vec{\mathbf{r}}(t)$ in the inertial frame and $\vec{\mathbf{v}}' = (d\vec{\mathbf{r}}'/dt)_{rot}$ is the derivative of the position vector $\vec{\mathbf{r}}'(t)$. Eq. (31.4.8) is the rotational version of Eq. (31.2.2). Keep in mind that at time t, the vectors $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}'(t)$ are instantaneously equal because they point from the origin to the moving object (although their decomposition into component vectors is different because the unit vectors in the two reference frames are different.) However the time derivatives are different.

Example 2: Moving tangentially on rotating platform

(a) Consider a platform that is rotating about the z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. Let O' denote a reference frame that is rotating with the platform. An object of mass m is moving in a circle of radius r on the platform with a constant tangential velocity $\vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}}$ in the inertial frame O, such that $v > r\omega$ (Figure 31.10a and Figure 31.10b). What is the velocity of the object $\vec{\mathbf{v}}'$ in the reference frame O'?

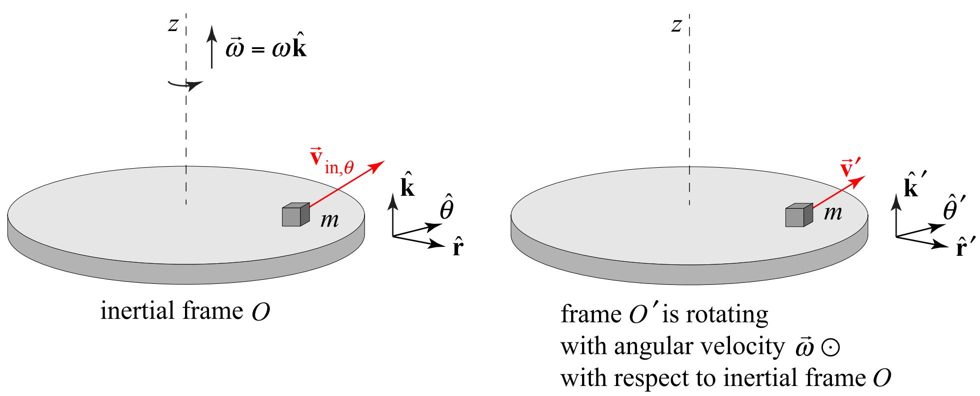


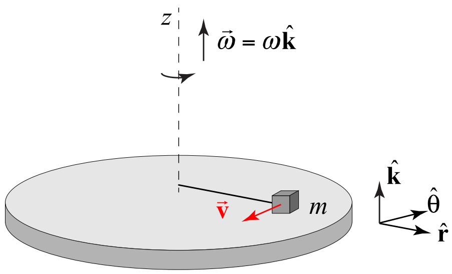
Figure 31.10a **Figure 31.10b**

Solution: In the instant shown in Figure 31.10a and Figure 31.10b, the unit vectors in the two frames are equal, and therefore Eq. (31.4.8) can be written as

$$\vec{\mathbf{v}}'(t) = \vec{\mathbf{v}}(t) - \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = v\hat{\boldsymbol{\theta}} - (\omega\,\hat{\mathbf{k}} \times r\hat{\mathbf{r}}) = (v - r\omega)\hat{\boldsymbol{\theta}}\,.$$

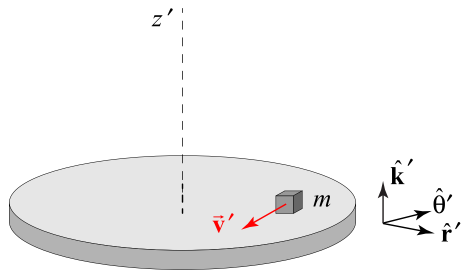
Note that if $v = r\omega$, then the object is at rest in O'.

(b) An object of mass m is moving in a circle of radius r on the platform with a constant tangential velocity $\vec{\mathbf{v}}' = -v'\hat{\boldsymbol{\theta}}'$ in the rotating frame O' (Figure 31.11a and Figure 31.11b). What is the velocity of the object $\vec{\mathbf{v}}$ in the reference frame O?



inertial frame O

Figure 31.11a



frame O' is rotating
with angular velocity $\vec{\omega}$
with respect to inertial frame O

Figure 31.11b

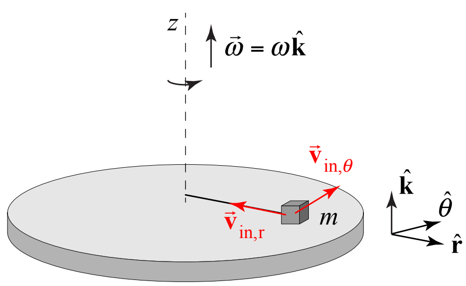
Solution: $\vec{\mathbf{v}}' = -v\hat{\boldsymbol{\theta}}$. The velocity $\vec{\mathbf{v}}$ in the reference frame O is given by

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = -v\hat{\boldsymbol{\theta}} + (\omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}}) = (-v + r\omega)\hat{\boldsymbol{\theta}}\,.$$

Note that if $v = r\omega$, then the object is at rest in O.

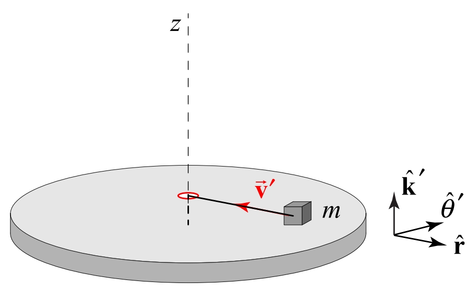
Example 3: Moving radially inward on rotating platform

Consider a platform that is rotating about the z-axis with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. Let O' denote a reference frame that is rotating with the platform. An object of mass m is connected to a string that is pulled radially inward along the surface of the platform at a constant speed v in O'. At the instant shown in Figure 31.12a and Figure 31.12b, the object is at a distance $r = r'$ from the center of the platform. What is the velocity of the object $\vec{\mathbf{v}}$ in the reference frame O?



inertial frame O

Figure 31.12a



frame O' is rotating
with angular velocity $\vec{\omega}$ ⊙
with respect to inertial frame O

Figure 31.12b

Solution: $\vec{\mathbf{v}}' = -v\hat{\mathbf{r}}'$. The velocity $\vec{\mathbf{v}}$ in the reference frame O is given by

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = -v\hat{\mathbf{r}} + (\omega\,\hat{\mathbf{k}} \times r\hat{\mathbf{r}}) = -v\hat{\mathbf{r}} + r\omega\,\hat{\boldsymbol{\theta}}\,.$$

31.4.2 Acceleration in a Rotating Reference Frame

The result in Eq. (31.4.8) that describes the transformation law for the time derivative of the position vector in the two reference frames O and O' holds for the derivative of any vector $\vec{\mathbf{C}}$.

Let $(d\vec{\mathbf{C}} / dt)_{in}$ denote the derivative of the vector $\vec{\mathbf{C}}$ in the inertial frame O, and let $(d\vec{\mathbf{C}} / dt)_{rot}$ denote the derivative of the vector in the rotating reference frame O'. Then

$$(d\vec{\mathbf{C}} / dt)_{in} = (d\vec{\mathbf{C}} / dt)_{rot} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{C}}\,. \tag{31.4.9}$$

(See Appendix 31.A.1 for a proof.) In particular the derivative of the velocity $\vec{\mathbf{v}}$ is then

$$(d\vec{\mathbf{v}} / dt)_{in} = (d\vec{\mathbf{v}} / dt)_{rot} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}\,. \tag{31.4.10}$$

Now $\vec{\mathbf{v}} = \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, therefore Eq. (31.4.10) becomes

$$\begin{aligned}
\vec{\mathbf{a}} &= (d\vec{\mathbf{v}} / dt)_{in} = (d(\vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) / dt)_{rot} + \vec{\boldsymbol{\omega}} \times (\vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) \\
\vec{\mathbf{a}} &= \vec{\mathbf{a}}' + (d(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) / dt)_{rot} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}).
\end{aligned}\;. \tag{31.4.11}$$

where $\vec{\mathbf{a}} = (d\vec{\mathbf{v}} / dt)_{in}$ is the acceleration of the particle has seen in the inertial frame O and $\vec{\mathbf{a}}' = (d\vec{\mathbf{v}}' / dt)_{rot}$ is the acceleration of the particle has seen in the inertial frame O' We have assumed that $\vec{\omega}$ is constant and therefore

$$(d(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) / dt)_{rot} = \vec{\boldsymbol{\omega}} \times (d\vec{\mathbf{r}} / dt)_{rot} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}'\;\;. \tag{31.4.12}$$

So Eq. (31.4.11) becomes the transformation law for the acceleration of an object in the two reference frames O and O' is given by

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}' + 2\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})\,. \tag{31.4.13}$$

