Chapter 28 Fluid Dynamics

Chapter 28 Fluid Dynamics

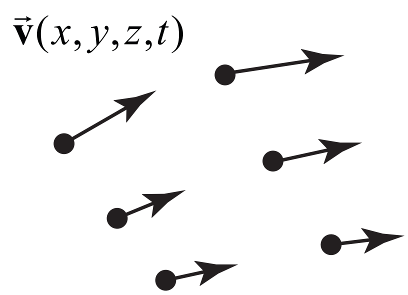
28.1 Ideal Fluids

An ideal fluid is a fluid that is incompressible and has no internal resistance to flow (zero viscosity). In addition, ideal fluid particles undergo no rotation about their center of mass (irrotational). An ideal fluid can flow in a circular pattern, but the individual fluid particles are irrotational. Real fluids exhibit all of these properties to some degree, but we shall often model fluids as ideal in order to approximate the behavior of real fluids. When we do so, one must be extremely cautious in applying results associated with ideal fluids to non-ideal fluids. For a non-ideal fluid, the differential equations describing the motion of the fluid are quite complicated and beyond the scope of this discussion.

28.2 Velocity Vector Field

The flow of a fluid like water consists of the movement of individual particles, (water molecules). These particles interact with each other through forces. Because the number of particles is very large applying the laws of motion to each individual particle in the fluid would be an extremely difficult computation problem.

To circumvent this problem, describe the state of a moving fluid by specifying the velocity of the fluid at each point in space and at each instant in time. Using Cartesian coordinates, a point in space-time is specified by the ordered triple (x, y, z) for the special location and the variable t to describe the instant in time. The distribution of fluid velocities is then given by the vector function $\vec{\mathbf{v}}(x, y, z, t)$. This represents the velocity of the fluid at the point (x, y, z) at the instant t. The quantity $\vec{\mathbf{v}}(x, y, z, t)$ is called the **velocity vector field**. It can be thought of at each instant in time as a collection of vectors, one for each point in space whose direction and magnitude describes the direction and magnitude of the velocity of the fluid at that point (Figure 28.1). This description of the velocity vector field of the fluid refers to fixed points in space and not to moving particles in the fluid.



Velocity vector field at time t

Figure 28.1: Velocity vector field for fluid flow at time t

The three functions $v_x(x,y,z,t)$, $v_y(x,y,z,t)$, and $v_z(x,y,z,t)$ to describe the components of the velocity vector field

$$\vec{\mathbf{v}}(x,y,z,t) = v_x(x,y,z,t)\,\hat{\mathbf{i}} + v_y(x,y,z,t)\,\hat{\mathbf{j}} + v_z(x,y,z,t)\hat{\mathbf{k}}\ . \qquad (28.2.1)$$

The three component functions are scalar fields. The velocity vector field is in general quite complicated for a three-dimensional time dependent flow.

For most flows, the velocity field varies in time. A **steady flow** is a model in which the velocity field does not change in time,

$$\frac{\partial \vec{\mathbf{v}}(x,y,z,t)}{\partial t} = \vec{\mathbf{0}} \qquad \text{(steady flow)}\ . \qquad (28.2.2)$$

For steady flows the velocity field is independent of time,

$$\vec{\mathbf{v}}(x,y,z) = v_x(x,y,z)\,\hat{\mathbf{i}} + v_y(x,y,z)\,\hat{\mathbf{j}} + v_z(x,y,z)\hat{\mathbf{k}} \quad \text{(steady flow)}\ . \qquad (28.2.3)$$

although the velocities may still vary in space (non-uniform steady flow).

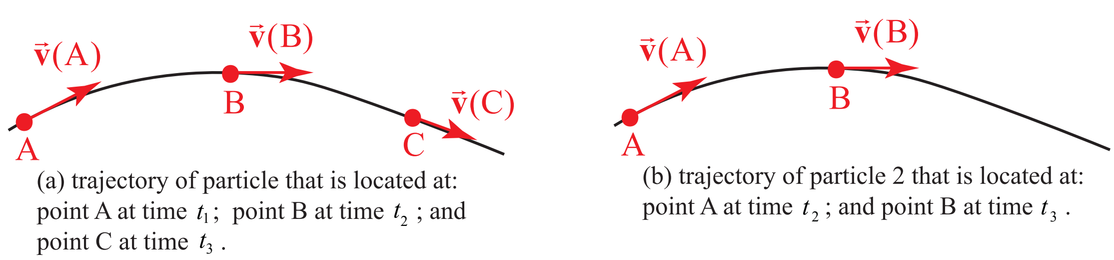


Figure 28.2: (a) trajectory of particle 1, (b) trajectory of particle 2

Let's trace the motion of particles in an ideal fluid undergoing steady flow during a succession of intervals of duration Δt. Consider particle 1 located at point A with coordinates (x_A, y_A, z_A). At the instant t_1, particle 1 has velocity $\vec{\mathbf{v}}(x_A, y_A, z_A) = \vec{\mathbf{v}}(A)$. During the time $[t_1, t_2]$, where $t_2 = t_1 + \Delta t_1$, the particle moves to point B arriving there at the instant t_2. At point B, the particle now has velocity $\vec{\mathbf{v}}(x_B, y_B, z_B) = \vec{\mathbf{v}}(B)$. During the next interval $[t_2, t_3]$, where $t_3 = t_2 + \Delta t$, particle 1 will move to point C arriving there at instant t_3, where it has velocity $\vec{\mathbf{v}}(x_C, y_C, z_C) = \vec{\mathbf{v}}(C)$. (Figure 28.2(a)). Because the flow has been assumed to be steady, at instant t_2, a different particle, particle 2, is now located at point A but it has the same velocity $\vec{\mathbf{v}}(x_A, y_A, z_A)$ as particle 1 had at point A and hence will arrive at point B at the end of the next interval, at the instant t_3 (Figure 28.2(b)). In this way every particle that lies on the trajectory that our first particle traces out in time will follow the same trajectory. This trajectory is called a *streamline*. The particles in the fluid will not have the same velocities at points along a streamline because we have not assumed that the velocity field is uniform.

