Chapter 22 Three Dimensional Rotations and Gyroscopes

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Chapter 22 Three Dimensional Rotations and Gyroscopes

Hypothesis: The earth, having once received a rotational movement around an axis, which agrees with its axis on the figure or only differs from it slightly, will always conserve this uniform movement, and its axis of rotation will always remain the same and will be directed toward the same points of the sky, unless the earth should be subjected to external forces which might cause some change either in the speed of rotational movement or in the position of the axis of rotation.[1]

Leonhard Euler

22.1 Introduction to Three Dimensional Rotations

Most of the examples and applications we have considered concerned the rotation of rigid bodies about a fixed axis. However, there are many examples of rigid bodies that rotate about an axis that is changing its direction. A turning bicycle wheel, a gyroscope, the earth's precession about its axis, a spinning top, and a coin rolling on a table are all examples of this type of motion. These motions can be very complex and difficult to analyze. However, for each of these motions we know that if there a non-zero torque about a point S, then the angular momentum about S must change in time, according to the rotational equation of motion,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \tag{22.1.1}$$

We also know that the angular momentum about S of a rotating body is the sum of the orbital angular momentum about S and the spin angular momentum about the center of mass.

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_S^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}. \tag{22.1.2}$$

For fixed axis rotation the spin angular momentum about the center of mass is just

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\vec{\boldsymbol{\omega}}_{\text{cm}}. \tag{22.1.3}$$

where $\vec{\boldsymbol{\omega}}_{\text{cm}}$ is the angular velocity about the center of mass and is directed along the fixed axis of rotation.

22.1.1 Angular Velocity for Three Dimensional Rotations

[1] L. Euler, Recherches sur la precession des equinoxes et sur la nutation de l'axe de la terre (Research concerning the precession of the equinoxes and of the nutation of the earth's axis), *Memoires de l'academie des sciences de Berlin* 5, 1751, pp. 289-325

When the axis of rotation is no longer fixed, the angular velocity will no longer point in a fixed direction.

For an object that is rotating with angular coordinates $(\theta_x, \theta_y, \theta_z)$ *about each respective Cartesian axis, the* ***angular velocity*** *of an object that is rotating about each axis is defined to be*

$$\begin{aligned}\vec{\boldsymbol{\omega}} &= \frac{d\theta_x}{dt}\hat{\mathbf{i}} + \frac{d\theta_y}{dt}\hat{\mathbf{j}} + \frac{d\theta_z}{dt}\hat{\mathbf{k}} \\ &= \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}\end{aligned} \tag{22.1.4}$$

This definition is the result of a property of very small (infinitesimal) angular rotations in which the order of rotations does matter. For example, consider an object that undergoes a rotation about the x-axis, $\vec{\boldsymbol{\omega}}_x = \omega_x\hat{\mathbf{i}}$, and then a second rotation about the y-axis, $\vec{\boldsymbol{\omega}}_y = \omega_y\hat{\mathbf{j}}$. Now consider a different sequence of rotations. The object first undergoes a rotation about the y-axis, $\vec{\boldsymbol{\omega}}_y = \omega_y\hat{\mathbf{j}}$, and then undergoes a second rotation about the x-axis, $\vec{\boldsymbol{\omega}}_x = \omega_x\hat{\mathbf{i}}$. In both cases the object will end up in the same position indicated that $\vec{\boldsymbol{\omega}}_x + \vec{\boldsymbol{\omega}}_y = \vec{\boldsymbol{\omega}}_y + \vec{\boldsymbol{\omega}}_x$, a necessary condition that must be satisfied in order for a physical quantity to be a vector quantity.

Example 22.1 Angular Velocity of a Rolling Bicycle Wheel

A bicycle wheel of mass m and radius R rolls without slipping about the z-axis. An axle of length b passes through its center. The bicycle wheel undergoes two simultaneous rotations. The wheel circles around the z-axis with angular speed Ω and associated angular velocity $\vec{\boldsymbol{\Omega}} = \Omega_z\hat{\mathbf{k}}$ (Figure 22.1). Because the wheel is rotating without slipping, it is spinning about its center of mass with angular speed ω_{spin} and associated angular velocity $\vec{\boldsymbol{\omega}}_{\text{spin}} = -\omega_{\text{spin}}\hat{\mathbf{r}}$.

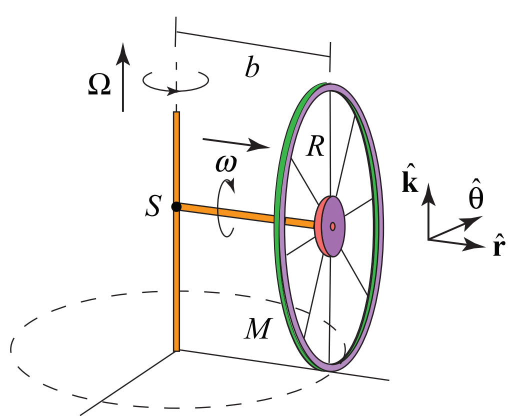


Figure 22.1 Example 22.1

The angular velocity of the wheel is the sum of these two vector contributions

$$\vec{\boldsymbol{\omega}} = \Omega\,\hat{\mathbf{k}} - \omega_{\text{spin}}\hat{\mathbf{r}}\ . \tag{22.1.5}$$

Because the wheel is rolling without slipping, $v_{\text{cm}} = b\Omega = \omega_{\text{spin}} R$ and so $\omega_{\text{spin}} = b\Omega / R$. The angular velocity is then

$$\vec{\boldsymbol{\omega}} = \Omega\,(\hat{\mathbf{k}} - (b/R)\hat{\mathbf{r}})\ . \tag{22.1.6}$$

The orbital angular momentum about the point S where the axle meets the axis of rotation (Figure 22.1), is then

$$\vec{\mathbf{L}}_S^{\text{orbital}} = bmv_{\text{cm}}\hat{\mathbf{k}} = mb^2\Omega\,\hat{\mathbf{k}}\ . \tag{22.1.7}$$

The spin angular momentum about the center of mass is more complicated. The wheel is rotating about both the z-axis and the radial axis. Therefore

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_z\Omega\,\hat{\mathbf{k}} + I_r\omega_{\text{spin}}(-\hat{\mathbf{r}})\,. \tag{22.1.8}$$

Therefore the angular momentum about S is the sum of these two contributions

$$\begin{aligned}\vec{\mathbf{L}}_S &= mb^2\Omega\,\hat{\mathbf{k}} + I_z\Omega\,\hat{\mathbf{k}} + I_r\omega_{\text{spin}}(-\hat{\mathbf{r}})\\ &= (mb^2\Omega + I_z\Omega)\,\hat{\mathbf{k}} - I_r(b\,\Omega/R)\hat{\mathbf{r}}.\end{aligned} \tag{22.1.9}$$

Comparing Eqs. (22.1.6) and (22.1.9), we note that the angular momentum about S is not proportional to the angular velocity.

22.2 Gyroscope

A toy gyroscope of mass m consists of a spinning flywheel mounted in a suspension frame that allows the flywheel's axle to point in any direction. One end of the axle is supported on a pylon a distance d from the center of mass of the gyroscope.

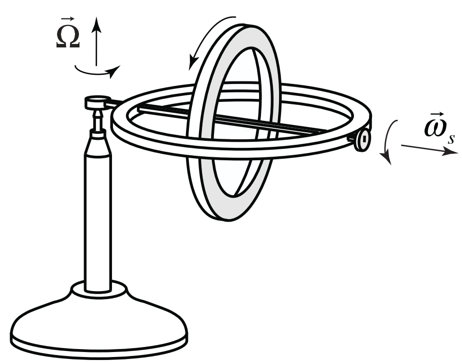


Figure 22.2a Toy Gyroscope

Choose polar coordinates so that the axle of the gyroscope flywheel is aligned along the r-axis and the vertical axis is the z-axis (Figure 22.2 shows a schematic representation of the gyroscope).

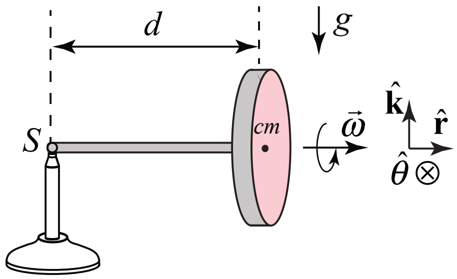


Figure 22.2 A toy gyroscope.

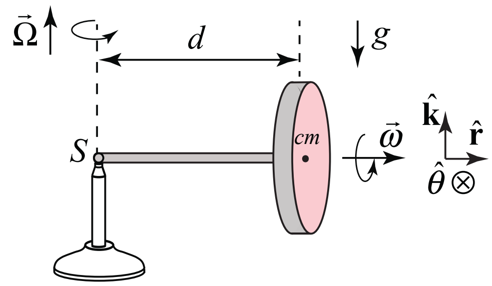


Figure 22.3 Angular rotations

The flywheel is spinning about its axis with a ***spin angular velocity***,

$$\vec{\boldsymbol{\omega}}_s = \omega_s \hat{\mathbf{r}} , \qquad (22.2.1)$$

where ω_s is the radial component and $\omega_s > 0$ for the case illustrated in Figure 22.2.

When we release the gyroscope it undergoes a very surprising motion. Instead of falling downward, the center of mass rotates about a vertical axis that passes through the contact point S of the axle with the pylon with a ***precessional angular velocity***

$$\vec{\boldsymbol{\Omega}} = \Omega_z \hat{\mathbf{k}} = \frac{d\theta}{dt}\hat{\mathbf{k}} , \qquad (22.2.2)$$

where $\Omega_z = d\theta / dt$ is the z-component and $\Omega_z > 0$ for the case illustrated in Figure 22.3. Therefore the angular velocity of the flywheel is the sum of these two contributions

$$\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}_s + \vec{\boldsymbol{\Omega}} = \omega_s \hat{\mathbf{r}} + \Omega_z \hat{\mathbf{k}} . \qquad (22.2.3)$$

We shall study the special case where the magnitude of the precession component $|\Omega_z|$ of the angular velocity is much less than the magnitude of the spin component $|\omega_s|$ of the spin angular velocity, $|\Omega_z| << |\omega_s|$, so that the magnitude of the angular velocity $|\vec{\boldsymbol{\omega}}| \simeq |\omega_s|$ and Ω_z and ω_s are nearly constant. These assumptions are collectively called the ***gyroscopic approximation***.