31.4.3 Newton's Second Law in Rotating Reference Frames

Let $\vec{\mathbf{F}}_{phy}$ denote the sum of the physical forces acting on a particle. Recall that in an inertial reference frame O, Newton's Second Law is given by

$$\vec{\mathbf{F}}_{phy} = m\vec{\mathbf{a}}\,. \tag{31.4.14}$$

In a non-inertial rotating reference frame O', the Second Law becomes, using Eq. (31.4.13),

$$\vec{\mathbf{F}}_{phy} = m(\vec{\mathbf{a}}' + 2\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})). \qquad (31.4.15)$$

Rewrite Eq. (31.4.15) as

$$\vec{\mathbf{F}}_{phy} - 2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' - m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) = m\vec{\mathbf{a}}'. \qquad (31.4.16)$$

Define two ‘fictitious forces’, the ***centrifugal fictitious force***:

$$\vec{\mathbf{F}}_{centrifugal} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})), \qquad (31.4.17)$$

and the ***Coriolis fictitious force***:

$$\vec{\mathbf{F}}_{coriolis} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' \ . \qquad (31.4.18)$$

Then the modified Newton’s Second law in the rotating frame becomes

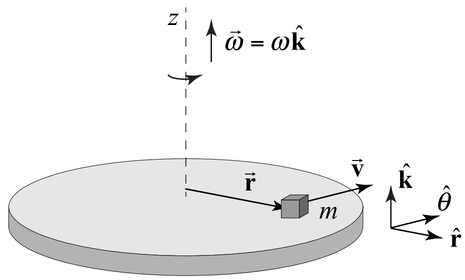
$$\vec{\mathbf{F}}_{physical} + \vec{\mathbf{F}}_{coriolis} + \vec{\mathbf{F}}_{centrifugal} = m\vec{\mathbf{a}}'. \qquad (31.4.19)$$

Eq. (31.4.19) will be the starting point for analyzing the motion of particles in a rotating reference frame.

The centrifugal force $\vec{\mathbf{F}}_{centrifugal}$ is perpendicular to both terms in the cross product $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, and therefore is perpendicular to the axis of rotation. It is a simple exercise to show that it is also pointing in the radially outward direction from the axis of rotation.The Coriolis force $\vec{\mathbf{F}}_{coriolis}$ is perpendicular to $\vec{\boldsymbol{\omega}}$ and the velocity $\vec{\mathbf{v}}'$ of the particle in the rotating frame.

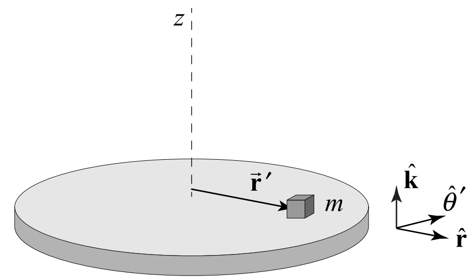
Because of these two fictitious forces, the motion of particles in rotating reference frames like the earth are far more complicated to analyze. For example, air molecules moving along the surface or water molecules in the ocean of the earth experience both of these fictitious forces as seen in the earth rotating reference frame. Thus the study of atmospheric physics, ocean physics on the earth, and the study of extraterrestrial spinning objects like stars, planets, or rotating gas clouds require an understanding of the centrifugal and the Coriolis forces.

Example 4: Mass at Rest on a Rotating Platform



inertial frame O

Figure 31.13a



rotating non-inertial frame O'

Figure 31.13b

A platform is rotating about the z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O (Figure 31.13a). Choose a set of cylindrical unit vectors $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{k}})$. An object of mass m that lies on the platform a distance r from the center is rotating with the platform, hence in the reference frame O, the object has angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ and velocity $\vec{\mathbf{v}} = r\omega\hat{\theta}$. The force keeping the object from moving on the platform is a radially inward static friction force $\vec{\mathbf{f}}_s = -f_s\hat{\mathbf{r}}$. The object is accelerating towards the center with $\vec{\mathbf{a}} = -r\omega^2\hat{\mathbf{r}}$. Newton's Second Law in the inertial reference frame O is then

$$-f_s\hat{\mathbf{r}} = -mr\omega^2\hat{\mathbf{r}} \Rightarrow f_s = mr\omega^2 \ . \qquad (31.4.20)$$

Let O' denote a reference frame that is rotating with the platform (Figure 31.13b). The object is at rest in the rotating frame O', $\vec{\mathbf{v}}' = \vec{\mathbf{0}}$, and therefore the Coriolis force is zero. Choose a set of cylindrical unit vectors $(\hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\mathbf{k}})$. The centrifugal force is non-zero and points in the outward radial direction and is given by

$$\vec{\mathbf{F}}_{centrifugal} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}')) = -m(\omega\,\hat{\mathbf{k}} \times (\omega\,\hat{\mathbf{k}} \times r\,\hat{\mathbf{r}}')) = -m(\omega\,\hat{\mathbf{k}} \times r\omega\hat{\theta}') = mr\omega^2\hat{\mathbf{r}}' \ .$$

The acceleration of the object is zero in O', and the modified Newton's Second Law Eq. (31.4.19) is then

$$\vec{\mathbf{f}}_s + \vec{\mathbf{F}}_{centrifugal} = \vec{\mathbf{0}} \ .$$

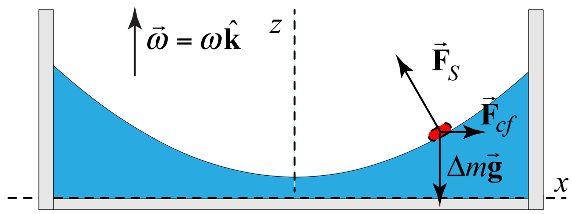
Using our results above, the static friction force is then

$$-f_s\hat{\mathbf{r}}' + mr\omega^2\hat{\mathbf{r}}' = \vec{\mathbf{0}} \Rightarrow f_s = mr\omega^2 \ ,$$

in agreement with Eq. (31.4.20).

Example 5: Rotating Water Bucket

In an inertial reference frame O, consider a water bucket that is rotating about the vertical z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$. The rotational motion of the bucket is transformed to the fluid contained within and after a period of time, the fluid is rotating with the same angular velocity $\vec{\boldsymbol{\omega}}$ and the surface of the fluid takes on a concave shape shown in Figure 31.14.



rotating non-inertial frame O'

Figure 31.14

In a reference frame O' rotating with the bucket, the water is in static equilibrium. The forces acting on a small surface element of mass Δm, located at the point (r,z), are the gravitational force $\Delta m\vec{\mathbf{g}}$, the fictitious centrifugal force $\vec{\mathbf{F}}_{cent}$, and a hydrostatic force $\vec{\mathbf{F}}_S$ that the rest of the fluid exerts on the fluid element (Figure 31.14b). Choose a cylindrical coordinate system with unit vectors $(\hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\mathbf{k}})$ as shown in Figure 31.15.

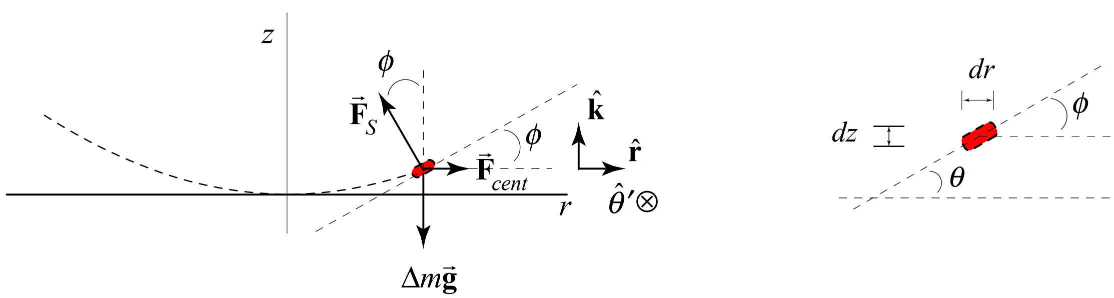


Figure 31.15

The tangent line to the surface element makes an angle ϕ with respect to the horizontal axis such that the slope is given by

$$\frac{dz}{dr} = \tan\phi \tag{31.4.21}$$

The centrifugal force is given by

$$\vec{\mathbf{F}}_{cent} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}')) = -m(\omega\,\hat{\mathbf{k}} \times (\omega\,\hat{\mathbf{k}} \times (r\,\hat{\mathbf{r}}' + z\hat{\mathbf{k}})) = -m(\omega\,\hat{\mathbf{k}} \times r\omega\,\hat{\theta}') = mr\omega^2\hat{\mathbf{r}}'\,.$$