A set of streamlines for an ideal fluid undergoing steady flow in which there are no sources or sinks for the fluid is shown in Figure 28.3.

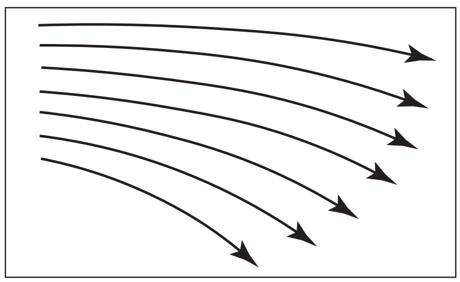


Figure 28.3: Set of streamlines for an ideal fluid flow

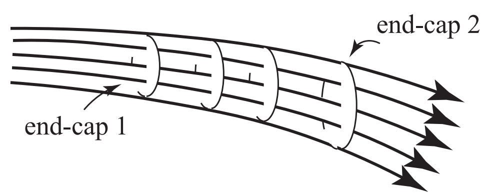


Figure 28.4: Flux Tube associated with set of streaml

28.3 Mass Continuity Equation

Let $\vec{\mathbf{v}}(x, y, z, t)$ denote the vector field associated with the fluid flow. Let $\rho(x, y, z, t)$ denote the density of the fluid at each point in space and at each instant in time. The density function is an example of a **scalar field** because there is only one number with appropriate units associated with each point in space at each instant in time.

A set of closely separated streamlines that form a *flow tube* are shown in Figure 28.4. The flow tube has two open surfaces (end-caps 1 and 2) of areas A_1 and A_2, respectively, that are perpendicular to the velocity of the fluid.

All fluid particles that enter end-cap 1 must follow their respective streamlines, therefore they all leave end-cap 2. If the streamlines that form the tube are sufficiently close together, then the velocity of the fluid in the vicinity of each end-cap surfaces can be assumed to be uniform.

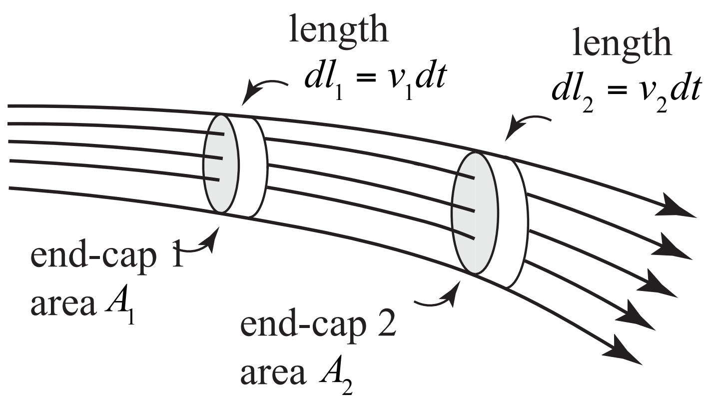


Figure 28.5: Mass flow through flux tube

Let v_1 denote the speed of the fluid near end-cap 1 and v_2 denote the speed of the fluid near end-cap 2. Let ρ_1 denote the density of the fluid near end-cap 1 and ρ_2 denote the density of the fluid near end-cap 2. The amount of mass that enters and leaves the tube in a time interval dt can be calculated as follows (Figure 28.5): consider a small volume of space of cross-sectional area A_1 and length $dl_1 = v_1 dt$ near end-cap 1. The mass that enters the tube in time interval dt is

$$dm_1 = \rho_1 dV_1 = \rho_1 A_1 dl_1 = \rho_1 A_1 v_1 dt \ . \tag{28.2.4}$$

In a similar fashion, consider a small volume of space of cross-sectional area A_2 and length $dl_2 = v_2 dt$ near end-cap 2. The mass that leaves the tube in the time interval dt is then

$$dm_2 = \rho_2 dV_2 = \rho_2 A_2 dl_2 = \rho_2 A_2 v_2 dt \ . \tag{28.2.5}$$

An equal amount of mass that enters end-cap 1 in the time interval dt must leave end-cap 2 in the same time interval, thus $dm_1 = dm_2$. Therefore using Eqs. (28.2.4) and (28.2.5), we have that $\rho_1 A_1 v_1 dt = \rho_2 A_2 v_2 dt$. Dividing through by dt implies that

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad \text{(steady flow)} \ . \tag{28.2.6}$$

Eq. (28.2.6) generalizes to any cross sectional area A of the thin tube, where the density is ρ, and the speed is v,

$$\rho A v = \text{constant} \quad \text{(steady flow)} \ . \tag{28.2.7}$$

Eq. (28.2.6) is referred to as the *mass continuity equation for steady flow*. For an incompressible fluid, Eq. (28.2.6) becomes

$$A_1 v_1 = A_2 v_2 \quad \text{(incompressable fluid, steady flow)} . \qquad (28.2.8)$$

Consider the steady flow of an incompressible fluid with streamlines and closed surface formed by a streamline tube shown in Figure 28.5. According to Eq. (28.2.8), when the spacing of the streamlines increases, the speed of the fluid must decrease. Therefore the speed of the fluid is greater entering end-cap 1 then when it is leaving end-cap 2. When representing fluid flow by streamlines, regions in which the streamlines are widely spaced have lower speeds than regions in which the streamlines are closely spaced.