The force diagram for the gyroscope is shown in Figure 22.4. The gravitational force acts at the center of the mass and is directed downward, $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{k}}$. There is also a contact force, $\vec{\mathbf{F}}^c$, between the end of the axle and the pylon. It may seem that the contact force, $\vec{\mathbf{F}}^c$, has only an upward component, $\vec{\mathbf{F}}^v = F_z\,\hat{\mathbf{k}}$, but as we shall soon see there must also be a radial inward component to the contact force, $\vec{\mathbf{F}}^r = F_r\,\hat{\mathbf{r}}$, with $F_r < 0$, because the center of mass undergoes circular motion.

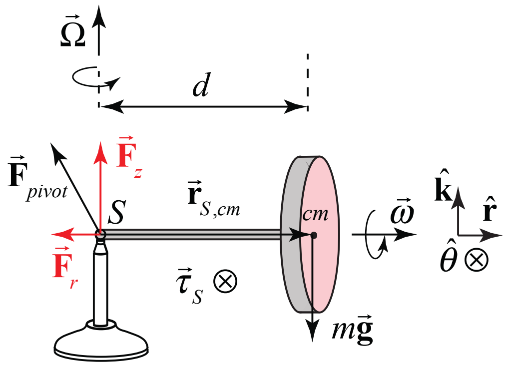


Figure 22.4 Force and torque diagram for the gyroscope

The reason that the gyroscope does not fall down is that the vertical component of the contact force exactly balances the gravitational force

$$F_z - mg = 0\ . \tag{22.2.4}$$

What about the torque about the contact point S? The contact force acts at S so it does not contribute to the torque about S; only the gravitational force contributes to the torque about S (Figure 22.5b). The direction of the torque about S is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{gravity}} = d\,\hat{\mathbf{r}} \times mg(-\hat{\mathbf{k}}) = d\,mg\hat{\boldsymbol{\theta}}, \tag{22.2.5}$$

and is in the positive $\hat{\boldsymbol{\theta}}$-direction. However we know that if there a non-zero torque about S, then the angular momentum about S must change in time, according to

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \tag{22.2.6}$$

The angular momentum about the point S of the gyroscope is given by

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_S^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}. \tag{22.2.7}$$

The orbital angular momentum about the point S is

$$\vec{\mathbf{L}}_S^{\text{orbital}} = \vec{\mathbf{r}}_{S,cm} \times m\vec{\mathbf{v}}_{cm} = d\,\hat{\mathbf{r}} \times md\Omega_z\,\hat{\boldsymbol{\theta}} = md^2\Omega_z\,\hat{\mathbf{k}}. \tag{22.2.8}$$

The magnitude of the orbital angular momentum about S is nearly constant and the direction does not change. Therefore

$$\frac{d}{dt}\vec{\mathbf{L}}_S^{\text{orbital}} = \vec{\mathbf{0}}. \qquad (22.2.9)$$

The spin angular momentum includes two terms. Recall that the flywheel undergoes two separate rotations about different axes. It is spinning about the flywheel axis with spin angular velocity $\vec{\boldsymbol{\omega}}_s$. As the flywheel precesses around the pivot point, the flywheel rotates about the z-axis with precessional angular velocity $\vec{\boldsymbol{\Omega}}$ (Figure 22.5). The spin angular momentum therefore is given by

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_r \omega_s \hat{\mathbf{r}} + I_z \Omega_z \hat{\mathbf{k}}, \qquad (22.2.10)$$

where I_r is the moment of inertia with respect to the flywheel axis and I_z is the moment of inertia with respect to the z-axis. If we assume the axle is massless and the flywheel is uniform with radius R, then $I_r = (1/2)mR^2$. By the perpendicular axis theorem $I_r = I_z + I_y = 2I_z$, hence $I_z = (1/4)mR^2$.

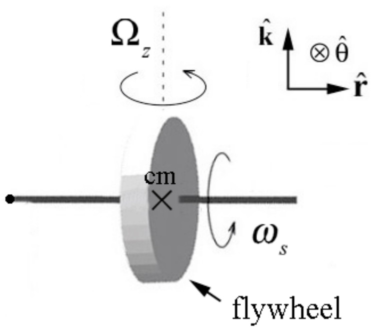


Figure 22.5: Rotations about center of mass of flywheel

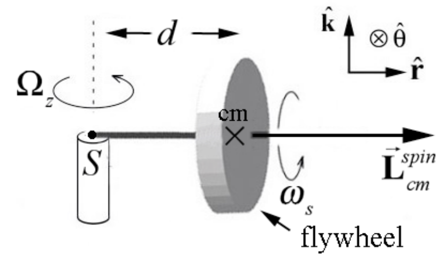


Figure 22.6 Spin angular momentum.

Recall that the gyroscopic approximation holds when $|\Omega_z| << |\omega_s|$, which implies that $I_z\Omega_z << I_r\omega_s$, and therefore we can ignore the contribution to the spin angular momentum from the rotation about the vertical axis, and so

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} \simeq I_{\text{cm}} \omega_s \hat{\mathbf{r}}. \qquad (22.2.11)$$

(The contribution to the spin angular momentum due to the rotation about the z-axis, $I_z\Omega_z \hat{\mathbf{k}}$, is nearly constant in both magnitude and direction so it does not change in time, $d(I_z\Omega_z \hat{\mathbf{k}})/dt \simeq \vec{\mathbf{0}}$.) Therefore the angular momentum about S is approximately

$$\vec{\mathbf{L}}_S \simeq \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}} \omega_s \hat{\mathbf{r}}. \qquad (22.2.12)$$

Our initial expectation that the gyroscope should fall downward due to the torque that the gravitational force exerts about the contact point S leads to a violation of the torque law. If the center of mass did start to fall then the change in the spin angular momentum, $\Delta\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$, would point in the negative z-direction and that would contradict the vector aspect of Eq. (22.2.6). Instead of falling down, the angular momentum about the center of mass, $\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$, must change direction such that the direction of $\Delta\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$ is in the same direction as torque about S (Eq. (22.2.5)), the positive $\hat{\boldsymbol{\theta}}$-direction.

Recall that in our study of circular motion, we have already encountered several examples in which the direction of a constant magnitude vector changes. We considered a point object of mass m moving in a circle of radius r. When we choose a coordinate system with an origin at the center of the circle, the position vector $\vec{\mathbf{r}}$ is directed radially outward. As the mass moves in a circle, the position vector has a constant magnitude but changes in direction. The velocity vector is given by

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}(r\,\hat{\mathbf{r}}) = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = r\omega_z\hat{\boldsymbol{\theta}} \tag{22.2.13}$$

and has direction that is perpendicular to the position vector (tangent to the circle), (Figure 22.7a)).

Figure 22.7 (a) Rotating position and velocity vector; (b) velocity and acceleration vector for uniform circular motion

For uniform circular motion, the magnitude of the velocity is constant but the direction constantly changes and we found that the acceleration is given by (Figure 22.7b)

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}(v_\theta\hat{\boldsymbol{\theta}}) = v_\theta\frac{d\theta}{dt}(-\hat{\mathbf{r}}) = r\omega_z\omega_z(-\hat{\mathbf{r}}) = -r\omega_z^2\hat{\mathbf{r}}\,. \tag{22.2.14}$$

Note that we used the facts that

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \frac{d\theta}{dt}\hat{\boldsymbol{\theta}},\\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\frac{d\theta}{dt}\hat{\mathbf{r}}\end{aligned}, \tag{22.2.15}$$

in Eqs. (22.2.13) and (22.2.14). We can apply the same reasoning to how the spin angular changes in time (Figure 22.8).

The time derivative of the spin angular momentum is given by

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d\vec{\mathbf{L}}^{\text{spin}}_{\text{cm},\omega_s}}{dt} = \left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm},\omega_s}\right| \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} = \left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm},\omega_s}\right| \Omega_z\,\hat{\boldsymbol{\theta}} = I_r \omega_s \Omega_z \hat{\boldsymbol{\theta}}\,. \qquad (22.2.16)$$

where $\Omega_z = d\theta / dt$ is the z-component and $\Omega_z > 0$. The center of mass of the flywheel rotates about a vertical axis that passes through the contact point S of the axle with the pylon with a precessional angular velocity

$$\vec{\boldsymbol{\Omega}} = \Omega_z \hat{\mathbf{k}} = \frac{d\theta}{dt}\hat{\mathbf{k}}\,, \qquad (22.2.17)$$

Substitute Eqs. (22.2.16) and (22.2.5) into Eq. (22.2.6) yielding

$$d\,mg\hat{\boldsymbol{\theta}} = \left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm}}\right| \Omega_z \hat{\boldsymbol{\theta}}\,. \qquad (22.2.18)$$

Solving Equation (22.2.18) for the z-component of the precessional angular velocity of the gyroscope yields

$$\Omega_z = \frac{d\,mg}{\left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm}}\right|} = \frac{d\,mg}{I_{\text{cm}}\,\omega_{\text{s}}}\,. \qquad (22.2.19)$$

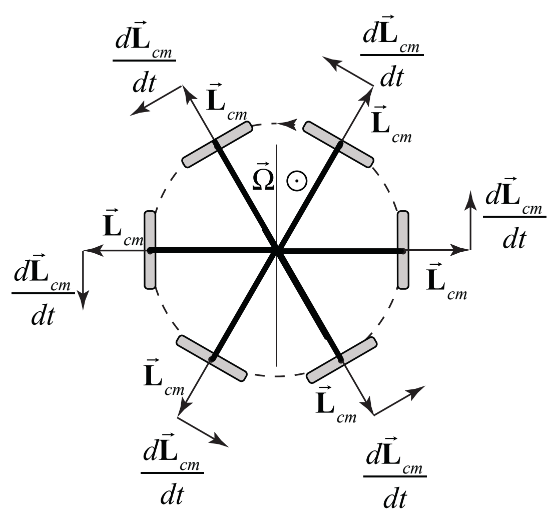


Figure 22.8 Time changing direction of the spin angular momentum

22.3 Why Does a Gyroscope Precess?

Why does a gyroscope precess? We now understand that the torque is causing the spin angular momentum to change but the motion still seems mysterious. We shall try to understand why the angular momentum changes direction by first examining the role of force and impulse on a single rotating particle and then generalize to a rotating disk.