Newton's modified Second Law is the $\hat{\mathbf{r}}'$-direction is given by

$$-F_s \sin\phi + mr\omega^2 = 0\ , \tag{31.4.22}$$

and in the $\hat{\mathbf{k}}$-direction is given by

$$F_s \cos\phi - mg = 0 \ . \qquad (31.4.23)$$

Eqs. (31.4.22) and (31.4.23) can be solved for $\tan\phi$:

$$\tan\phi = \frac{r\omega^2}{g} \ . \qquad (31.4.24)$$

Therefore the slope of the surface at the point (r,z) is given by

$$\frac{dz}{dr} = \frac{r\omega^2}{g} \ . \qquad (31.4.25)$$

Separate variables and form an integral equation

$$\int_{z=0}^{z} dz = \frac{\omega^2}{g} \int_{r=0}^{r} r dr \qquad (31.4.26)$$

which upon integration yields the equation for the surface of the fluid

$$z = \frac{1}{2}\frac{\omega^2}{g} r^2 \ . \qquad (31.4.27)$$

31.5 Motion on the Earth

31.5.1 Introduction

In an inertial reference frame O fixed with respect to the distant stars, the earth is rotating with a period of 23 hours, 53 minutes and 4 seconds corresponding to an angular speed $\omega = \dfrac{2\pi \text{ rad}}{85984 \text{ s}} = 7.307 \times 10^{-5} \text{ rad/sec}$. Choose the positive z-direction to point in the direction of the angular velocity $\vec{\omega}$. In a non-inertial reference frame O' that is rotating with the earth, consider a point located on the surface of the earth at latitude λ. Choose a spherical coordinate system with coordinates (r,θ,ϕ) with associated unit vectors, $(\hat{\mathbf{r}},\hat{\theta},\hat{\phi})$, as shown in Figure 31.17.

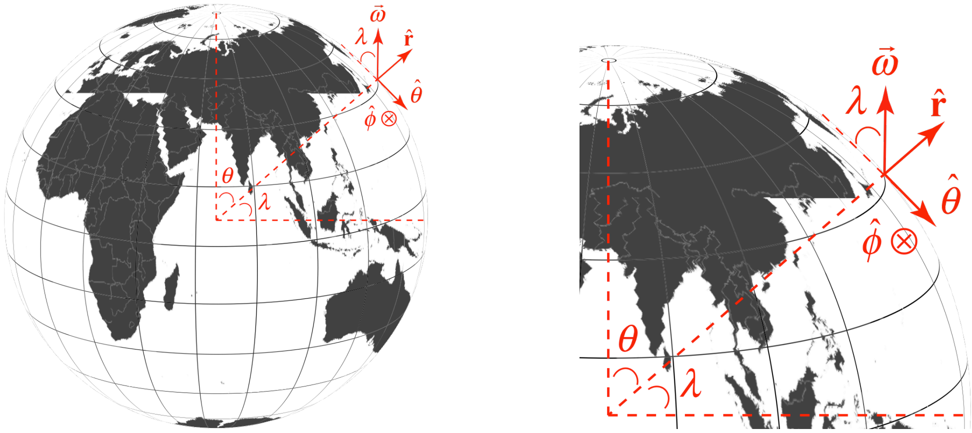


Figure 31.17

31.5.2 Centrifugal Fictitious Force on Earth

At the latitude λ, the angular velocity vector can be written as the vector sum (Figure 31.17)

$$\vec{\boldsymbol{\omega}} = \omega \sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\,\hat{\theta} = \vec{\omega}_{\perp} + \vec{\omega}_{\parallel} \ . \qquad (31.5.1)$$

where $\vec{\omega}_{\perp} = \omega \sin\lambda\,\hat{\mathbf{r}}$ is the component of the angular velocity perpendicular to the surface of the earth and $\vec{\omega}_{\parallel} = -\omega\cos\lambda\,\hat{\theta}$ is the component of the angular velocity tangent to the surface of the earth.

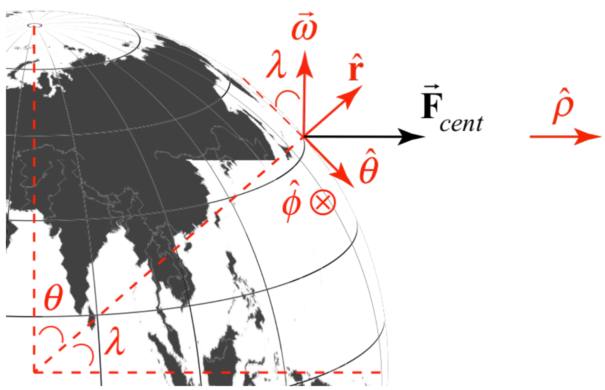


Figure 31.18

Example 6 The Centrifugal Force and Apparent Gravitation Acceleration

(a) Show that in the rotating reference frame the centrifugal force points radially away from the axis of rotation. In particular show that

$$\vec{\mathbf{F}}_{centrifugal} = mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\rho}} \;, \qquad (31.5.2)$$

where $\hat{\boldsymbol{\rho}} = \cos\boldsymbol{\lambda}\,\hat{\mathbf{r}} + \sin\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}}$ is the unit vector pointing radially away from the rotation axis (Figure 31.18).

(b) The acceleration due to gravity measured in an earthbound rotating coordinate system is denoted by $\vec{\mathbf{g}}$. However, because of the earth's rotation, $\vec{\mathbf{g}}$ is different from the true acceleration due to gravity $\vec{\mathbf{g}}_0 = -g_0\hat{\mathbf{r}}$, where $g_0 = GM_E / R_E^{\,2}$. Assuming that the earth is perfectly round, with radius R_e and angular velocity Ω_e, find $g = \left|\vec{\mathbf{g}}\right|$ as a function of latitude λ. (Assuming the earth to be round is actually not justified; the contributions to the variation of g due to the polar flattening is comparable to the effect calculated here.)

Solution:

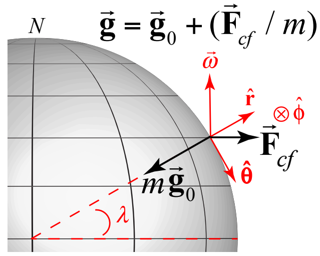
(a) Choose coordinates in the rotating frame as shown in Figure 31.18. At the latitude λ, the angular velocity vector is given by $\vec{\boldsymbol{\omega}} = \omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\,\hat{\theta}$. The position vector is $\vec{\mathbf{r}} = R_E\,\hat{\mathbf{r}}$. Note that $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}$. Therefore the centrifugal fictitious force is given by

$$\begin{aligned}
&\vec{\mathbf{F}}_{centrifugal} \equiv \vec{\mathbf{F}}_{cf} = -m(\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}))\\
&= -m((\boldsymbol{\omega}\sin\boldsymbol{\lambda}\,\hat{\mathbf{r}} - \boldsymbol{\omega}\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\times((\boldsymbol{\omega}\sin\boldsymbol{\lambda}\,\hat{\mathbf{r}} - \boldsymbol{\omega}\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\times R_E\,\hat{\mathbf{r}}))\\
&= -m(\boldsymbol{\omega}\sin\boldsymbol{\lambda}\,\hat{\mathbf{r}} - \boldsymbol{\omega}\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\times\boldsymbol{\omega}\cos\boldsymbol{\lambda}R_E\,\hat{\boldsymbol{\phi}})\\
&= m(\boldsymbol{\omega}^2\sin\boldsymbol{\lambda}\cos\boldsymbol{\lambda}R_E\,\hat{\boldsymbol{\theta}} + \boldsymbol{\omega}^2\cos^2\boldsymbol{\lambda}R_E\,\hat{\mathbf{r}})\\
&= mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}(\cos\boldsymbol{\lambda}\,\hat{\mathbf{r}} + \sin\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\\
&= mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\rho}}
\end{aligned}$$

where we used the fact that $\hat{\boldsymbol{\rho}} = \cos\boldsymbol{\lambda}\,\hat{\mathbf{r}} + \sin\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}}$. The centrifugal force points radially away from the axis of rotation as we expect. For future use note that in spherical coordinates the centrifugal force is given by

$$\vec{\mathbf{F}}_{centrifugal} = mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}(\cos\boldsymbol{\lambda}\,\hat{\mathbf{r}} + \sin\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}}) \qquad (31.5.3)$$

(b) The force diagram on the object in the rotating frame is shown in the figure below.