28.4 Bernoulli's Principle

Consider the case of an ideal fluid that undergoes steady flow. Let $P(x, y, z, t)$ denote the pressure scalar field at each point in space and at each instant in time. The *equation of state* relates pressure, density, and speed of the flow at different points in the fluid.

A steady horizontal flow is shown in the overhead view in Figure 28.6. The flow is represented by streamlines along with a flow tube. Consider the motion of a fluid particle along one streamline passing through points A and B in Figure 28.6. The cross-sectional area of the flow tube at point A is less than the cross-sectional area of the flow tube at point B.

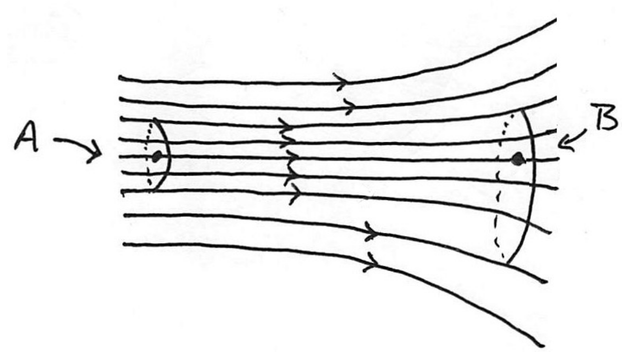


Figure 28.6 Overhead view of steady horizontal flow: in regions where spacing of the streamlines increases, the speed of the fluid must decrease

According to Eq. (28.2.8), the particle located at point A has a greater speed than the particle located at point B. Therefore a particle traveling along the streamline from point A to point B must decelerate. Because the streamline is horizontal, the force responsible is due to pressure differences in the fluid. Thus, for this steady horizontal flow in regions of lower speed there must be greater pressure than in regions of higher speed.

Now suppose the steady flow of the ideal fluid is not horizontal, with the y-axis representing the vertical direction. A side view of the streamlines and flow tube for this steady flow are shown in Figure 28.7.

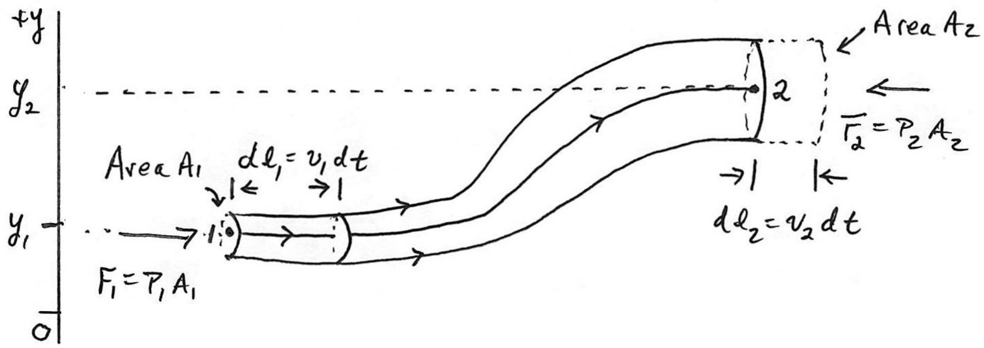


Figure 28.7: Side view of a non-horizontal steady flow

Define the system by the mass contained in the flow tube shown in Figure 28.7. The external forces acting on the system are due to the pressure acting at the two ends of the flow tube and the gravitational force. Consider a streamline passing through points 1 and 2 at opposite ends of the flow tube. Assume that the flow tube is narrow enough such that the velocity of the fluid is uniform on the cross-sectional areas of the tube at points 1 and 2. At point 1, denote the speed of a fluid particle by v_1, the cross-sectional area by A_1, the fluid pressure by P_1, and the height of the center of the cross-sectional area by y_1. At point 2, denote the speed of a fluid particle by v_2, the cross-sectional area by A_2, the fluid pressure by P_2, and the height of the center of the cross-sectional area by y_2.

At the left end of the flow, tube in a time interval dt, a particle at point 1 travels a distance $dl_1 = v_1 dt$. Therefore a small volume $dV_1 = A_1 dl_1 = A_1 v_1 dt$ of fluid is displaced at the right end of the flow tube. In a similar fashion, at particle at point 2, travels a distance $dl_2 = v_2 dt$. Therefore a small volume of fluid $dV_2 = A_2 dl_2 = A_2 v_2 dt$ is also displaced to the right in the flow tube during the time interval dt. Because the fluid is assumed to be incompressible, by Eq.(28.2.8), these volume elements are equal, $dV \equiv dV_1 = dV_2$.