22.3.1 Deflection of a Particle by a Small Impulse

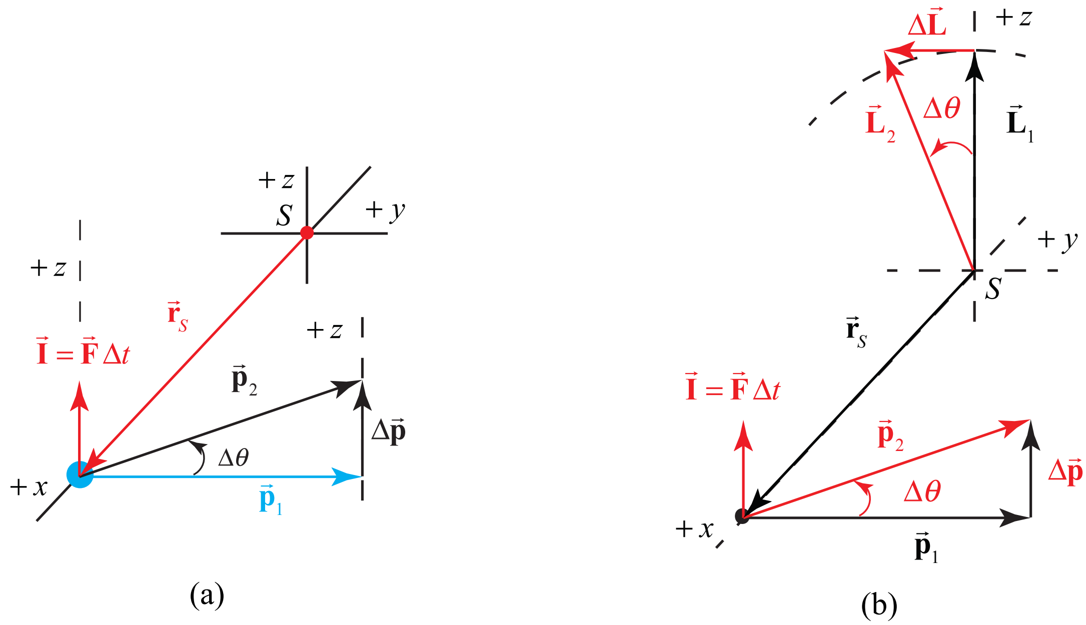


Figure 22.9 (a) Deflection of a particle by a small impulse, (b) change in angular momentum about origin

We begin by first considering how a particle with momentum $\vec{\mathbf{p}}_1$ undergoes a deflection due to a small impulse (Figure 22.9a). If the impulse $|\vec{\mathbf{I}}| << |\vec{\mathbf{p}}_1|$, the primary effect is to rotate the momentum $\vec{\mathbf{p}}_1$ about the x-axis by a small angle θ, with $\vec{\mathbf{p}}_2 = \vec{\mathbf{p}}_1 + \Delta\vec{\mathbf{p}}$. The application of $\vec{\mathbf{I}}$ causes a change in the angular momentum $\vec{\mathbf{L}}_{O,1}$ about the origin S, according to the torque equation, $\Delta\vec{\mathbf{L}}_S = \vec{\boldsymbol{\tau}}_{\text{ave},S}\Delta t = (\vec{\mathbf{r}}_S \times \vec{\mathbf{F}}_{\text{ave}})\Delta t$. Because $\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{F}}_{\text{ave}}\Delta t$, we have that $\Delta\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{I}}$. As a result, $\Delta\vec{\mathbf{L}}_S$ rotates about the x-axis by a small angle θ, to a new angular momentum $\vec{\mathbf{L}}_{S,2} = \vec{\mathbf{L}}_{S,1} + \Delta\vec{\mathbf{L}}_S$. Note that although $\vec{\mathbf{L}}_S$ is in the z-direction, $\Delta\vec{\mathbf{L}}_S$ is in the negative y-direction (Figure 22.9b).

22.3.2 Effect of Small Impulse on Tethered Object

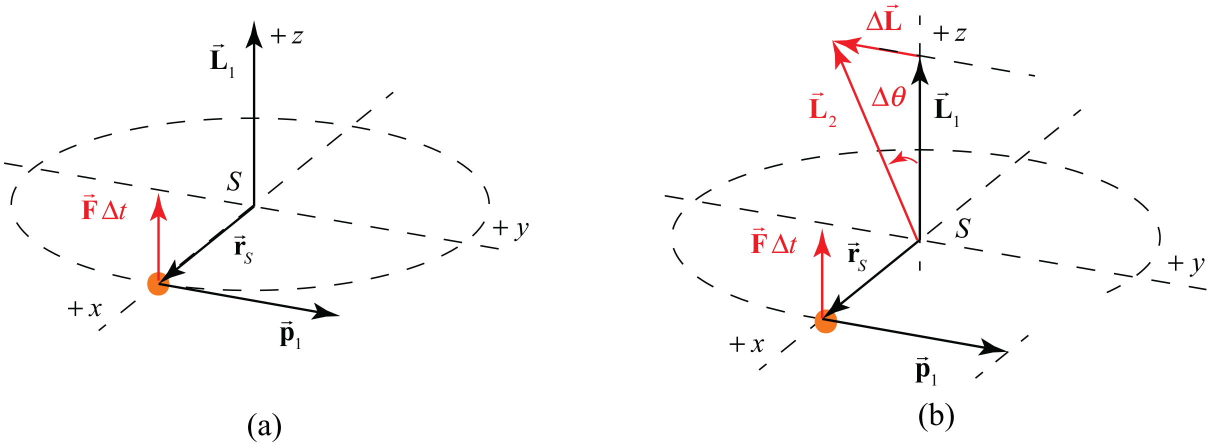


Figure 22.10a Small impulse on object undergoing circular motion, (b) change in angular momentum

Now consider an object that is attached to a string and is rotating about a fixed point S with momentum $\vec{\mathbf{p}}_1$. The object is given an impulse $\vec{\mathbf{I}}$ perpendicular to $\vec{\mathbf{r}}_S$ and to $\vec{\mathbf{p}}_1$. Neglect gravity. As a result $\Delta\vec{\mathbf{L}}_S$ rotates about the x-axis by a small angle θ (Figure 22.10a). Note that although $\vec{\mathbf{I}}$ is in the z-direction, $\Delta\vec{\mathbf{L}}_S$ is in the negative y- direction (Figure 22.10b). Note that although $\vec{\mathbf{I}}$ is in the z-direction, the plane in which the ball moves also rotates about the x-axis by the same angle (Figure 22.11).

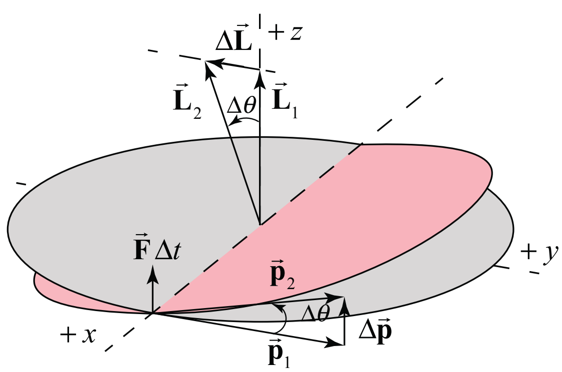


Figure 22.11 Plane of object rotates about x-axis

Example 22.2 Effect of Large Impulse on Tethered Object

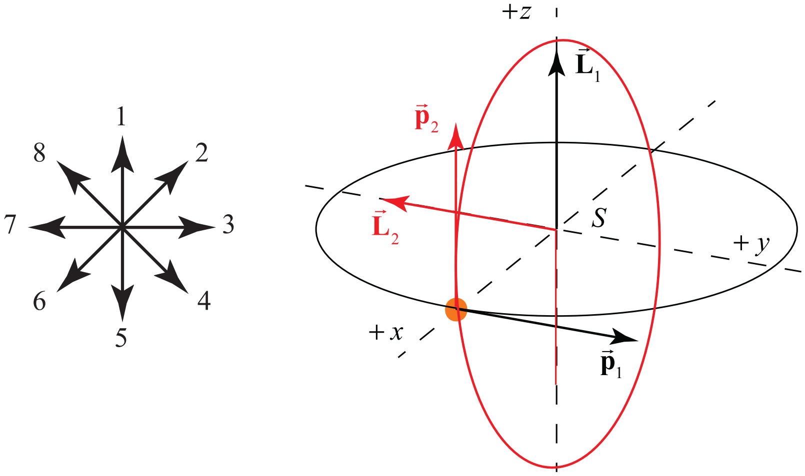


Figure 22.12 Example 22.2

What impulse, $\vec{\mathbf{I}}$, must be given to the ball in order to rotate its orbit by 90 degrees as shown without changing its speed (Figure 21.12)?

Solution: h. The impulse $\vec{\mathbf{I}}$ must halt the momentum $\vec{\mathbf{p}}_1$ and provide a momentum $\vec{\mathbf{p}}_2$ of equal magnitude along the z-direction such that $\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}}$.

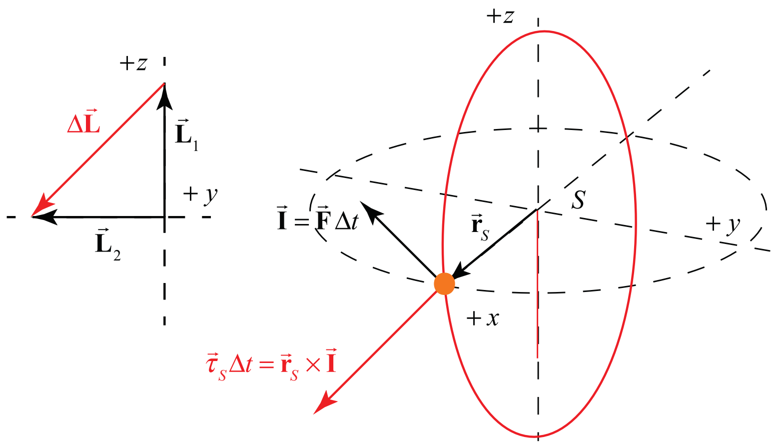


Figure 22.13 Impulse and torque about S

The angular impulse about S must be equal to the change in angular momentum about S

$$\vec{\boldsymbol{\tau}}_S \Delta t = \vec{\mathbf{r}}_S \times \vec{\mathbf{I}} = (\vec{\mathbf{r}}_S \times \Delta\vec{\mathbf{p}}) = \Delta\vec{\mathbf{L}}_S \qquad (22.3.1)$$

The change in angular momentum, $\Delta\vec{\mathbf{L}}_S$, due to the torque about S, cancels the z-component of $\vec{\mathbf{L}}_S$ and adds a component of the same magnitude in the negative y-direction (Figure 22.13).