The acceleration due to gravity measured in an earthbound rotating coordinate system is denoted by

$$\vec{\mathbf{g}}_{eff} = \vec{\mathbf{g}}_0 + (\vec{\mathbf{F}}_{cf} / m) = (R_E\omega^2\cos^2\lambda - g_0)\hat{\mathbf{r}} + R_E\omega^2\cos\lambda\sin\lambda\hat{\theta} \ . \qquad (31.5.4)$$

The magnitude of $\vec{\mathbf{g}}_{eff}$ is then

$$\begin{aligned} g_{eff} &= \left|\vec{\mathbf{g}}_{eff}\right| = ((R_E\omega^2\cos^2\lambda - g_0)^2 + (R_E\omega^2\cos\lambda\sin\lambda)^2)^{1/2} \\ &= ((R_E\omega^2\cos^2\lambda)^2 - 2g_0R_E\omega^2\cos^2\lambda + g_0{}^2 + (R_E\omega^2\cos\lambda\sin\lambda)^2)^{1/2} \\ &= g_0\left[\left(\frac{R_E\omega^2\cos^2\lambda}{g_0}\right)^2 - 2\frac{R_E\omega^2}{g_0}\cos^2\lambda + 1 + \left(\frac{R_E\omega^2\cos\lambda\sin\lambda}{g_0}\right)^2\right]^{1/2} \end{aligned}$$

To simplify the calculation, let $y = R_E\omega^2 / g_0$. (Note that $y = R_E\omega^2 / g_0 = R_E{}^3\omega^2 / GM_E$). Then

$$\begin{aligned} g_{eff} &= g_0\left[\left(y\cos^2\lambda\right)^2 - 2y\cos^2\lambda + 1 + \left(y\cos\lambda\sin\lambda\right)^2\right]^{1/2} \\ &= g_0\left[1 - (2y - y^2)\cos^2\lambda\right]^{1/2} \end{aligned}$$

At the latitude of MIT, $\lambda = 42.36^\circ\mathrm{N}$. The mean radius of the earth is $R_E = 6.371\times10^6\,\mathrm{m}$, the angular speed $\omega = 7.307\times10^{-5}\ \mathrm{rad\cdot s^{-1}}$, the mass of the earth $M_E = 5.972\times10^{24}\,\mathrm{kg}$ and the universal gravitational constant is $G = 6.674\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}$. Then

$$g_0 = (6.674\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}})(5.972\times10^{24}\mathrm{kg}) / (6.371\times10^6)^2 = 9.82\ \mathrm{m\cdot s^{-2}}$$

and

$$\begin{aligned} y &= (6.371\times10^6\mathrm{m})^3(7.307\times10^{-5}\ \mathrm{rad\cdot s^{-1}})^2 / (6.674\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}})(5.972\times10^{24}\mathrm{kg}) \\ &= 3.461\times10^{-3}. \end{aligned}$$

Therefore

$$g_{eff} = g_0\left[\left(y\cos^2\lambda\right)^2 - 2y\cos^2\lambda + 1 + \left(y\cos\lambda\sin\lambda\right)^2\right]^{1/2}$$
$$= g_0\left[1-(2y-y^2)\cos^2\lambda\right]^{1/2} = 9.801\,\mathrm{m\cdot s^{-2}}.$$

The actual value of the acceleration due to gravitation at the latitude of MIT based on the International Gravity Formula IGF) 1980 from the parameters of the Geodetic Reference System 1980 (GRS80), which determines the gravity from the position of latitude, is $g_{eff} = 9.80381\,\mathrm{m\cdot s^{-2}}$.

31.5.3 Coriolis Fictitious Force

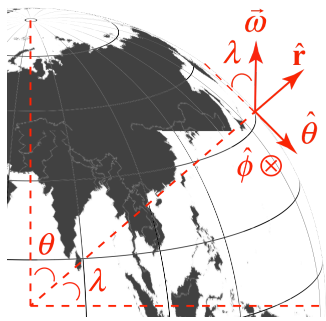


Figure 31.19

Consider a particle traveling in the northern hemisphere tangent to the surface of the earth with velocity (in the earth rotating reference frame) $\vec{\mathbf{v}} = v_\theta\hat{\theta} + v_\phi\hat{\phi}$, where $(\hat{\mathbf{r}},\hat{\theta},\hat{\phi})$ are unit vectors in the rotating frame, (v_θ, v_ϕ) are the components of the velocity with speed $v = (v_\theta{}^2 + v_\phi{}^2)^{1/2}$. The Coriolis force is given by

$$\vec{\mathbf{F}}_{\text{cor}} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}} = -2m(\vec{\boldsymbol{\omega}}_\perp + \vec{\boldsymbol{\omega}}_{||})\times\vec{\mathbf{v}} = -2m\vec{\boldsymbol{\omega}}_\perp\times\vec{\mathbf{v}} - 2m\vec{\boldsymbol{\omega}}_{||}\times\vec{\mathbf{v}} \qquad (31.5.5)$$

The contribution from the term $-2m\vec{\boldsymbol{\omega}}_\perp\times\vec{\mathbf{v}}$ is tangent to the surface of the earth, perpendicular to the velocity, and has magnitude $2m\omega_\perp v = 2m\omega v\left|\sin\lambda\right|$. The contribution from the term $-2m\vec{\boldsymbol{\omega}}_{||}\times\vec{\mathbf{v}}$ is perpendicular to the surface of the earth, and has magnitude $2m\omega\left|\cos\lambda\right|$. This term is quite small compared to the gravitational force and we shall usually ignore its contribution to the fictitious force acting on particles that are moving tangential to the surface of the earth. The full vector expression for the Coriolis force is given by

$$\begin{aligned}\vec{\mathbf{F}}_{coriolis} &= -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}} = -2m(\omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\theta})\times(v_\theta\hat{\theta} + v_\phi\hat{\phi}) \\ &= 2m\omega\sin\lambda(-v_\theta\hat{\phi} + v_\phi\hat{\theta}) + 2m\omega\cos\lambda v_\phi\hat{\mathbf{r}}\end{aligned} \qquad (31.5.6)$$

The component of the Coriolis force tangential to the surface of the earth is given by

$$\vec{\mathbf{F}}_{\text{cor},\parallel} = 2m\omega\sin\lambda(-v_\theta\hat{\phi} + v_\phi\hat{\theta}) \tag{31.5.7}$$

with magnitude

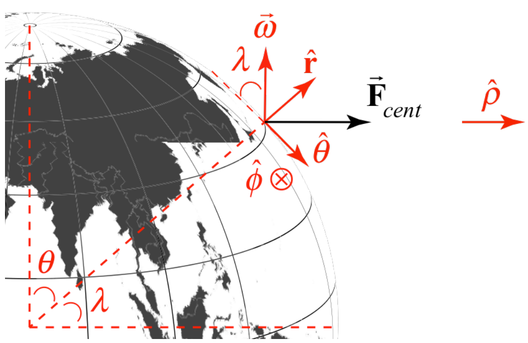
$$\left|\vec{\mathbf{F}}_{\text{cor},\parallel}\right| = 2m\omega\sin\lambda(v_\theta{}^2 + v_\phi{}^2)^{1/2} = 2m\omega\sin\lambda v \tag{31.5.8}$$

in agreement with our discussion above. The component perpendicular to the surface of the earth is given by

$$\vec{\mathbf{F}}_{\text{cor},\perp} = 2m\omega\cos\lambda v_\phi\hat{\mathbf{r}}\,. \tag{31.5.9}$$

Example 7: Direction of Coriolis Force in Northern Hemisphere

Consider a particle moving in the northern hemisphere at north latitude λ. Note that $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$.



a) If the particle is moving along a longitude line towards the North Pole with velocity $\vec{\mathbf{v}} = -v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

b) If the particle is moving along a longitude line away from the North Pole with velocity $\vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

Solution:

a) In the northern hemisphere the angular velocity of the earth is given by $\vec{\boldsymbol{\omega}} = \omega\sin\lambda\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$. The Coriolis force acting on a particle that is moving along a longitude line towards the North Pole is given by

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}_{\perp} \times \vec{\mathbf{v}} = -2m(\omega \sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}) \times (-v\hat{\boldsymbol{\theta}}) = 2m\omega \sin\lambda v \hat{\boldsymbol{\phi}}\,.$$

It points in the positive $\hat{\phi}$-direction, which is east.

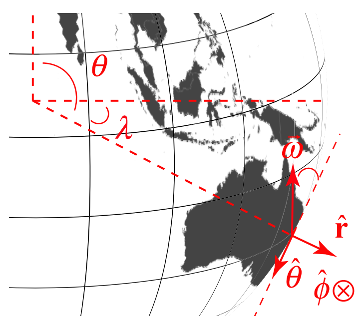
b) The Coriolis force acting on a particle that is moving along a longitude line away from the North Pole is given by

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}_{\perp} \times \vec{\mathbf{v}} = -2m(\omega \sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}) \times (v\hat{\boldsymbol{\theta}}) = -2m\omega \sin\lambda v \hat{\boldsymbol{\phi}}\,.$$

It points due west, in the negative $\hat{\phi}$-direction.

Example 8: Direction of Coriolis Force in Southern Hemisphere

Consider a particle moving in the southern hemisphere at south latitude $\lambda > 0$. Note that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$.



a) If the particle is moving along a longitude line away from the South Pole with velocity $\vec{\mathbf{v}} = -v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

b) If the particle is moving along a longitude line towards the South Pole with velocity $\vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

Solution:

a) In the southern hemisphere the angular velocity of the earth is given by $\vec{\boldsymbol{\omega}} = -\omega\sin\lambda\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$. The Coriolis force acting on an object moving along a longitude line away from the South Pole is

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} = -2m(-\omega \sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}) \times (-v\hat{\boldsymbol{\theta}}) = -2m\omega v \sin\lambda\hat{\boldsymbol{\phi}}\,.$$

It points in the negative $\hat{\phi}$-direction, which is west.