There is a force of magnitude $F_1 = P_1 A_1$ in the direction of the flow arising from the fluid pressure at the left end of the tube acting on the mass element that enters the tube. The work done displacing the mass element is then

$$dW_1 = F_1 dl_1 = P_1 A_1 dl_1 = P_1 dV \ . \tag{28.3.1}$$

There is also a force of magnitude $F_2 = P_2 A_2$ in the direction opposing the flow arising from the fluid pressure at the right end of the tube. The work done opposing the displacement of the mass element leaving the tube is then

$$dW_1 = -F_2 dl_2 = -P_2 A_2 dl_2 = -P_2 dV \ . \tag{28.3.2}$$

Therefore the external work done by the force associated with the fluid pressure is the sum of the work done at each end of the tube

$$dW^{ext} = dW_1 + dW_2 = (P_1 - P_2)dV \ . \tag{28.3.3}$$

In a time interval dt, the work done by the gravitational force is equal to

$$dW^g = -dm\, g(y_2 - y_1) = -\rho dVg(y_2 - y_1) \ . \tag{28.3.4}$$

The assumption that the fluid is ideal means that there are no frictional losses due to viscosity. The change in the potential energy of the mass in the flow tube (the system) is then

$$dU = -W^g = \rho dVg(y_2 - y_1) \ . \tag{28.3.5}$$

At time t, the kinetic energy of the system is the sum of the kinetic energy of the small mass element of volume $dV = A_1 dl_1$ moving with speed v_1 and the rest of the mass in the flow tube. At time $t + dt$, the kinetic energy of the system is the sum of the kinetic energy of the small mass element of volume $dV = A_2 dl_2$ moving with speed v_2 and the rest of the mass in the flow tube. The change in the kinetic energy of the system is due to the mass elements at the two ends. Therefore

$$dK = \frac{1}{2}dm_2 {v_2}^2 - \frac{1}{2}dm_1 {v_1}^2 = \frac{1}{2}\rho dV({v_2}^2 - {v_1}^2) \ . \tag{28.3.6}$$

The work-energy theorem $dW^{ext} = dU + dK$ for system is then

$$(P_1 - P_2)dV = \frac{1}{2}\rho dV({v_2}^2 - {v_1}^2) + \rho g(y_2 - y_1)dV \ . \tag{28.3.7}$$

Divide each term in Eq. (28.3.7) by the volume dV and rearrange terms, yielding

$$P_1 + \rho g y_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \rho g y_2 + \frac{1}{2}\rho {v_2}^2 \ . \tag{28.3.8}$$

Because points 1 and 2 were arbitrarily chosen, drop the subscripts and write Eq. (28.3.8) as

$$P + \rho g y + \frac{1}{2}\rho v^2 = \text{constant} \quad \text{(ideal fluid, steady flow)} \ . \tag{28.3.9}$$

Eq. (28.3.9) is known as **Bernoulli's Equation**.

28.5 Worked Examples: Bernoulli's Equation

Example 28.1 Venturi Meter

Figure 28.8 shows a Venturi Meter, a device used to measure the speed of a fluid in a pipe. A fluid of density ρ_f is flowing through a pipe. A U-shaped tube partially filled with mercury of density ρ_{Hg} lies underneath the points 1 and 2.

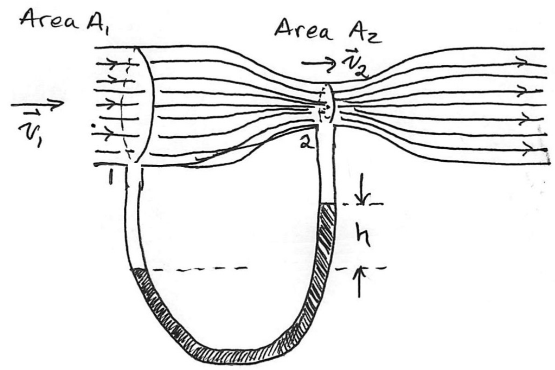


Figure 28.8: Venturi Meter

The cross-sectional areas of the pipe at points 1 and 2 are A_1 and A_2 respectively. Determine an expression for the flow speed at the point 1 in terms of the cross-sectional areas A_1 and A_2, and the difference in height h of the liquid levels of the two arms of the U-shaped tube.

Solution:

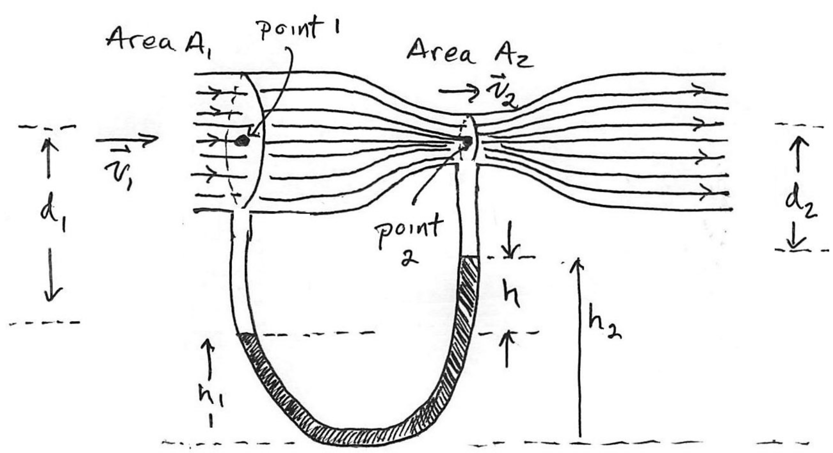


Figure 28.8: Coordinate system for Venturi tube

We shall assume that the pressure and speed are constant in the cross-sectional areas A_1 and A_2. We also assume the fluid is incompressible so the density ρ_f is constant throughout the tube. The two points 1 and 2 lie on the streamline passing through the midpoint of the tube so they are at the same height. Using $y_1 = y_2$ in Eq. (28.3.8), the pressure and flow speeds at the two points 1 and 2 are related by

$$P_1 + \frac{1}{2}\rho_f v_1^{\,2} = P_2 + \frac{1}{2}\rho_f v_2^{\,2} \ . \tag{28.3.10}$$