22.3.3 Effect of Small Impulse Couple on Baton

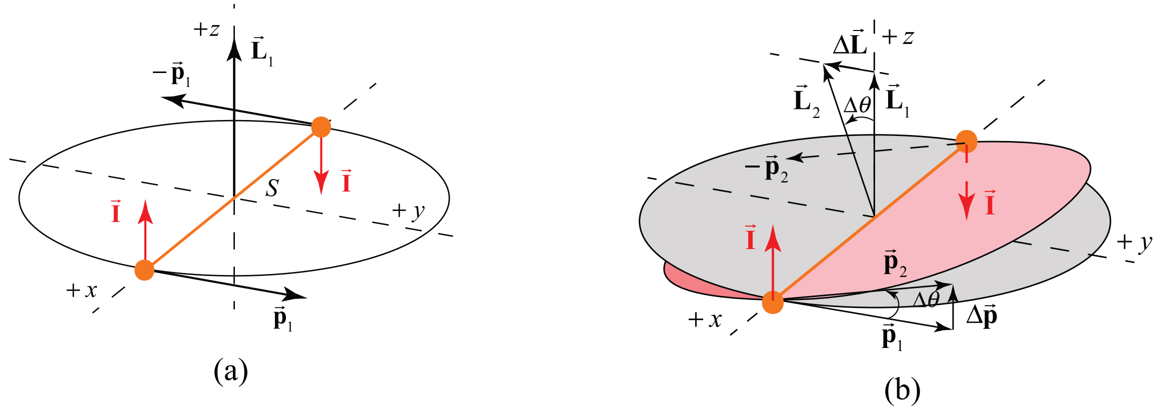


Figure 22.14 (a) and (b)

Now consider two equal masses at the ends of a massless rod, which spins about its center. We apply an impulse couple to insure no motion of the center of mass. Again note that the impulse couple is applied in the z-direction (Figure 22.14a). The resulting torque about S lies along the negative y-direction and the plane of rotation tilts about the x-axis (Figure 22.14b).

22.3.4 Effect of Small Impulse Couple on Massless Shaft of Baton

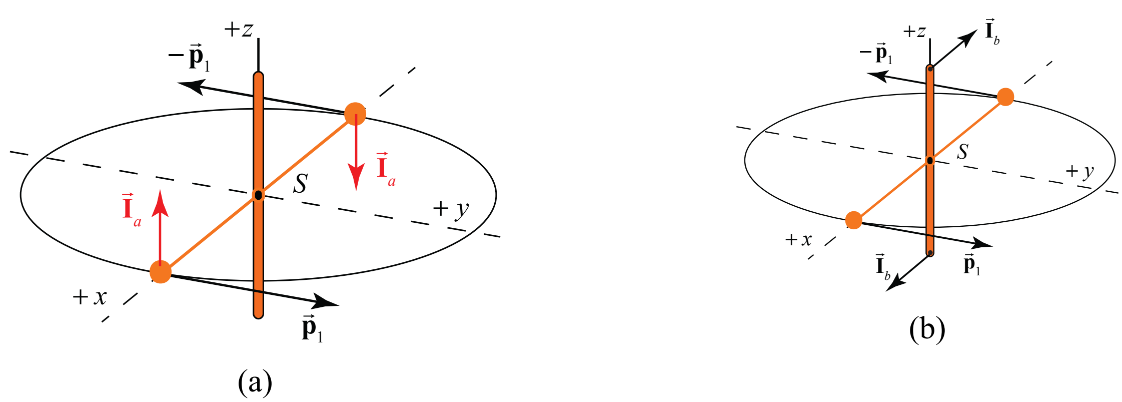


Figure 22.15 Apply impulse couple to (a) objects and (b) shaft

Instead of applying the impulse couple $\vec{\mathbf{I}}_a$ to the masses (Figure 21.15a), one could apply the same impulse couple $\vec{\mathbf{I}}_b = \vec{\mathbf{I}}_a$ to the vertical massless shaft that is connected to the baton (Figure 22.15b) to achieve the same result.

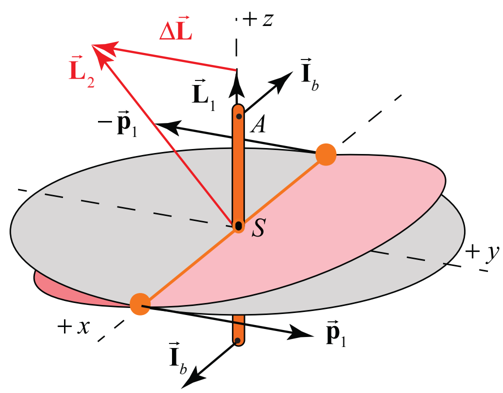


Figure 22.16 Twisting shaft causes shaft and plane to rotate about x-axis

Twisting the shaft around the y-axis causes the shaft and the plane in which the baton moves to rotate about the x-axis.

22.3.5 Effect of a Small Impulse Couple on a Rotating Disk

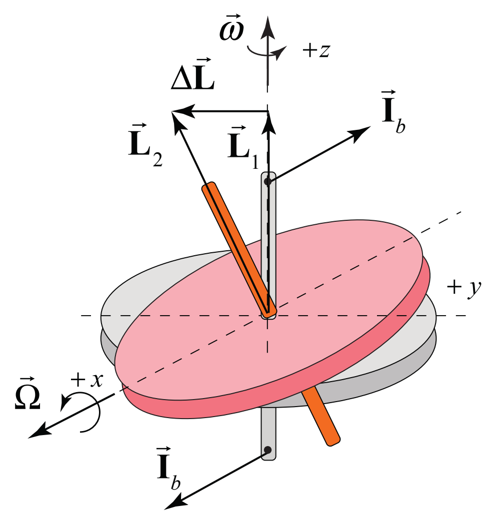


Figure 22.17 Impulse couple causes a disk to rotate about the x-axis.

Now let's consider a rotating disk. The plane of a rotating disk and its shaft behave just like the plane of the rotating baton and its shaft when one attempts to twist the shaft about the y-axis. The plane of the disk rotates about the x-axis (Figure 22.17). This unexpected result is due to the large pre-existing angular momentum about S, $\vec{\mathbf{L}}_1$, due to the spinning disk. It does not matter where along the shaft the impulse couple is applied, as long as it creates the same torque about S.

22.3.6 Effect of a Force Couple on a Rotating Disk

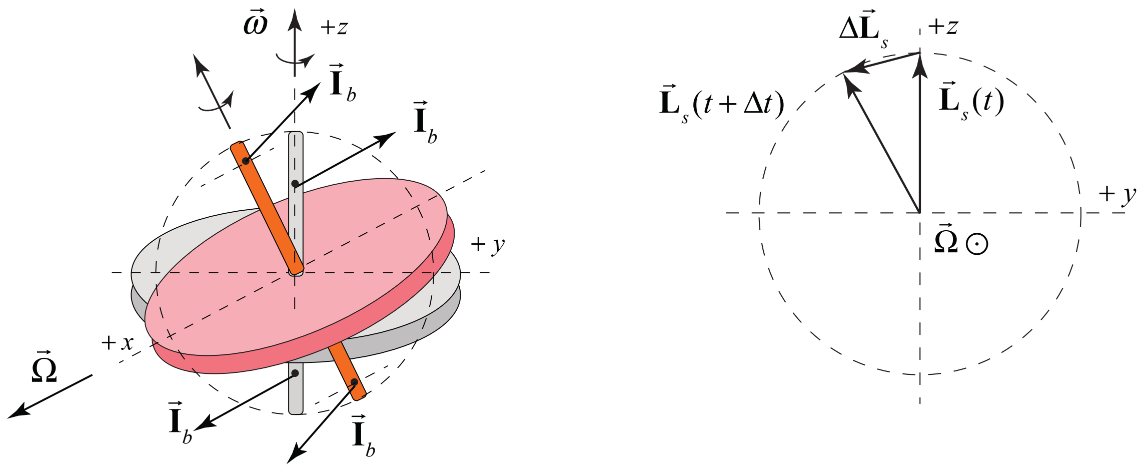


Figure 22.18 A series of small impulse couples causes the tip of the shaft to execute circular motion about the x-axis

A series of small impulse couples, or equivalently a continuous force couple (with force $\vec{\mathbf{F}}$), causes the tip of the shaft to execute circular motion about the x-axis (Figure 22.18). The magnitude of the angular momentum about S changes according to $\left|d\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{L}}_S\right| \Omega\, dt = I\omega\, \Omega\, dt$. Recall that torque and changing angular momentum about S are related by $\vec{\boldsymbol{\tau}}_S = d\vec{\mathbf{L}}_S / dt$. Therefore $\left|\vec{\boldsymbol{\tau}}_S\right| = \left|\vec{\mathbf{L}}_S\right| \Omega = I\omega\, \Omega$. The precession rate of the shaft is the ratio of the magnitude of the torque to the angular momentum $\Omega = \left|\vec{\boldsymbol{\tau}}_S\right| / \left|\vec{\mathbf{L}}_S\right| = \left|\vec{\boldsymbol{\tau}}_S\right| / I\omega$.

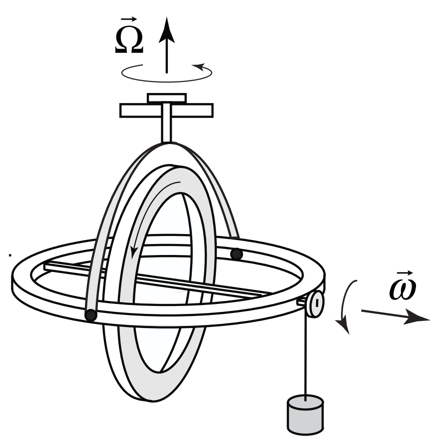


Figure 22.19 Precessing gyroscope with hanging object

Thus we can explain the motion of a precessing gyroscope in which the torque about the center of mass is provided by the force of gravity on the hanging object (Figure 22.19).

22.3.7 Effect of a Small Impulse Couple on a Non-Rotating Disc

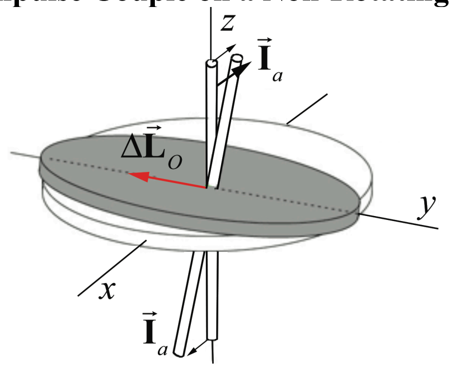


Figure 22.20 Impulse couple on non-rotating disk causes shaft to rotate about negative y-axis.

If the disk is not rotating to begin with, $\Delta\vec{\mathbf{L}}_S$ is also the final $\vec{\mathbf{L}}_S$. The shaft moves in the direction it is pushed (Figure 22.20).

22.4 Worked Examples

Example 22.3 Tilted Toy Gyroscope

A wheel is at one end of an axle of length d. The axle is pivoted at an angle ϕ with respect to the vertical. The wheel is set into motion so that it executes uniform precession; that is, the wheel's center of mass moves with uniform circular motion with z-component of precessional angular velocity Ω_z. The wheel has mass m and moment of inertia I_{cm} about its center of mass. Its spin angular velocity $\vec{\boldsymbol{\omega}}_s$ has magnitude ω_s and is directed as shown in Figure 22.21. Assume that the gyroscope approximation holds, $|\Omega_z| << \omega_s$. Neglect the mass of the axle. What is the z-component of the precessional angular velocity Ω_z? Does the gyroscope rotate clockwise or counterclockwise about the vertical axis (as seen from above)?