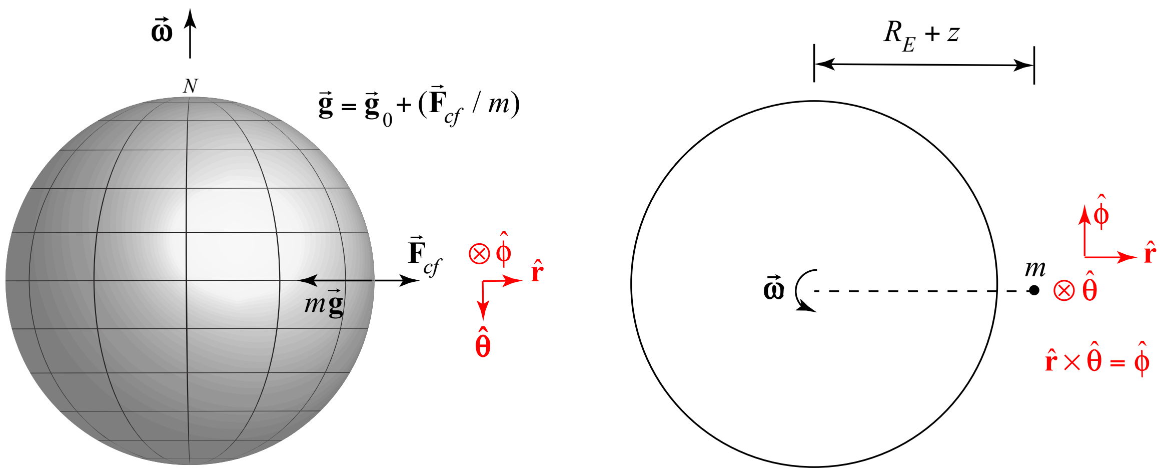
b) The Coriolis force on an object moving along a longitude line towards the South Pole is

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} = -2m(-\omega \sin\lambda \hat{\mathbf{r}} - \omega \cos\lambda \hat{\boldsymbol{\theta}}) \times (v\hat{\boldsymbol{\theta}}) = +2m\omega v \sin\lambda \hat{\boldsymbol{\phi}} ,$$

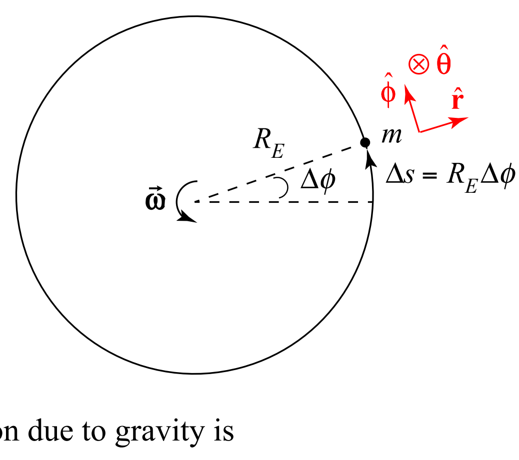
which is due east in the positive $\hat{\phi}$-direction.

Example 9: Tangential Deflection of a Freely Falling Object

An object is released from rest at a height $r_0 = R_E + z$ from the center of Earth directly above the equator. Choose a spherical coordinate system, as shown in the figure below.



In the figure on the right, north is pointing out of the plane of the figure, $\hat{\boldsymbol{\phi}}$ is pointing in the east direction, and the angular velocity of Earth is $\vec{\boldsymbol{\omega}} = -\omega\hat{\boldsymbol{\theta}}$. The problem is to determine the tangential deflection of the object due to the Coriolis force , $\Delta s = R_E \Delta\phi$



The apparent acceleration due to gravity is

$$\vec{\mathbf{g}} = \vec{\mathbf{g}}_0 + (\vec{\mathbf{F}}_{cf} / m) = (r\omega^2 - g_0)\hat{\mathbf{r}} = -g\hat{\mathbf{r}} \quad . \qquad (31.5.10)$$

where we set $\lambda = 0$ in Eq. (31.5.4). When the object is in free fall, the velocity is

$$\vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} \ . \tag{31.5.11}$$

The acceleration of the object is given by

$$\vec{\mathbf{a}} = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right]\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2}\right)\hat{\boldsymbol{\phi}} \ . \tag{31.5.12}$$

The Coriolis force acting on the object is

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} = -2m\left[-\omega\hat{\boldsymbol{\theta}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}\right)\right] = -2m\omega\frac{dr}{dt}\hat{\boldsymbol{\phi}} + 2m\omega r\frac{d\phi}{dt}\hat{\mathbf{r}} \ . \tag{31.5.13}$$

Newton's second law in the radial direction is then

$$m\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right] = m\left(-g + 2\omega r\frac{d\phi}{dt}\right) . \tag{31.5.14}$$

Newton's second law in the tangential direction is then

$$m\left(2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2}\right) = -2m\omega\frac{dr}{dt} \ . \tag{31.5.15}$$

(See Appendix 31.B.1 for derivation of acceleration.)

We shall make a few simplifying assumptions: (1) $R_E >> z$, therefore approximate $r \approx R_E$; (2) $\dfrac{d\phi}{dt} << \omega$, therefore the radial equation is approximately

$$\frac{d^2 r}{dt^2} = -g \ , \tag{31.5.16}$$

and the tangential equation is approximately

$$R_E\frac{d^2\phi}{dt^2} = -2\omega\frac{dr}{dt} \ . \tag{31.5.17}$$

We can solve these equations for the coordinates (r,θ) of the falling object. Integrating the radial equation noting that initially the object was at rest yields

$$\frac{dr}{dt} = -gt \quad . \tag{31.5.18}$$

Integrating again with initial position $r_0 = R_E + z$ yields

$$r = (R_E + z) - \frac{1}{2}gt^2 \quad . \tag{31.5.19}$$

Substitute Eqs. (31.5.18) and (31.5.19) into Eq. (31.5.17) and after rearranging terms

$$\frac{d^2\phi}{dt^2} = \frac{2\omega gt}{R_E} \quad . \tag{31.5.20}$$

Integration then yields

$$\frac{d\phi}{dt} = \frac{\omega gt^2}{R_E} \quad . \tag{31.5.21}$$

A second integration yields

$$\phi = \frac{\omega gt^3}{3R_E} \quad . \tag{31.5.22}$$

The object reaches Earth at time $t = t_f$ when $r(t_f) = r_E$. With these substitutions, Eq. (31.5.19) becomes

$$R_E = (R_E + z) - \frac{1}{2}gt_f^2 \quad , \tag{31.5.23}$$

which we can solve for t_f:

$$t_f = \left(\frac{2z}{g}\right)^{1/2} \quad . \tag{31.5.24}$$

The tangential displacement of the object when it hits Earth is then

$$\Delta s = R_E\Delta\phi = R_E\phi(t_f) = \frac{\omega gt_f^3}{3} = \frac{\omega g}{3}\left(\frac{2z}{g}\right)^{3/2} \quad . \tag{31.5.25}$$

31.6 Trajectories of a Particle in an Inertial and Rotating Frame

Consider an object that is moving at constant velocity in an inertial reference frame O. The trajectory of that object is a straight line. Now consider a platform that is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ that lies beneath that object such that the object passes over the center of the platform. Let O' denote the non-inertial reference frame fixed to the platform i.e. O' is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ with respect to O. Choose cylindrical coordinates (r,θ,z) in O'. Let $\vec{\mathbf{v}}' = (v_\theta\,\hat{\theta} + v_r\,\hat{\mathbf{r}})$ denote the velocity of the object along the trajectory in O'. (We are dropping the primes for coordinates and component functions in O' to simplify the notation). Note that when the object is moving inward in the inertial frame, $v_r < 0$ and $v_\theta < 0$, and when the object is moving outward, $v_r > 0$ and $v_\theta < 0$. In both cases the tangential velocity in the rotating frame is negative. The Coriolis force is given by

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}' = -2m\omega\,\hat{\mathbf{k}}\times(v_\theta\,\hat{\boldsymbol{\theta}} + v_r\,\hat{\mathbf{r}}) = 2m\omega v_\theta\hat{\mathbf{r}} - 2m\omega v_r\,\hat{\boldsymbol{\theta}} = F_{\text{cor},r}\hat{\mathbf{r}} + F_{\text{cor},\theta}\hat{\boldsymbol{\theta}} \quad (31.6.1)$$