We can rewrite Eq. (28.3.10) as

$$P_1 - P_2 = \frac{1}{2}\rho_f (v_2^{\,2} - v_1^{\,2}) \ . \tag{28.3.11}$$

Let h_1 and h_2 denote the heights of the liquid level in the arms of the U-shaped tube directly beneath points 1 and 2 respectively. Pascal's Law relates the pressure difference between the two arms of the U-shaped tube according to in the left arm of the U-shaped tube according to

$$P_{bottom} = P_1 + \rho_f g d_1 + \rho_{Hg} g h_1 \ . \tag{28.3.12}$$

In a similar fashion, the pressure at point 2 is given by

$$P_{bottom} = P_2 + \rho_f g d_2 + \rho_{Hg} g h_2 \ . \tag{28.3.13}$$

Therefore, setting Eq. (28.3.12) equal to Eq. (28.3.13), we determine that the pressure difference on the two sides of the U-shaped tube is

$$P_1 - P_2 = \rho_f g(d_2 - d_1) + \rho_{Hg} g(h_2 - h_1). \tag{28.3.14}$$

From Figure 28.8, $d_2 + h_2 = d_1 + h_1$, therefore $d_2 - d_1 = h_1 - h_2 = -h$. We can rewrite Eq. (28.3.14) as

$$P_1 - P_2 = (\rho_{Hg} - \rho_f) g h. \tag{28.3.15}$$

Substituting Eq. (28.3.11) into Eq. (28.3.15) yields

$$\frac{1}{2}\rho_f (v_2^{\,2} - v_1^{\,2}) = (\rho_{Hg} - \rho_f) g h \ . \tag{28.3.16}$$

The mass continuity condition (Eq.(28.2.8)) implies that $v_2 = (A_1 / A_2) v_1$ and so we can rewrite Eq. (28.3.16) as

$$\frac{1}{2}\rho_f((A_1 / A_2)^2 - 1)v_1^{\,2} = (\rho_{Hg} - \rho_f)gh \ . \qquad (28.3.17)$$

We can now solve Eq. (28.3.17) for the speed of the flow at point 1;

$$v_1 = \sqrt{\frac{2(\rho_{Hg} - \rho_f)gh}{\rho_f((A_1 / A_2)^2 - 1)}} \ . \qquad (28.3.18)$$

Example 28.2 Water Pressure

A cylindrical water tower of diameter $3.0\,\text{m}$ supplies water to a house. The level of water in the water tower is $35\,\text{m}$ above the point where the water enters the house through a pipe that has an inside diameter $5.1\,\text{cm}$. The intake pipe delivers water at a maximum rate of $2.0\times10^{-3}\ \text{m}^3\cdot\text{s}^{-1}$. The pipe is connected to a narrower pipe leading to the second floor that has an inside diameter $2.5\,\text{cm}$. What is the pressure and speed of the water in the narrower pipe at a point that is a height $5.0\ \text{m}$ above the level where the pipe enters the house?

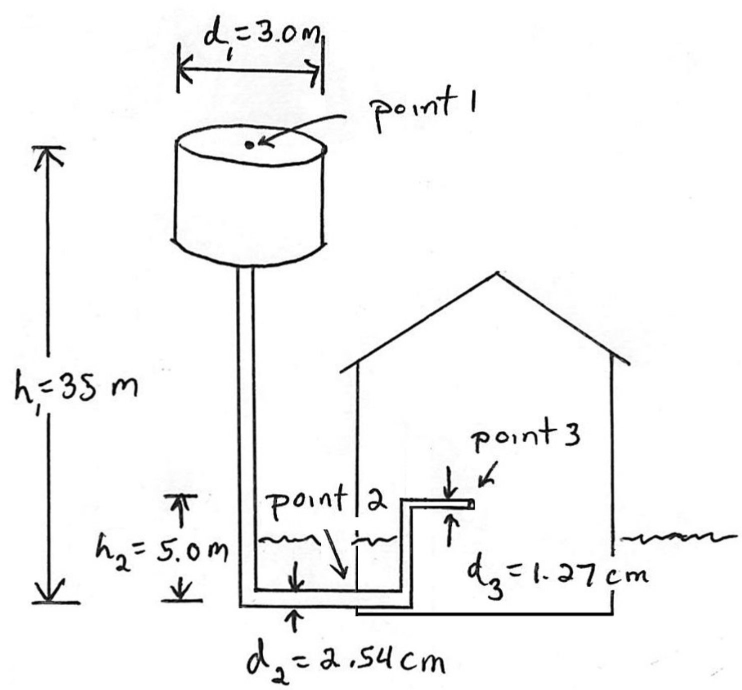


Figure 28.9: Example 28.2 (not to scale)

Solution: We shall assume that the water is an ideal fluid and that the flow is a steady flow and that the level of water in the water tower is constantly maintained. Let's choose three points, point 1 at the top of the water in the tower, point 2 where the water just enters the house, and point 3 in the narrow pipe at a height $h_2 = 5.0\text{ m}$ above the level where the pipe enters the house.