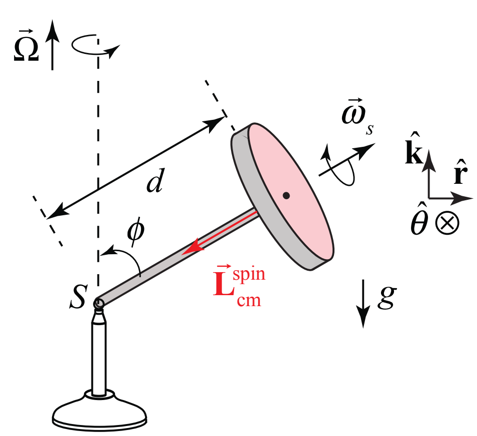


Figure 22.21 Example 22.3

Solution: The gravitational force acts at the center of mass and is directed downward, $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{k}}$. Let S denote the contact point between the pylon and the axle. The contact force between the pylon and the axle is acting at S so it does not contribute to the torque about S. Only the gravitational force contributes to the torque. Let's choose cylindrical coordinates. The torque about S is

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}^g = (d\sin\phi\,\hat{\mathbf{r}} + d\cos\phi\,\hat{\mathbf{k}}) \times mg(-\hat{\mathbf{k}}) = mgd\sin\phi\,\hat{\boldsymbol{\theta}}, \quad (22.4.1)$$

which is into the page in Figure 22.21. Because we are assuming that $\left|\Omega_z\right| << \omega_s$, we only consider contribution from the spinning about the flywheel axle to the spin angular momentum,

$$\vec{\boldsymbol{\omega}}_s = -\omega_s \sin\phi\,\hat{\mathbf{r}} - \omega_s \cos\phi\,\hat{\mathbf{k}} \quad (22.4.2)$$

The spin angular momentum has a vertical and radial component,

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = -I_{\text{cm}}\omega_s \sin\phi\,\hat{\mathbf{r}} - I_{\text{cm}}\omega_s \cos\phi\,\hat{\mathbf{k}}. \quad (22.4.3)$$

We assume that the spin angular velocity ω_s is constant. As the wheel precesses, the time derivative of the spin angular momentum arises from the change in the direction of the radial component of the spin angular momentum,

$$\frac{d}{dt}\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = -I_{\text{cm}}\omega_s \sin\phi \frac{d\hat{\mathbf{r}}}{dt} = -I_{\text{cm}}\omega_s \sin\phi \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}. \quad (22.4.4)$$

where we used the fact that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}. \quad (22.4.5)$$

The z-component of the angular velocity of the flywheel about the vertical axis is defined to be

$$\Omega_z \equiv \frac{d\theta}{dt}. \quad (22.4.6)$$

Therefore the rate of change of the spin angular momentum is then

$$\frac{d}{dt}\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = -I_{\text{cm}}\omega_s \sin\phi\,\Omega_z\,\hat{\boldsymbol{\theta}}. \quad (22.4.7)$$

The torque about S induces the spin angular momentum about S to change,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt}. \tag{22.4.8}$$

Now substitute Equation (22.4.1) for the torque about S, and Equation (22.4.7) for the rate of change of the spin angular momentum into Equation (22.4.8), yielding

$$mgd\sin\phi\,\hat{\boldsymbol{\theta}} = -I_{\text{cm}}\omega_s \sin\phi\,\Omega_z\,\hat{\boldsymbol{\theta}}. \tag{22.4.9}$$

Solving Equation (22.2.18) for the z-component of the precessional angular velocity of the gyroscope yields

$$\Omega_z = -\frac{d\,mg}{I_{\text{cm}}\,\omega_s}. \tag{22.4.10}$$

The z-component of the precessional angular velocity is independent of the angle ϕ. Because $\Omega_z < 0$, the direction of the precessional angular velocity, $\vec{\Omega} = \Omega_z \hat{\mathbf{k}}$, is in the negative z-direction. That means that the gyroscope precesses in the clockwise direction when seen from above (Figure 21.22).

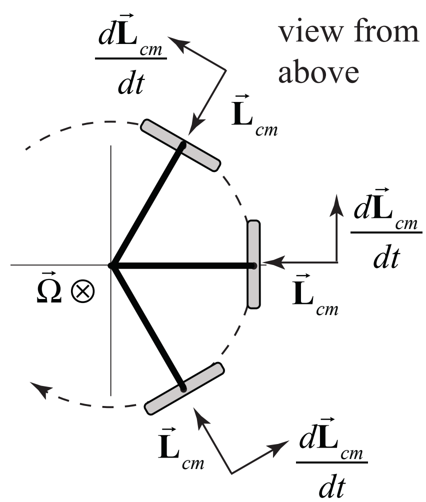


Figure 21.22 Precessional angular velocity of tilted gyroscope as seen from above

Both the torque and the time derivative of the spin angular momentum point in the $\hat{\boldsymbol{\theta}}$-direction indicating that the gyroscope will precess clockwise when seen from above in agreement with the calculation that $\Omega_z < 0$.

Example 22.4 Gyroscope on Rotating Platform

A gyroscope consists of an axle of negligible mass and a disk of mass M and radius R mounted on a platform that rotates with angular speed Ω. The gyroscope is spinning with angular speed ω. Forces F_a and F_b act on the gyroscopic mounts. What are the magnitudes of the forces F_a and F_b (Figure 22.22)? You may assume that the moment of inertia of the gyroscope about an axis passing through the center of mass normal to the plane of the disk is given by I_{cm}.

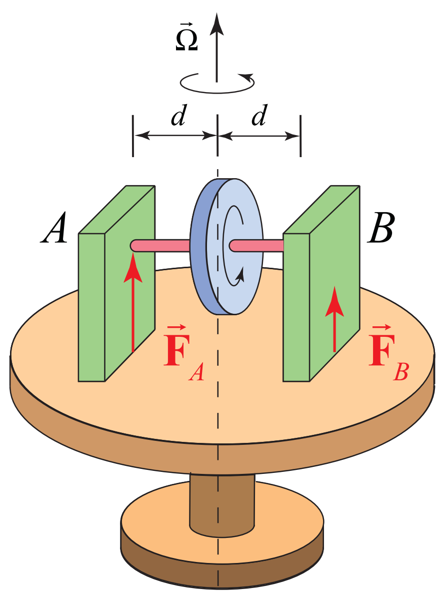


Figure 22.22 Example 22.4

Solution: Figure 22.23 shows a choice of coordinate system and force diagram on the gyroscope.

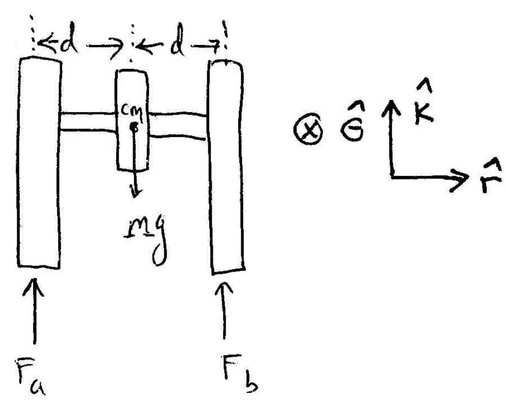


Figure 22.23 Free-body force diagram

The vertical forces sum to zero since there is no vertical motion

$$F_a + F_b - Mg = 0 \tag{22.4.11}$$

Using the coordinate system depicted in the Figure 22.23, torque about the center of mass is

$$\vec{\boldsymbol{\tau}}_{\text{cm}} = d(F_a - F_b)\hat{\boldsymbol{\theta}} \tag{22.4.12}$$

The spin angular momentum is (gyroscopic approximation)

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} \simeq I_{\text{cm}}\omega\,\hat{\mathbf{r}} \tag{22.4.13}$$

Looking down on the gyroscope from above (Figure 2.23), the radial component of the angular momentum about the center of mass is rotating counterclockwise.

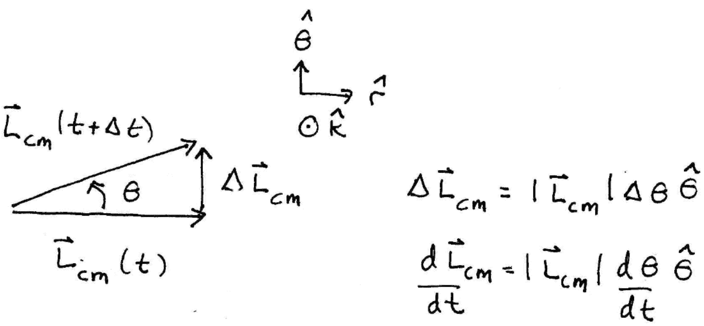


Figure 22.24 Change in angular momentum

During a very short time interval Δt, the change in the spin angular momentum is $\Delta\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\omega\Delta\theta\,\hat{\boldsymbol{\theta}}$, (Figure 22.24). Taking limits we have that

$$\frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{\Delta t} = \lim_{\Delta t \to 0} I_{\text{cm}}\omega\frac{\Delta\theta}{\Delta t}\hat{\boldsymbol{\theta}} = I_{\text{cm}}\omega\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \tag{22.4.14}$$

We can now apply the torque law

$$\vec{\boldsymbol{\tau}}_{\text{cm}} = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt}. \tag{22.4.15}$$

Substitute Eqs. (22.4.12) and (22.4.14) into Eq. (22.4.15) and just taking the component of the resulting vector equation yields

$$d(F_a - F_b) = I_{\text{cm}}\omega\,\Omega_z. \tag{22.4.16}$$

We can divide Eq. (22.4.16) by the quantity d yielding

$$F_a - F_b = \frac{I_{\text{cm}}\omega\,\Omega_z}{d}. \tag{22.4.17}$$

We can now use Eqs. (22.4.17) and (22.4.11) to solve for the forces F_a and F_b,

$$F_a = \frac{1}{2}\left(Mg + \frac{I_{\text{cm}} \omega \, \Omega_z}{d} \right) \tag{22.4.18}$$

$$F_b = \frac{1}{2}\left(Mg - \frac{I_{\text{cm}} \omega \, \Omega_z}{d} \right). \tag{22.4.19}$$

Note that if $\Omega_z = Mgd \, / \, I_{\text{cm}}\omega$ then $F_b = 0$ and one could remove the right hand support in the Figure 22.22. The simple pivoted gyroscope that we already analyzed Section 22.2 satisfied this condition. The forces we just found are the forces that the mounts must exert on the gyroscope in order to cause it to move in the desired direction. It is important to understand that the gyroscope is exerting equal and opposite forces on the mounts, i.e. the structure that is holding it. This is a manifestation of Newton's Third Law.