Thus when the object is moving inward with $v_r < 0$ and $v_\theta < 0$, the $\hat{\boldsymbol{\theta}}$-component of the Coriolis force is positive, $F_{\text{cor},\theta} > 0$, and the radial component of the Coriolis force is negative $F_{\text{cor},r} < 0$. When the object is moving outward with $v_r > 0$ and $v_\theta < 0$, the $\hat{\boldsymbol{\theta}}$-component of the Coriolis force is negative, $F_{\text{cor},\theta} < 0$ and the radial component of the Coriolis force remains negative $F_{\text{cor},r} < 0$, gradually increasing in magnitude as $|v_\theta|$ gradually increases. There is also a centrifugal force in the radial direction

$$\vec{\mathbf{F}}_{cf} = -m(\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}) = -m(\omega\,\hat{\mathbf{k}}\times(\omega\,\hat{\mathbf{k}}\times r\hat{\mathbf{r}})) = -m(\omega\,\hat{\mathbf{k}}\times r\omega\,\hat{\boldsymbol{\theta}}) = mr\omega^2\hat{\mathbf{r}}\,. \quad (31.6.2)$$

In the inertial frame, O, the object is moving with a constant velocity, therefore $\vec{\mathbf{F}}^{total}_{\text{phy}} = \vec{\mathbf{0}}$. Newton's Second Law, $\vec{\mathbf{F}}^{total}_{phy} + \vec{\mathbf{F}}_{cor} + \vec{\mathbf{F}}_{cf} = m\vec{\mathbf{a}}'$, applied to the object in the rotating frame O' is then

$$(2m\omega v_\theta + mr\omega^2)\hat{\mathbf{r}} - 2m\omega v_r\,\hat{\boldsymbol{\theta}} = m\vec{\mathbf{a}}' \quad (31.6.3)$$

Recall that in polar coordinates, the expression for the acceleration of an object is

$$\vec{\mathbf{a}}' = (dv_r / dt - r(d\theta / dt)^2)\hat{\mathbf{r}} + (2v_r(d\theta / dt) + r(d^2\theta / dt^2))\hat{\boldsymbol{\theta}}\ , \quad (31.6.4)$$

where $v_r = dr / dt$ and $dv_r / dt = d^2r / dt^2$. (See Appendix 31.B for a derivation). The equations of notion in the rotating frame are

in the radial direction:

$$2\omega v_\theta + r\omega^2 = dv_r / dt - r(d\theta / dt)^2 \quad (31.6.5)$$

and in the tangential direction:

$$-2\omega v_r = 2v_r(d\theta/dt) + r(d^2\theta/dt^2) \ . \tag{31.6.6}$$

Let's consider the case in which the initial conditions are given by $(d\theta/dt)_0 = -\omega$ and $v_{r,0} = (dr/dt)_0 = v_{in}$. Then there is a unique solution to Eqs. (31.6.5) and (31.6.6) given by

$$d\theta/dt = -\omega \ . \tag{31.6.7}$$

Using that result in Eq. (31.6.5), implies $dv_r/dt = 0$: the radial component of the velocity in O' is constant. This is the condition that the radially component of the Coriolis force and the centrifugal force are equal to the centripetal acceleration. In Figure 31.20, we show the orbit in the two frames under these special conditions.

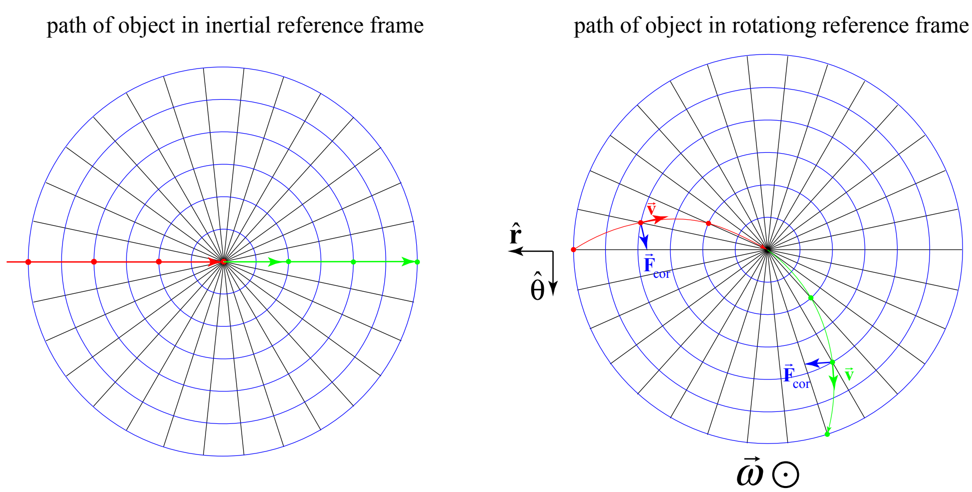


Figure 31.20

The object is moving across a rotating platform at a constant speed v_{in}. The object traverses the platform in time $T_{transit} = 2R/v_{in}$. In Figure 31.20, the platform is rotating with angular speed $\omega = 2\pi/5T_{transit}$ hence with period $T_{rot} = 2\pi/\omega = 5T_{transit}$. In the inertial reference frame, as the object travels $\Delta s = (1/3)R$ (the distance between two adjacent circles), the platform rotates $\Delta\theta_{platform} = 12^\circ$. During each of the these intervals, $\Delta t = (1/3)R/v_{in}$, in the reference frame rotating with the platform, the object appears to decrease it's angular position by $\Delta\theta_{object} = -12^\circ$.

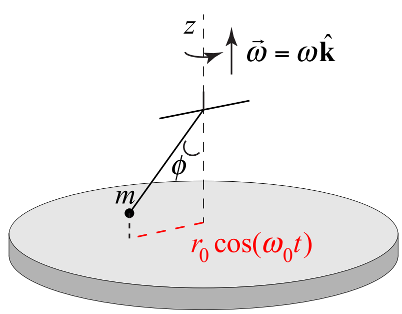
The velocity $\vec{\mathbf{v}}'$ of the object O' is no longer constant. The tangent line at any point on trajectory in O' (red line moving inward, green line moving outward) indicates the direction of the velocity $\vec{\mathbf{v}}'$. The direction of $\vec{\mathbf{v}}'$ at various points along the trajectory in O' is shown in Figure 13. Initially, in the frame O the object is moving radially inward.

Because the platform is rotating, an observer on the platform also observes that the particle is moving in the negative $\hat{\boldsymbol{\theta}}$-direction. Hence the velocity $\vec{\mathbf{v}}'$ at the initial position in O' has component inward and also in the negative $\hat{\boldsymbol{\theta}}$-direction. As the object moves inward in O', the $\hat{\boldsymbol{\theta}}$-component of the velocity becomes less negative indicating that there is a positive angular acceleration in the $\hat{\boldsymbol{\theta}}$-direction. The observer in O' attributes this angular acceleration to the Coriolis force $\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}'$, which is perpendicular to the velocity $\vec{\mathbf{v}}'$. As the object moves outward, the $\hat{\boldsymbol{\theta}}$-component of the velocity decreases (becomes more negative) indicating that there is a negative $\hat{\boldsymbol{\theta}}$-component to the acceleration.

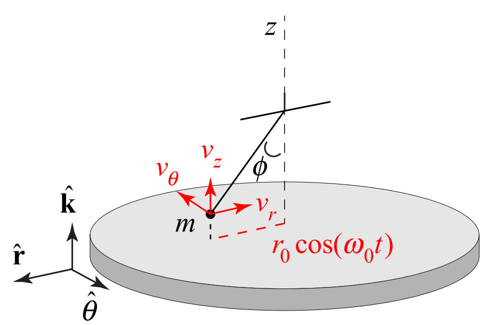
31.7 Simple Pendulum in Rotating Frames

Consider a simple pendulum of length l with a bob of mass m. In inertial space (non-rotating frame) the pendulum will undergo small oscillations with angular frequency $\omega_0 = \sqrt{g/l}$. If the pendulum is placed on a rotating platform or on the surface of Earth at latitude λ, and undergoes small oscillations, in the rotating frame the plane of oscillation will precess in the opposite direction of the rotation due to the Coriolis force.

Example 10 Pendulum on a Rotating Platform



inertial frame O



rotating non-inertial frame O'

Figure 31.21a **Figure 31.21b**

A simple pendulum consists of a bob of mass m at the end of a string of length l. Choose polar coordinates on the rotating platform. Suppose the bob is released from rest at a small angle ϕ_0 with respect to the vertical axis. In the frame O, the bob undergoes linear simple harmonic motion with distance from the center varying in time according to $\vec{\mathbf{r}}(t) = r(t)\hat{\mathbf{r}} + z(t)\hat{\mathbf{k}}$, where $\omega_0 = \sqrt{g/l}$, $r(t) = l\sin(\phi_0)\cos(\phi(t))$, $z(t) = l(1-\cos\phi(t))$, and $\phi(t) = \phi_0 \cos(\omega_0 t)$. Choose cylindrical coordinates (r,θ,z) in the rotating frame O' with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ with respect to O. The motion of the bob is no longer in the radial direction because the platform is rotating underneath the bob.