We begin by applying Bernoulli's Equation to the flow from the water tower at point 1, to where the water just enters the house at point 2. Bernoulli's equation (Eq. (28.3.8)) tells us that

$$P_1 + \rho g y_1 + \frac{1}{2}\rho v_1^{\,2} = P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^{\,2} \ . \qquad (28.3.19)$$

We assume that the speed of the water at the top of the tower is negligibly small due to the fact that the water level in the tower is maintained at the same height and so we set $v_1 = 0$. The pressure at point 2 is then

$$P_2 = P_1 + \rho g (y_1 - y_2) - \frac{1}{2}\rho v_2^{\,2} \ . \qquad (28.3.20)$$

In Eq. (28.3.20) we use the value for the density of water $\rho = 1.0\times10^3\text{ kg}\cdot\text{m}^{-3}$, the change in height is $(y_1 - y_2) = 35\text{ m}$, and the pressure at the top of the water tower is $P_1 = 1\text{ atm}$. The rate R that the water flows at point 1 satisfies $R = A_1 v_1 = \pi(d_1/2)^2 v_1$. Therefore, the speed of the water at point 1 is

$$v_1 = \frac{R}{\pi(d_1/2)^2} = \frac{2.0\times10^{-3}\text{ m}^3\cdot\text{s}^{-1}}{\pi(1.5\text{ m})^2} = 2.8\times10^{-4}\text{ m}\cdot\text{s}^{-1}, \qquad (28.3.21)$$

which is negligibly small and so we are justified in setting $v_1 = 0$. Similarly the speed of the water at point 2 is

$$v_2 = \frac{R}{\pi(d_2/2)^2} = \frac{2.0\times10^{-3}\text{ m}^3\cdot\text{s}^{-1}}{\pi(2.5\times10^{-2}\text{ m})^2} = 1.0\text{ m}\cdot\text{s}^{-1}, \qquad (28.3.22)$$

We can substitute Eq. (28.3.21) into Eq. (28.3.22), yielding

$$v_2 = (d_1^{\,2} / d_2^{\,2}) v_1, \qquad (28.3.23)$$

a result which we will shortly find useful. Therefore the pressure at point 2 is

$$P_2 = 1.01\times10^5\text{ Pa} + (1.0\times10^3\text{ kg}\cdot\text{m}^{-3})(9.8\text{ m}\cdot\text{s}^{-2})(35\text{ m}) - \frac{1}{2}(1.0\times10^3\text{ kg}\cdot\text{m}^{-3})(1.0\text{ m}\cdot\text{s}^{-1})^2$$

$$P_2 = 1.01\times10^5\text{ Pa} + 3.43\times10^5\text{ Pa} - 5.1\times10^2\text{ Pa} = 4.4\times10^5\text{ Pa}.$$

(28.3.24)

The dominant contribution is due to the height difference between the top of the water tower and the pipe entering the house. The quantity $(1/2)\rho v_2^{\,2}$ is called the ***dynamic pressure*** due to the fact that the water is moving**.** The amount of reduction in pressure due to the fact that the water is moving at point 2 is given by

$$\frac{1}{2}\rho v_2^2 = \frac{1}{2}(1.0\times 10^3 \text{ kg}\cdot\text{m}^{-3})(1.0 \text{ m}\cdot\text{s}^{-1})^2 = 5.1\times 10^3 \text{ Pa}, \qquad (28.3.25)$$

which is much smaller than the contributions from the other two terms.

We now apply Bernoulli's Equation to the points 2 and 3,

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 = P_3 + \frac{1}{2}\rho v_3^2 + \rho g y_3 \ . \qquad (28.3.26)$$

Therefore the pressure at point 3 is

$$P_3 = P_2 + \frac{1}{2}\rho(v_2^2 - v_3^2) + \rho g(y_2 - y_3) \ . \qquad (28.3.27)$$

The change in height $y_2 - y_3 = -5.0 \text{ m}$. The speed of the water at point 3 is

$$v_3 = \frac{R}{\pi(d_3/2)^2} = \frac{2.0\times 10^{-3} \text{ m}^3\cdot\text{s}^{-1}}{\pi(1.27\times 10^{-2} \text{ m})^2} = 3.9 \text{ m}\cdot\text{s}^{-1}, \qquad (28.3.28)$$

Then the pressure at point 3 is

$$\begin{aligned} P_3 &= (4.4\times 10^5 \text{ Pa}) + \frac{1}{2}(1.0\times 10^3 \text{ kg}\cdot\text{m}^{-3})((1.0 \text{ m}\cdot\text{s}^{-1})^2 - (3.9 \text{ m}\cdot\text{s}^{-1})^2) \\ &\quad -(1.0\times 10^3 \text{ kg}\cdot\text{m}^{-3})(9.8 \text{ m}\cdot\text{s}^{-2})(5.0 \text{ m}) \\ &= (4.4\times 10^5 \text{ Pa}) - (7.1\times 10^3 \text{ Pa}) - 4.9\times 10^4 \text{ Pa} \\ &= 3.8\times 10^5 \text{ Pa} \end{aligned} \ . \qquad (28.3.29)$$

Because the speed of the water at point 3 is much greater than at point 2, the dynamic pressure contribution at point 3 is much larger than at point 2.

28.6 Laminar and Turbulent Flow

28.6.1 Introduction

During the flow of a fluid, different layers of the fluid may be flowing at different speeds relative to each other, one layer sliding over another layer. For example consider a fluid flowing in a long cylindrical pipe. For slow velocities, the fluid particles move along lines parallel to the wall. Far from the entrance of the pipe, the flow is steady (fully developed). This steady flow is called *laminar flow*. The fluid at the wall of the pipe is at rest with respect to the pipe. This is referred to as the *no-slip condition* and is experimentally holds for all points in which a fluid is in contact with a wall. The speed of the fluid increases towards the interior of the pipe reaching a maximum, $v_{\max}$, at the center. The velocity profile across a cross section of the pipe exhibiting fully developed flow is shown in Figure 28.10. This parabolic velocity profile has a non-zero *velocity gradient* that is normal to the flow.