Example 22.5 Grain Mill

In a mill, grain is ground by a massive wheel that rolls without slipping in a circle on a flat horizontal millstone driven by a vertical shaft. The rolling wheel has mass M , radius b and is constrained to roll in a horizontal circle of radius R at angular speed Ω (Figure 22.25). The wheel pushes down on the lower millstone with a force equal to twice its weight (normal force). The mass of the axle of the wheel can be neglected. What is the precessional angular frequency Ω ?

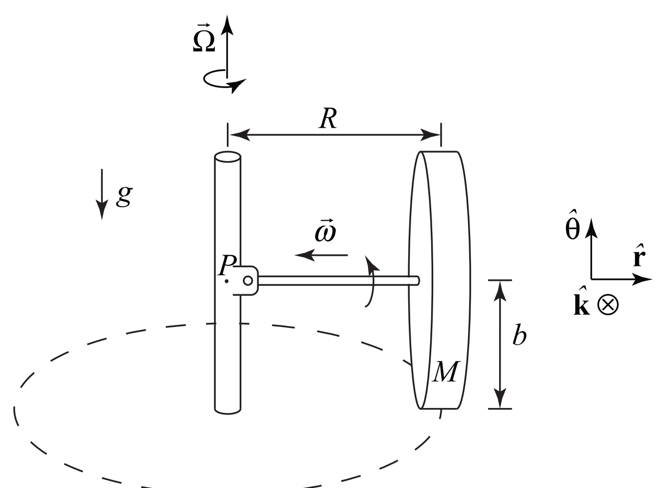


Figure 22.25 Example 22.5

Solution: Figure 22.5 shows the pivot point along with some convenient coordinate axes. For rolling without slipping, the speed of the center of mass of the wheel is related to the angular spin speed by

$$v_{cm} = b\omega \, . \tag{22.4.20}$$

Also the speed of the center of mass is related to the angular speed about the vertical axis associated with the circular motion of the center of mass by

$$v_{cm} = R\Omega . \qquad (22.4.21)$$

Therefore equating Eqs. (22.4.20) and (22.4.21) we have that

$$\omega = \Omega R / b . \qquad (22.4.22)$$

Assuming a uniform millwheel, $I_{\text{cm}} = (1/2)Mb^2$, the magnitude of the horizontal component of the spin angular momentum about the center of mass is

$$L_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\omega = \frac{1}{2}Mb^2\omega = \frac{1}{2}\Omega\, MRb . \qquad (22.4.23)$$

The horizontal component of $\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$ is directed inward, and in vector form is given by

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = -\frac{\Omega\, MRb}{2}\hat{\mathbf{r}} . \qquad (22.4.24)$$

The axle exerts both a force and torque on the wheel, and this force and torque would be quite complicated. That's why we consider the forces and torques on the axle/wheel combination. The normal force of the wheel on the ground is equal in magnitude to $N_{\text{W,G}} = 2mg$ so the third-law counterpart; the normal force of the ground on the wheel has the same magnitude $N_{\text{G,W}} = 2mg$. The joint (or hinge) at point P therefore must exert a force $\vec{\mathbf{F}}_{\text{H,A}}$ on the end of the axle that has two components, an inward force $\vec{\mathbf{F}}_2$ to maintain the circular motion and a downward force $\vec{\mathbf{F}}_1$ to reflect that the upward normal force is larger in magnitude than the weight (Figure 22.26).

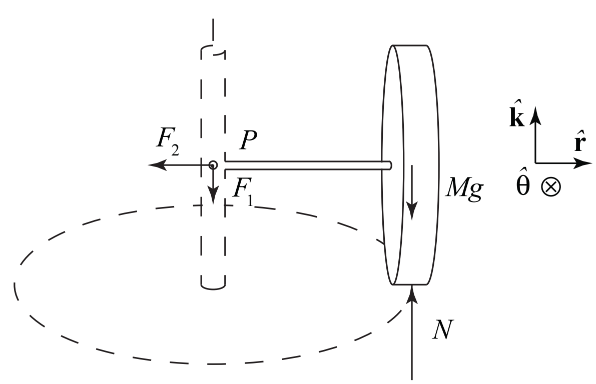


Figure 22.26 Free-body force diagram on wheel

About point P, $\vec{\mathbf{F}}_{\text{H,A}}$ exerts no torque. The normal force exerts a torque of magnitude $N_{\text{G,W}} R = 2mgR$, directed out of the page, or, in vector form, $\vec{\tau}_{P,N} = -2mgR\hat{\boldsymbol{\theta}}$. The weight exerts a toque of magnitude mgR, directed into the page, or, in vector form, $\vec{\tau}_{P,mg} = mgR\hat{\boldsymbol{\theta}}$. The torque about P is then

$$\vec{\tau}_P = \vec{\tau}_{P,N} + \vec{\tau}_{P,mg} = -2mgR\hat{\boldsymbol{\theta}} + mgR\hat{\boldsymbol{\theta}} = -mgR\hat{\boldsymbol{\theta}}. \tag{22.4.25}$$

As the wheel rolls, the horizontal component of the angular momentum about the center of mass will rotate, and the inward-directed vector will change in the negative $\hat{\boldsymbol{\theta}}$-direction. The angular momentum about the point P has orbital and spin decomposition

$$\vec{\mathbf{L}}_P = \vec{\mathbf{L}}_P^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}. \tag{22.4.26}$$

The orbital angular momentum about the point P is

$$\vec{\mathbf{L}}_P^{\text{orbital}} = \vec{\mathbf{r}}_{P,cm} \times m\vec{\mathbf{v}}_{cm} = R\,\hat{\mathbf{r}} \times mb\Omega\,\hat{\boldsymbol{\theta}} = mRb\Omega_z\,\hat{\mathbf{k}}. \tag{22.4.27}$$

The magnitude of the orbital angular momentum about P is nearly constant and the direction does not change. Therefore

$$\frac{d\vec{\mathbf{L}}_P^{\text{orbital}}}{dt} = \vec{\mathbf{0}}. \tag{22.4.28}$$

Therefore the change in angular momentum about the point P is

$$\frac{d\vec{\mathbf{L}}_P}{dt} = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt} = \frac{d}{dt}\left(\frac{\Omega mRb}{2}(-\hat{\mathbf{r}})\right) = \frac{1}{2}\Omega mRb\Omega(-\hat{\boldsymbol{\theta}}), \tag{22.4.29}$$

where we used Eq. (22.4.24) for the magnitude of the horizontal component of the angular momentum about the center of mass. This is consistent with the torque about P pointing out of the plane of Figure 22.26. We can now apply the rotational equation of motion,

$$\vec{\boldsymbol{\tau}}_P = \frac{d\vec{\mathbf{L}}_P}{dt}. \tag{22.4.30}$$

Substitute Eqs.(22.4.25) and (22.4.29) into Eq. (22.4.30) yielding

$$mgR(-\hat{\boldsymbol{\theta}}) = \frac{1}{2}\Omega^2 mRb(-\hat{\boldsymbol{\theta}}). \tag{22.4.31}$$

We can now solve Eq. (22.4.31) for the angular speed about the vertical axis

$$\Omega = \sqrt{\frac{2g}{b}} \,. \tag{22.4.32}$$

22.5 Angular Momentum and the Moment of Inertia Tensor

Consider a rigid body rotating about the center of mass with angular velocity $\vec{\boldsymbol{\omega}}$. Choose an inertial Cartesian reference frame O with origin at the center of mass and coordinates (x_1, x_2, x_3) with unit vectors $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)$, then $\vec{\boldsymbol{\omega}} = \omega_1 \vec{\mathbf{e}}_1 + \omega_2 \vec{\mathbf{e}}_2 + \omega_3 \vec{\mathbf{e}}_3$. Divide the rigid body into N small pieces. Let $a = 1,...,N$ be an index labeling each small piece that has mass
m_a, located at a position $\vec{\mathbf{r}}_a = (x_{a,1}, x_{a,2}, x_{a,3})$ and moving with velocity $\vec{\mathbf{v}}_a = \vec{\omega} \times \vec{\mathbf{r}}_a$.

The angular momentum about the center of mass of the N small pieces is given by

$$\vec{\mathbf{L}}_{cm} = \sum_{a=1}^{N} \vec{\mathbf{r}}_a \times m_a \vec{\mathbf{v}}_a = \sum_{a=1}^{N} m_a \vec{\mathbf{r}}_a \times (\vec{\omega} \times \vec{\mathbf{r}}_a) \,.$$

Note that the triple cross product in vector notation is given by

$$\vec{\mathbf{r}}_a \times (\vec{\omega} \times \vec{\mathbf{r}}_a) = (\vec{\mathbf{r}}_a \cdot \vec{\mathbf{r}}_a)\vec{\omega} - (\vec{\mathbf{r}}_a \cdot \vec{\omega})\vec{\mathbf{r}}_a$$

The angular momentum is then

$$\vec{\mathbf{L}}_{cm} = \sum_{a=1}^{N} m_a ((\vec{\mathbf{r}}_a \cdot \vec{\mathbf{r}}_a)\vec{\omega} - (\vec{\mathbf{r}}_a \cdot \vec{\omega})\vec{\mathbf{r}}_a) \,.$$

Example: The 1-component of the angular momentum is

$$\begin{aligned}
\vec{\mathbf{L}}_{cm,1} &= \sum_{a=1}^{N} m_a ((\vec{\mathbf{r}}_a \cdot \vec{\mathbf{r}}_a)\omega_1 - (\vec{\mathbf{r}}_a \cdot \vec{\omega}) x_{a,1}) \\
&= \sum_{a=1}^{N} m_a ((x_{a,1}^2 + x_{a,2}^2 + x_{a,3}^2)\omega_1 - (x_{a,1}\omega_1 + x_{a,2}\omega_2 + x_{a,3}\omega_3) x_{a,1}) \\
&= \sum_{a=1}^{N} m_a ((x_{a,2}^2 + x_{a,3}^2)\omega_1 - (x_{a,1}x_{a,2}\omega_2 + x_{a,1}x_{a,3}\omega_3))
\end{aligned}$$

Define

$$I_{11} = \sum_{a=1}^{N} m_a ((x_{a,2}^2 + x_{a,3}^2)$$

$$I_{12} = -\sum_{a=1}^{N} m_a x_{a,1} x_{a,2}$$

$$I_{13} = -\sum_{a=1}^{N} m_a x_{a,1} x_{a,3}$$

The quantities I_{12} and I_{13} are called the *products of inertia.* The 1-component of the angular momentum is then

$$\vec{\mathbf{L}}_{cm,1} = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \ . \tag{33}$$

Similarly the other components are given by

$$\vec{\mathbf{L}}_{cm,2} = I_{22}\omega_2 + I_{21}\omega_1 + I_{23}\omega_3 \ , \tag{34}$$

$$\vec{\mathbf{L}}_{cm,3} = I_{33}\omega_3 + I_{31}\omega_1 + I_{32}\omega_2 \ , \tag{35}$$

These three equations can be collected into a matrix form

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{36}$$

The matrix

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \tag{37}$$

is called *the moment of inertia tensor*.