Let $\vec{\mathbf{T}}$ denote the tension force of the string on the bob. The effective gravity is

$$\begin{aligned}\vec{\mathbf{g}}_{eff} &= \vec{\mathbf{g}} - m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})) \\ &= -g\hat{\mathbf{k}} - m(\omega\hat{\mathbf{k}} \times (\omega\hat{\mathbf{k}} \times (r\hat{\mathbf{r}} + z\hat{\mathbf{k}})) \\ &= -g\hat{\mathbf{k}} - m(\omega\hat{\mathbf{k}} \times r\omega\hat{\boldsymbol{\theta}}) = -g\hat{\mathbf{k}} + mr\omega^2\hat{\mathbf{r}}\end{aligned} \quad . \tag{31.7.1}$$

Newton's Second Law on the bob is then

$$m\vec{\mathbf{a}}_{rot} = m\vec{\mathbf{g}}_{eff} + \vec{\mathbf{T}} - 2m(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}_{rot}) \ . \tag{31.7.2}$$

The velocity of the bob is given by

$$\vec{\mathbf{v}}_{rot} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dz}{dt}\hat{\mathbf{k}} \ . \tag{31.7.3}$$

The angular rotational velocity is given by $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$. The Coriolis force is then

$$\begin{aligned}\vec{\mathbf{F}}_{cor} &= -2m((\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}_{rot}) \\ &= -2m\left[\omega\hat{\mathbf{k}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dz}{dt}\hat{\mathbf{k}}\right)\right] \\ &= -2m\left(\omega\frac{dr}{dt}\hat{\boldsymbol{\theta}} - \omega r\frac{d\theta}{dt}\hat{\mathbf{r}}\right)\end{aligned} \ . \tag{31.7.4}$$

The acceleration of an object in the plane is given by

$$\vec{\mathbf{a}}' = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\boldsymbol{\theta}} \ . \tag{31.7.5}$$

(For the derivation of the acceleration see Appendix 31.B.1.)

For small oscillations, the contribution of the centrifugal force is very small which we shall neglect. Therefore Newton's Second Law in the $\hat{\boldsymbol{\theta}}$-direction in the plane is thus

$$m\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) = -2m\omega\frac{dr}{dt} \ . \tag{31.7.6}$$

One possible solution occurs when $d\theta / dt = \text{constant}$, hence $d^2\theta / dt^2 = 0$. Then Eq. (31.7.6) becomes

$$\frac{d\theta}{dt} = -\omega \sin\lambda \ . \tag{31.7.7}$$

The pendulum precesses in the negative $\hat{\boldsymbol{\theta}}$-direction. With a period

$$T = \frac{2\pi}{|d\theta / dt|} = \frac{2\pi}{\omega} \tag{31.7.8}$$

Example 11: Foucault Pendulum on Earth.

The analysis for a pendulum of length l with a bob of mass m that is located on the surface of Earth at latitude λ and undergoing small oscillations is slightly more complicated due to the variation of the Coriolis force with latitude. In this case the plane of oscillation of the pendulum will precess in a plane that is tangential to the surface of Earth with a period that depends on the latitude.

The analysis will be very similar to the analysis on the rotating platform. On the tangent plane, choose local cylindrical coordinates (r,θ,z) with unit vectors $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\mathbf{k}})$ where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors that lie in the tangent plane, and $\hat{\mathbf{k}}$ points perpendicular to the tangent plane. (Note this is not a spherical coordinate system.)

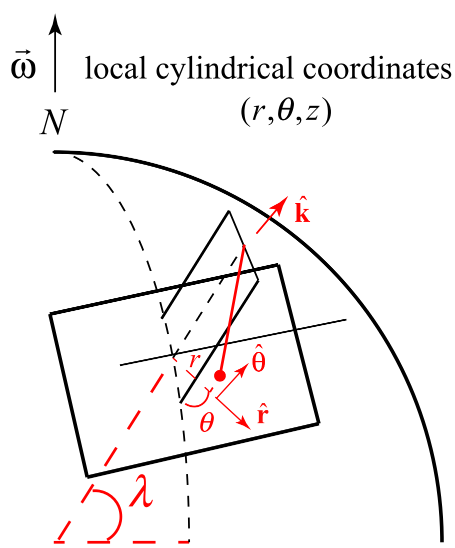


Figure 31. 22

For small angles of oscillation, the pendulum undergoes simple harmonic motion with the period $T = 2\pi\sqrt{l / g_{eff}}$, where (see Equation (31.5.4) is

$$\vec{\mathbf{g}}_{eff} = \vec{\mathbf{g}}_0 + (\vec{\mathbf{F}}_{cf} / m) \ . \tag{31.7.9}$$

Recall that the centrifugal force points radially outward form the rotation axis. Let $\hat{\boldsymbol{\rho}}$ be a unit vector pointing radially outward from the axis of rotation. Then the centrifugal force

$$(\vec{\mathbf{F}}_{cf} / m) = R_E \omega^2 \cos\lambda \hat{\boldsymbol{\rho}} \ . \tag{31.7.10}$$

Because this is proportional to ω^2, it will be smaller compared to the Coriolis force which is proportional to ω and we can neglect its effect.

Let $\vec{\mathbf{T}}$ denote the tension force of the string on the bob. Newton's Second Law on the bob is then

$$m\vec{\mathbf{a}}_{rot} = m\vec{\mathbf{g}}_0 + \vec{\mathbf{T}} - 2m(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}_{rot}) \ . \tag{31.7.11}$$

where we neglected the centrifugal force. The velocity of the bob in local coordinates is given by

$$\vec{\mathbf{v}}_{rot} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dz}{dt}\hat{\mathbf{k}} \ . \tag{31.7.12}$$

In local coordinates, the angular rotational velocity in the northern hemisphere is given by $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}_{||} + \vec{\boldsymbol{\omega}}_{\perp} = \vec{\boldsymbol{\omega}}_{||} + \omega\sin\lambda\hat{\mathbf{k}}$, where $\vec{\boldsymbol{\omega}}_{||}$ lies in the tangent plane and we have assumed that $\hat{\mathbf{k}}$ is approximately radially outward from the surface. Only $\vec{\boldsymbol{\omega}}_{\perp} = \omega\sin\lambda\hat{\mathbf{k}}$ contributes a non-negligible component of the Coriolis force in the tangential plane. The Coriolis force is

$$\begin{aligned}
\vec{\mathbf{F}}_{cor} &= -2m((\vec{\boldsymbol{\omega}}_{\perp} + \vec{\boldsymbol{\omega}}_{||}) \times \vec{\mathbf{v}}_{rot}) \\
&= -2m\left[\omega\sin\lambda\hat{\mathbf{k}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dz}{dt}\hat{\mathbf{k}}\right)\right] - 2m((\vec{\boldsymbol{\omega}}_{||} \times \vec{\mathbf{v}}_{rot}) \ . \qquad (31.7.13) \\
&= -2m\left(\omega\sin\lambda\frac{dr}{dt}\hat{\boldsymbol{\theta}} + \omega\sin\lambda r\frac{d\theta}{dt}\hat{\mathbf{r}}\right) - 2m((\vec{\boldsymbol{\omega}}_{||} \times \vec{\mathbf{v}}_{rot})
\end{aligned}$$

The last term in the Coriolis force has two terms

$$-2m((\vec{\boldsymbol{\omega}}_{||} \times \vec{\mathbf{v}}_{rot}) = -2m\left[\vec{\boldsymbol{\omega}}_{||} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right)\right] - 2m\left(\vec{\boldsymbol{\omega}}_{||} \times \frac{dz}{dt}\hat{\mathbf{k}}\right).$$

The first term, $-2m\left[\vec{\boldsymbol{\omega}}_{||} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right)\right]$, is perpendicular to the tangent plane. The second term, $-2m\left(\vec{\boldsymbol{\omega}}_{||} \times \frac{dz}{dt}\hat{\mathbf{k}}\right)$, lies in the tangent plane, but we shall neglect it because dz / dt is small.

The acceleration of an object in the local coordinates on the tangent plane is

$$\vec{\mathbf{a}}' = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\boldsymbol{\theta}} \ . \tag{31.7.14}$$

Therefore Newton's Second Law in the $\hat{\boldsymbol{\theta}}$-direction in the plane is thus

$$m\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) = -2m\omega\sin\lambda\frac{dr}{dt} \ . \tag{31.7.15}$$

One possible solution occurs when $\dfrac{d\theta}{dt} = \text{constant}$, hence $\dfrac{d^2\theta}{dt^2} = 0$. Then Eq. (31.7.15) becomes

$$\frac{d\theta}{dt} = -\omega\sin\lambda \ . \tag{31.7.16}$$

The pendulum precesses in the negative $\hat{\boldsymbol{\theta}}$-direction. With a period

$$T = \frac{2\pi}{\left|d\theta/dt\right|} = \frac{2\pi}{\omega\sin\lambda} = \frac{24\text{ h}}{\sin\lambda} \tag{31.7.17}$$

When $\lambda = 90^\circ$, $T = 24\text{ h}$. The significance of this result is that Earth rotates underneath the pendulum, so the plane of the pendulum's oscillation is not changing with respect to inertial space.