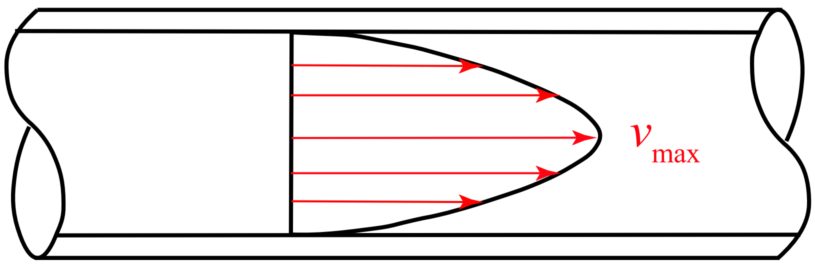


Figure 28.10 Steady laminar flow in a pipe with a non-zero velocity gradient

28.6.2 Viscosity

Due to the cylindrical geometry of the pipe, cylindrical layers of fluid are sliding with respect to one another resulting in tangential forces between layers. The tangential force per area is called a *shear stress*. The *viscosity* of a fluid is a measure of the resistance to this sliding motion of one layer of the fluid with respect to another layer. A *perfect fluid* has no tangential forces between layers. A fluid is called *Newtonian* if the shear forces per unit area are proportional to the velocity gradient. For a Newtonian fluid undergoing laminar flow in the cylindrical pipe, the shear stress, σ_S, is given by

$$\sigma_S = \eta \frac{dv}{dr} , \qquad (28.3.30)$$

where η is the constant of proportionality and is called the *absolute viscosity*, r is the radial distance form the central axis of the pipe, and dv/dr is the velocity gradient normal to the flow.

The SI units for viscosity are $\text{poise} = 10^{-1}\ \text{Pa}\cdot\text{s}$. Some typical values for viscosity for fluids at specified temperatures are given in Table 1.

Table 1: Coefficients of absolute viscosity

fluid	Coefficient of absolute viscosity η
oil	$1-10$ poise
Water at 0°	1.79×10^{-2} poise
Water at 100°	0.28×10^{-2} poise
Air at 20°	1.81×10^{-4} poise

At a certain flow rate, this resistance suddenly increases and the fluid particles no longer follow straight lines but appear to move randomly although the average motion is still along the axis of the pipe. This type of flow is called *turbulent flow*. Osbourne Reynolds was the first to experimentally measure these two types of flow. He was able to characterize the transition between these two types of flow by a parameter called the *Reynolds number* that depends on the average velocity of the fluid in the pipe, the diameter, and the viscosity of the fluid. The transition point between flows corresponds to a value of the Reynolds number that is associated with a sudden increase in the friction between layers of the fluid. Much after Reynolds initial observations, it was experimentally noted that a small disturbance in the laminar flow could rapidly grow and produce turbulent flow.

Example 28.3 Couette Flow

Consider the flow of a Newtonian fluid between two very long parallel plates, each plate of width w, length s, and separated by a distance d. The upper plate moves with a constant relative speed v_0 with respect to the lower plate, (Figure 28.11).

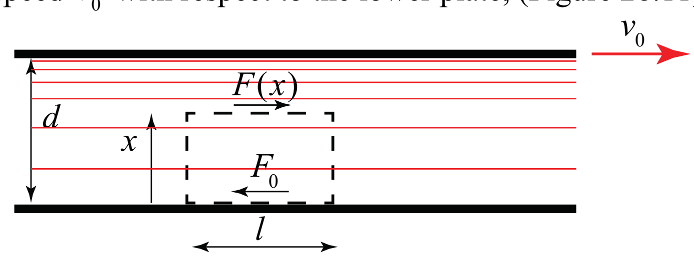


Figure 28.11 Laminar flow between two plates moving with relative speed v_0

Choose a reference frame in which the lower plate, located on the plane at $x = 0$, is at rest. Choose a volume element of length l and cross sectional area A, with one side in contact with the plate at rest, and the other side located a distance x from the lower plate. The velocity gradient in the direction normal to the flow is dv/dx. The shear force on the volume element due to the fluid above the element is given by

$$F(x) = \eta A \frac{dv}{dx} \quad . \qquad (28.3.31)$$

The shear force is balanced by the shear force F_0 of the lower plate on the element, such that $F(x) = F_0$. Hence

$$F_0 = \eta A \frac{dv}{dx} \ . \tag{28.3.32}$$

The velocity of the fluid at the lower plate is zero. The integral version of this differential equation is then

$$\frac{1}{\eta A} \int_{x'=0}^{x'=x} F_0 \, dx' = \int_{v'=0}^{v'=v(x)} dv' \ . \tag{28.3.33}$$

Integration yields

$$\frac{F_0}{\eta A} x = v(x) \ . \tag{28.3.34}$$

The velocity of the fluid at the upper plate is v_0, therefore the constant shear stress is given by

$$\frac{F_0}{A} = \frac{\eta v_0}{d} \ , \tag{28.3.35}$$

hence the velocity profile is

$$v(x) = \frac{v_0}{d} x \ . \tag{28.3.36}$$

This type of flow is known as *Couette flow*.

Example 28.4 Laminar flow in a cylindrical pipe.