In general for a rigid body, the direction of the angular momentum does not coincide with the direction of the angular velocity.

Principal Axes Theorem: It is always possible for an arbitrary rigid body to find a set of orthogonal axes such that the moment of inertia tensor has only diagonal components. This set of axes are called the *principal axes*, and the moment of inertia tensor is then

$$I = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \quad (prinicpal\ axes)\,.$$

Alternative derivation:

A vector cross product can be written in index notation as

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}})_k = \sum_{i,j=1}^{3} \varepsilon_{ijk} A_i B_j$$

where

$$\varepsilon_{ijk} = \begin{Bmatrix} +1 \; for\; any\; even\; permutation\; of\; i=1, j=2, k=3 \\ 0 \; if\; any\; two\; indices\; are\; equal \\ -1 \; for\; any\; odd\; permutation\; of\; i=1, j=2, k=3 \end{Bmatrix}.$$

Example:

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}})_3 = \sum_{i,j=1}^{3} \varepsilon_{ij3} A_i B_j = \varepsilon_{123} A_1 B_2 + \varepsilon_{213} A_2 B_1 = A_1 B_2 - A_2 B_1 .$$

Then the kth-component of the angular momentum can be written as

$$\begin{aligned} \vec{\mathbf{L}}_{cm,k} &= \sum_{a=1}^{N} m_a \vec{\mathbf{r}}_a \times (\vec{\omega} \times \vec{\mathbf{r}}_a) = \sum_{a=1}^{N} m_a \sum_{i,j=1}^{3} \varepsilon_{ijk} r_{a,i} (\vec{\omega} \times \vec{\mathbf{r}}_a)_j \\ &= \sum_{a=1}^{N} m_a \left[\sum_{i,j=1}^{3} \varepsilon_{ijk} r_{a,i} \sum_{l,m=1}^{3} \varepsilon_{lmj} \omega_l r_{a,m} \right] \\ &= \sum_{a=1}^{N} m_a \left[\sum_{i,j,l,m=1}^{3} \varepsilon_{ijk} \varepsilon_{lmj} r_{a,i} \omega_l r_{a,m} \right] \end{aligned}$$

Note that $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$. Therefore

$$\vec{\mathbf{L}}_{cm,k} = \sum_{a=1}^{N} m_a \left[\sum_{i,j,l,m=1}^{3} \varepsilon_{kij} \varepsilon_{lmj} r_{a,i} \omega_l r_{a,m} \right]$$

Also note that

$$\sum_{j=1}^{3} \varepsilon_{kij} \varepsilon_{lmj} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il}$$

where δ_{kl} is the *Kronecker delta function*

$$\delta_{kl} = \begin{Bmatrix} +1 \; if \; k = l \\ 0 \; if \; k \neq l \end{Bmatrix}$$

Example:

$$\sum_{j=1}^{3} \varepsilon_{23j} \varepsilon_{13j} = \delta_{21} \delta_{33} - \delta_{23} \delta_{31} = 0$$

$$\sum_{j=1}^{3} \varepsilon_{23j} \varepsilon_{32j} = \delta_{23} \delta_{32} - \delta_{22} \delta_{33} = -1$$

Hence the kth-component of the angular momentum is

$$\vec{\mathbf{L}}_{cm,k} = \sum_{a=1}^{N} m_a \left[\sum_{i=1}^{3} \sum_{l,m=1}^{3} (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) r_{a,i}\omega_l r_{a,m} \right]$$

$$= \sum_{a=1}^{N} m_a \left[\sum_{i=1}^{3} (r_{a,i}\omega_k r_{a,i} \right] - \sum_{a=1}^{N} m_a \left[\sum_{j=1}^{3} r_{a,j}\omega_j r_{a,k} \right]$$

Because $\omega_k = \sum_{j=1}^{3} \delta_{kj}\omega_j$, the *k*th-component of angular momentum is then

$$\vec{\mathbf{L}}_{cm,k} = \sum_{a=1}^{N} m_a \left[\sum_{i=1}^{3} (r_{a,i}\omega_k r_{a,i} - r_{a,i}\omega_i r_{a,k}) \right]$$

$$= \sum_{a=1}^{N} m_a \left[\sum_{j=1}^{3} r_a^2 \delta_{kj}\omega_j - \sum_{j=1}^{3} r_{a,j} r_{a,k}\omega_j \right]$$

$$= \sum_{a=1}^{N} \left[\sum_{j=1}^{3} (m_a(\delta_{kj} r_a^2 - r_{a,j} r_{a,k})\omega_j \right]$$

In the limit as $a \to \infty$ and $m_a \to 0$, the sum becomes an integral and

$$\vec{\mathbf{L}}_{cm,k} = \sum_{j=1}^{3} \int_{body} dm(\delta_{kj} r^2 - r_j r_k)\omega_j$$

where r_j is the jth component coordinate of the mass element dm, and r is the distance form the center of mass to the mass element.

Define the *moment of inertia tensor* by

$$I_{kj} = \int_{body} dm(\delta_{kj} r^2 - r_j r_k) .$$

Then

$$\vec{\mathbf{L}}_{cm,k} = \sum_{j=1}^{3} I_{kj}\omega_j$$

For example for a finite number of particles the 1-component of the angular momentum is

$$\vec{\mathbf{L}}_{cm,1} = \sum_{j=1}^{3} I_{1j}\omega_j = \sum_{j=1}^{3}\sum_{a} m_a ({x_{a,1}}^2 + {x_{a,2}}^2 + {x_{a,3}}^2)\delta_{1j}\omega_j - \sum_{j=1}^{3}\sum_{a} m_a (x_{a,j} x_{a,k})\omega_j$$

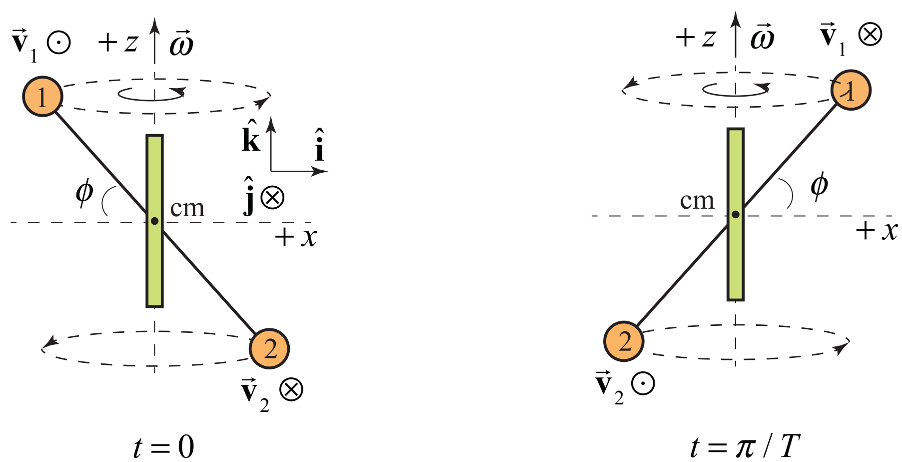
$$= \sum_{a} m_a ({x_{a,1}}^2 + {x_{a,2}}^2 + {x_{a,3}}^2)\omega_1 - \sum_{a} m_a (({x_{a,1}}^2\omega_1 + x_{a,2}x_{a,1}\omega_2 + x_{a,3}x_{a,1}\omega_3)$$

$$= \sum_{a} m_a (({x_{a,2}}^2 + {x_{a,3}}^2)\omega_1 - x_{a,2}x_{a,1}\omega_2 - x_{a,3}x_{a,1}\omega_2)$$

in agreement with our result above.

Example 22.5.1: Angular Momentum and Torque for a Rotating Skew Rod (without using principle axis theorem)

Consider a simple rigid body consisting of two particles of mass m separated by a rod of length $2l$ and negligible mass. The midpoint of the rod is attached to a vertical axis that rotates with angular velocity $\vec{\boldsymbol{\omega}} = \omega\,\hat{\mathbf{k}}$ about the z-axis. The rod is skewed from the vertical at an angle ϕ. Set time $t = 0$ when the rod is in the position shown in figure below left. At $t = \pi / \omega$ the rod has rotated to the position shown in the figure below right.



a) Find the direction and magnitude of the angular momentum about the center of mass at $t = 0$.

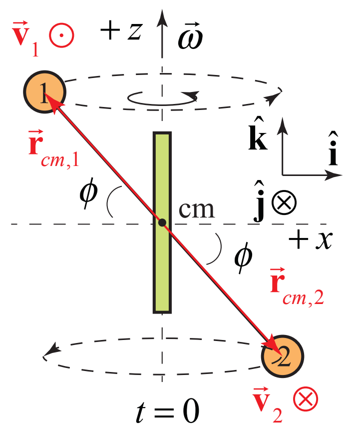
b) Find the direction and magnitude of the torque about the center of mass at time $t = 0$.