Appendix 31.A: Algebraic Derivation of Time Derivative of Vector in Rotating Reference Frame

The components of a vector $\vec{\mathbf{C}}(t)$ can be expressed in any coordinate system. However the time derivative of a vector will differ in inertial and rotating coordinate systems. Consider an inertial reference frame and a reference frame O' such that the origins and z and z' axes of O and O' coincide, and O' is rotating with **constant** angular frequency $\vec{\boldsymbol{\omega}} = (d\theta / dt)\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}}$ with respect to an inertial frame O.

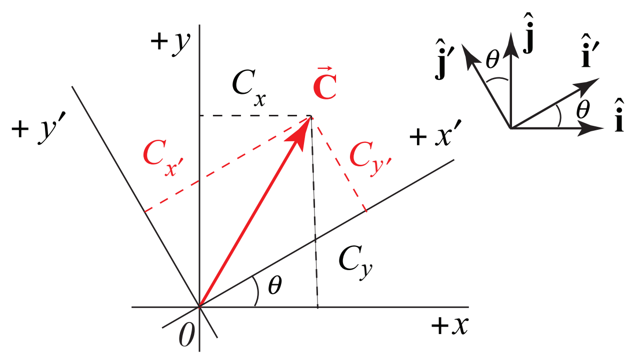


Figure 31.A.1

The vector expression for $\vec{\mathbf{C}}(t)$ in O is given by

$$\vec{\mathbf{C}}(t) = C_x(t)\hat{\mathbf{i}} + C_y(t)\hat{\mathbf{j}} \ , \tag{31.8.1}$$

and in O' by

$$\vec{\mathbf{C}}(t) = C_{x'}(t)\hat{\mathbf{i}}' + C_{y'}(t)\hat{\mathbf{j}}' \ . \tag{31.8.2}$$

The derivative of $\vec{\mathbf{C}}(t)$ in the inertial frame O is then

$$\left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{in} = \frac{d}{dt}(C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}}) \tag{31.8.3}$$

Because $C_x(t)\hat{\mathbf{i}} + C_y(t)\hat{\mathbf{j}} = C_{x'}(t)\hat{\mathbf{i}}' + C_{y'}(t)\hat{\mathbf{j}}'$, the derivative in the inertial frame is then

$$\begin{aligned}\left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{in} &= \frac{d}{dt}(C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}}) = \frac{d}{dt}(C_{x'}\hat{\mathbf{i}}' + C_{y'}\hat{\mathbf{j}}') \\ &= \frac{dC_{x'}}{dt}\hat{\mathbf{i}}' + C_{x'}\frac{d\hat{\mathbf{i}}'}{dt} + \frac{dC_{y'}}{dt}\hat{\mathbf{j}}' + C_{y'}\frac{d\hat{\mathbf{j}}'}{dt}\end{aligned} \tag{31.8.4}$$

Because the unit vectors $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$ are equal to

$$\begin{aligned}\hat{\mathbf{i}}' &= \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} \\ \hat{\mathbf{j}}' &= -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\end{aligned} \tag{31.8.5}$$

the derivatives are

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{i}}' &= \frac{d}{dt}(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = -\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}} = \frac{d\theta}{dt}\hat{\mathbf{j}}' \\ \frac{d}{dt}\hat{\mathbf{j}}' &= \frac{d}{dt}(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) = -\sin\theta\frac{d\theta}{dt}\hat{\mathbf{j}} - \cos\theta\frac{d\theta}{dt}\hat{\mathbf{i}} = -\frac{d\theta}{dt}\hat{\mathbf{i}}'\end{aligned} \tag{31.8.6}$$

The angular velocity is given by $\vec{\boldsymbol{\omega}} = \dfrac{d\theta}{dt}\hat{\mathbf{k}}$, therefore

$$\begin{aligned}\frac{d\hat{\mathbf{i}}'}{dt} &= \frac{d\theta}{dt}\hat{\mathbf{j}}' = \frac{d\theta}{dt}\hat{\mathbf{k}}\times\hat{\mathbf{i}}' = \vec{\boldsymbol{\omega}}\times\hat{\mathbf{i}}' \\ \frac{d\hat{\mathbf{j}}'}{dt} &= -\frac{d\theta}{dt}\hat{\mathbf{i}}' = \frac{d\theta}{dt}\hat{\mathbf{k}}\times\hat{\mathbf{j}}' = \vec{\boldsymbol{\omega}}\times\hat{\mathbf{j}}'\end{aligned} \tag{31.8.7}$$

Thus

$$\begin{aligned}&\left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{in} = \frac{dC_{x'}}{dt}\hat{\mathbf{i}}' + \frac{dC_{y'}}{dt}\hat{\mathbf{j}}' + \vec{\boldsymbol{\omega}}\times(C_{x'}\hat{\mathbf{i}}' + C_{y'}\hat{\mathbf{j}}') \\ &= \left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{rot} + \vec{\boldsymbol{\omega}}\times\vec{\mathbf{C}}\end{aligned} \tag{31.8.8}$$

Example 31.A.1: Let $\vec{\mathbf{r}}(t)$ be the position vector of an object, let $\vec{\mathbf{v}}(t) = (d\vec{\mathbf{r}}/dt)_{in}$ denote the velocity of the object in the inertial frame O, and let $\vec{\mathbf{v}}'(t) = (d\vec{\mathbf{r}}/dt)_{rot}$ denote the velocity of the object in the rotating frame O', Then the two velocities are related by

$$\mathbf{v} = \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}\ , \tag{31.8.9}$$

in agreement with Eq. (31.4.8).

Appendix 31.B Acceleration in Polar Coordinates

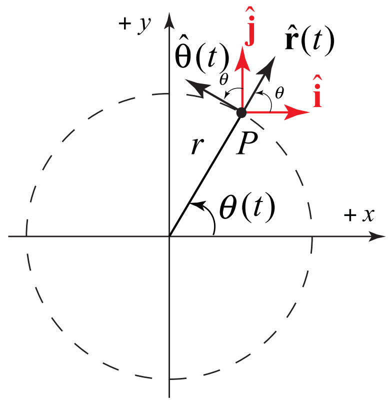


Figure 31.B.1

Let's now consider central motion in a plane that is non-circular. In polar coordinates, the key point is that the time derivative dr/dt of the position function r is no longer zero. The second derivative d^2r/dt^2 also may or may not be zero. In the following calculation we will drop all explicit references to the time dependence of the various quantities. The position vector is given by

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}\,. \tag{31.9.1}$$

Because $dr/dt \neq 0$, when we differentiate Eq. (31.9.1), we need to use the product rule

$$\vec{\mathbf{v}} = \frac{d\,\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\,\hat{\mathbf{r}}}{dt}\,. \tag{31.9.2}$$

At the point P, consider two sets of unit vectors $(\hat{\mathbf{r}}(t), \hat{\boldsymbol{\theta}}(t))$ and $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$, as shown in the figure above. The vector decomposition expression for $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is given by

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}\,, \tag{31.9.3}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\,. \tag{31.9.4}$$

The time derivative of the unit vectors are given by

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\,) = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\,. \tag{31.9.5}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}(\cos\theta(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}}\,) = -\frac{d\theta}{dt}\hat{\mathbf{r}}\,. \tag{31.9.6}$$

Substituting Eq. (31.9.5) into Eq. (31.9.2) yields

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}\,. \tag{31.9.7}$$

The velocity is no longer tangential but now has a radial component as well

$$v_r = \frac{dr}{dt}\,. \tag{31.9.8}$$

In order to determine the acceleration, we now differentiate Eq. (31.9.7), again using the product rule, which is now a little more involved:

$$\vec{\mathbf{a}} = \frac{d\,\vec{\mathbf{v}}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\,\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\,\hat{\boldsymbol{\theta}}}{dt}\,. \tag{31.9.9}$$

Now substitute Eqs. (31.9.5) and (31.9.6) for the time derivatives of the unit vectors in Eq. (31.9.9), and after collecting terms yields

$$\begin{aligned}\vec{\mathbf{a}} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}}\\ &= a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}\end{aligned}\,. \tag{31.9.10}$$

The radial and tangential components of the acceleration are now more complicated than then in the case of circular motion due to the non-zero derivatives of dr/dt and d^2r/dt^2. The radial component is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\,. \tag{31.9.11}$$

and the tangential component is

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\,. \tag{31.9.12}$$

The second term in the radial component of acceleration is called the centripetal acceleration. The first term in the tangential component of the acceleration, $2(dr/dt)(d\theta/dt)$ has a special name, the *Coriolis acceleration*,

$$a_{cor} = 2\frac{dr}{dt}\frac{d\theta}{dt}\,. \tag{31.9.13}$$