Let's consider a long cylindrical pipe of radius r_0 in which the fluid undergoes laminar flow with each fluid particle moves in a line parallel to the pipe axis. Choose a cylindrical volume element of length dl and radius r, centered along the pipe axis as shown in Figure 28.12. There is a pressure drop $dp < 0$ over the length of the volume element resulting in forces on each end cap. Denote the force on the left end cap by $F_L = p / A$ and the force on the right end cap by $F_R = (p + dp) / A$ on the right end cap, where $A = \pi r^2$ is the cross sectional area of the end cap.

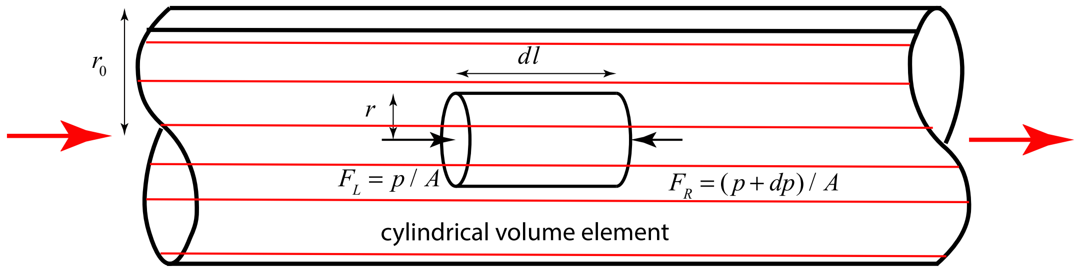


Figure 28.12 Volume element for steady laminar flow in a pipe

The forces on the volume element sum to zero and are due to the pressure difference and the shear stress; hence

$$F_L - F_R + \sigma_S 2\pi r dl = 0 \ . \tag{28.3.37}$$

Using our Newtonian model for the fluid (Eq. (28.3.30) and expressing the force in terms of pressure, Eq. (28.3.37) becomes

$$\frac{dp}{2\eta dl} r = \frac{dv}{dr} \ . \tag{28.3.38}$$

Eq. (28.3.38) can be integrated by the method of separation of variables with boundary conditions $v(r=0) = v_{\max}$ and $v(r = r_0) = 0$. (Recall that for laminar flow of a Newtonian fluid the velocity of a fluid is always zero at the surface of a solid.)

$$\frac{dp}{2\eta dl} \int_{r'=r}^{r'=r_0} r' \, dr' = \int_{v'=v(r)}^{v'(r=r_0)=0} dv' \ . \tag{28.3.39}$$

Integration then yields

$$v(r) = -\frac{dp}{4\eta dl}(r_0^{\ 2} - r^2) \ . \tag{28.3.40}$$

Recall that the pressure drop $dp < 0$. The maximum velocity at the center is then

$$v_{\max} = v(r=0) = -\frac{dp}{4\eta dl} r_0^{\ 2} \ . \tag{28.3.41}$$

To determine the flow rate through the pipe, choose a ring of radius r and thickness r, oriented normal to the flow. The flow through the ring is then

$$v(r) 2\pi r dr = -\frac{dp\pi}{2\eta dl}(r_0^{\ 2} - r^2) r dr \ . \tag{28.3.42}$$

Integrating over the cross sectional area of the pipe yields

$$Q = \int_{r=0}^{r=r_0} v(r) 2\pi r \, dr$$

$$Q = -\frac{dp\pi}{2\eta dl} \int_{r=0}^{r=r_0} (r_0^2 - r^2) r \, dr = -\frac{dp\pi}{2\eta dl} (r_0^2 r^2 / 2 - r^4 / 4) \Big|_{r=0}^{r=r_0} = \frac{\pi r_0^4}{8\eta dl} |dp| \qquad (28.3.43)$$

The average velocity is then

$$v_{ave} = \frac{Q}{\pi r_0^2} = -\frac{dp}{8\eta dl} r_0^2 \qquad (28.3.44)$$

Notice that the pressure difference and the volume flow rate are related by

$$|dp| = \frac{8\eta dl}{\pi r_0^4} Q \qquad (28.3.45)$$

which is equal to one half the maximum velocity at the center of the pipe. We can rewrite Eq. (28.3.45) in terms of the average velocity as

$$|dp| = \frac{8\eta dl}{\pi r_0^4} Q = \frac{64\eta dl}{v_{ave} 2d^2} v_{ave}^2 \qquad (28.3.46)$$

where $d = 2r_0$ is the diameter of the pipe. For a pipe of length l and pressure difference Δp, the *head loss* in a pipe is defined as the ratio

$$h_f = \frac{|\Delta p|}{\rho g} = \frac{64}{(\rho v_{ave} d / \eta)} \frac{v_{ave}^2}{2g} \frac{l}{d} \, , \qquad (28.3.47)$$

where we have extended Eq. (28.3.46) for the entire length of the pipe. Head loss is also written in terms of a loss coefficient k according to

$$h_f = k \frac{v_{ave}^2}{2g} \, , \qquad (28.3.48)$$

For a long straight cylindrical pipe, the loss coefficient can be written in terms of a factor f times an equivalent length of the pipe

$$k = f \frac{l}{d} \, . \qquad (28.3.49)$$

The factor f can be determined by comparing Eqs. (28.3.47)-(28.3.49) yielding

$$f = \frac{64}{(\rho v_{ave} d / \eta)} = \frac{64}{\mathrm{Re}} \ , \tag{28.3.50}$$

where Re is the Reynolds number and is given by

$$\mathrm{Re} = \rho v_{ave} d / \eta \ . \tag{28.3.51}$$