Solution:

a) We use $\vec{\mathbf{L}}_{cm} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$ for the angular momentum about the center of mass for each particle:

$$\vec{\mathbf{L}}_{cm} = \vec{\mathbf{r}}_1 \times \vec{\mathbf{p}}_1 + \vec{\mathbf{r}}_2 \times \vec{\mathbf{p}}_2 = \vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 + \vec{\mathbf{r}}_2 \times m\vec{\mathbf{v}}_2$$

Since each particle travels at an angular speed ω in a circular orbit of radius $\ell\cos\phi$, the speed of each particle is given by $v = \omega\ell\cos\phi$. We choose a coordinate system shown in the figure below



For particle 1: $\vec{\mathbf{r}}_1 = -l\cos\phi\,\hat{\mathbf{i}} + l\sin\phi\,\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}_1 = -l\cos\phi\omega\hat{\mathbf{j}}$. Thus

$$\vec{\mathbf{L}}_{cm,1} = \vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 = (-l\cos\phi\,\hat{\mathbf{i}} + l\sin\phi\hat{\mathbf{k}}) \times (-ml\cos\phi\omega\,\hat{\mathbf{j}}).$$

After calculating the cross products, we have that the angular momentum about the center of mass for particle 1 is

$$\vec{\mathbf{L}}_{cm,1} = ml^2\omega\cos\phi(\cos\phi\,\hat{\mathbf{k}} + \sin\phi\,\hat{\mathbf{i}}).$$

Note that for particle 2, $\vec{\mathbf{r}}_2 = -\vec{\mathbf{r}}_1$ and $\vec{\mathbf{v}}_2 = -\vec{\mathbf{v}}_1$, so

$$\vec{\mathbf{L}}_{cm,2} = \vec{\mathbf{r}}_2 \times m\vec{\mathbf{v}}_2 = \vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 = \vec{\mathbf{L}}_{cm,1}.$$

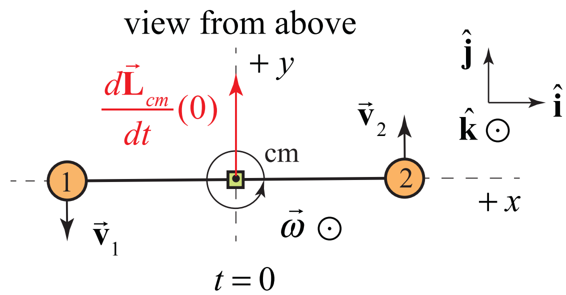
Thus the angular momentum about the center of mass at time $t = 0$ is given by

$$\vec{\mathbf{L}}_{cm}(0) = 2ml^2\omega\cos\phi(\cos\phi\,\hat{\mathbf{k}} + \sin\phi\,\hat{\mathbf{i}}) \tag{38}$$

The magnitude of the angular momentum about the center of mass is given by

$$L_{cm} = 2ml^2\omega\cos\phi(\cos^2\phi + \sin^2\phi) = 2ml^2\omega\cos\phi.$$

b) At $t=0$, the rod is rotating in the $x-y$ plane. The figure below shows the orientation of the rod as seen from above.
\



The z-component of the angular momentum about the center of mass is constant and the x-component of the angular momentum about the center of mass is changing in time as the rod rotates and is given by

$$\vec{\mathbf{L}}_{cm,x}(0) = 2ml^2\omega\cos\phi\sin\phi\,\hat{\mathbf{i}} \tag{39}$$

The time derivative of the angular momentum about the center of mass is perpendicular the angular momentum, points in the positive y-direction, and has a magnitude that is equal to $\left|\vec{\mathbf{L}}_{cm,x}(0)\right|\omega$. Therefore

$$\frac{d\vec{\mathbf{L}}_{cm}}{dt}(0) = \left|\vec{\mathbf{L}}_{cm,x}(0)\right|\omega\hat{\mathbf{j}} = 2ml^2\omega^2\cos\phi\sin\phi\,\hat{\mathbf{j}}.$$

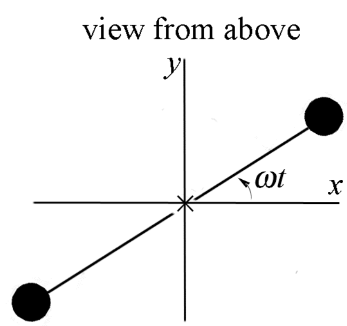
The torque about the center of mass is given by

$$\vec{\boldsymbol{\tau}}_{cm} = \frac{d\vec{\mathbf{L}}_{cm}}{dt}$$

Therefore at $t=0$, we that

$$\vec{\boldsymbol{\tau}}_{cm} = 2ml^2\omega^2\cos\phi\sin\phi\,\hat{\mathbf{j}}\ . \tag{40}$$

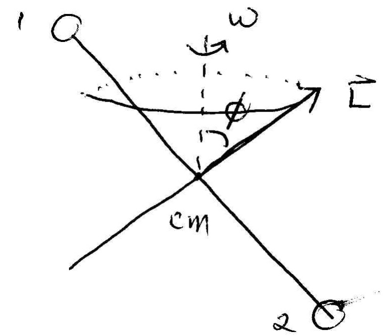
b) At t, in the figure below the rod has rotated by an angle $\theta = \omega t$ in the $x-z$ plane



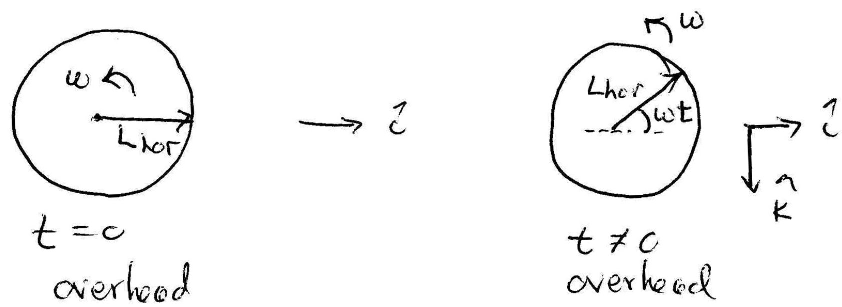
The angular momentum about the center of mass is then

$$\vec{\mathbf{L}}_{cm}(t) = 2ml^2\omega\cos\phi(\cos\phi\,\hat{\mathbf{k}} + \sin\phi(\cos(\omega t)\,\hat{\mathbf{i}} + \sin(\omega t)\hat{\mathbf{j}})\ . \tag{41}$$

c) As the rod rotates, the angular momentum vector is precessing at an angular speed ω. The horizontal component of the angular momentum (the part that rotates) is given by $L\sin\phi$.



At the instant shown in the figure below (shown from the overhead perspective), the horizontal component of the angular momentum about the center of mass points in the $\hat{\mathbf{i}}$-direction, and the direction of the change of the angular momentum is into the page ($\hat{\mathbf{k}}$-direction).



The time derivative of the angular momentum about the center of mass is given by

$$\frac{d\vec{\mathbf{L}}_{cm}(t)}{dt} = 2ml^2\omega^2\cos\phi(\sin\phi(-\sin(\omega t)\,\hat{\mathbf{i}} + \cos(\omega t)\hat{\mathbf{j}})$$

The torque as a function of time is given by

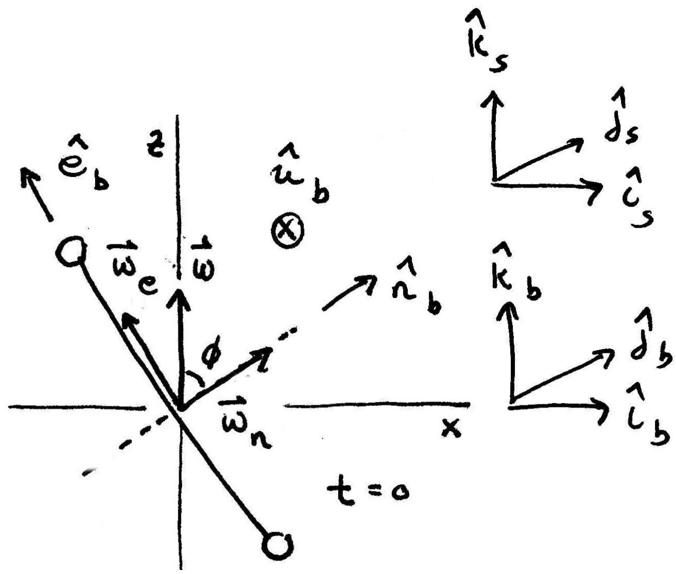
$$\vec{\boldsymbol{\tau}}_{cm}(t) = 2ml^2\omega^2\cos\phi(\sin\phi(-\sin(\omega t)\,\hat{\mathbf{i}} + \cos(\omega t)\hat{\mathbf{j}}) \qquad (42)$$

Example 22.5.2: Principal Axes and Angular Momentum for a Skewed Rod

a) What are the principal axes of a rotating skewed rod.
b) Find the components of the angular velocity about those axes.
c) Find the angular momentum about the center of the skewed rod.

Solution:

a) The principal axes are a set of axes that coincide with the symmetry axes of the body. The principal axes for the skewed rod are as follows: an axis along the length of the rod and two axes forming a plane perpendicular to the rod. At time $t = 0$, the position of the rod is shown in the figure below.

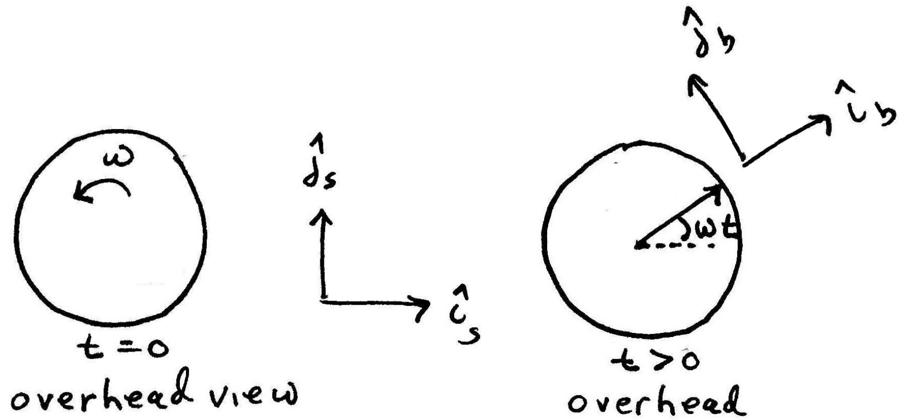


Choose three unit vectors that point along these principal axes as follows. The subscript b denotes axes associated with the body. Choose a unit vector $\hat{\mathbf{e}}_b$ that points from the origin to particle 1. Choose a second unit vector $\hat{\mathbf{n}}_b$ perpendicular to the rod lying in the plane formed by $\hat{\mathbf{e}}_b$ and $\vec{\omega}$, and perpendicular to $\hat{\mathbf{e}}_b$. Choose a third unit vector $\hat{\mathbf{u}}_b$ perpendicular to $\hat{\mathbf{n}}_b$ and $\hat{\mathbf{e}}_b$ pointing into the page of the figure above to complete the description of the principal axes. These axes are fixed to the body. Note that with respect to the x-y-z axis that are fixed in space, the body principal axes are rotating. Choose a Cartesian set of body axes $(\hat{\mathbf{i}}_b, \hat{\mathbf{j}}_b, \hat{\mathbf{k}}_b)$ --they are not the principal axes-- that at $t = 0$ coincide with a set of fixed spatial unit vectors $(\hat{\mathbf{i}}_s, \hat{\mathbf{j}}_s, \hat{\mathbf{k}}_s)$. The unit vectors associated with the principal axes are given by

$$\begin{aligned} \hat{\mathbf{e}}_b &= \sin\phi\, \hat{\mathbf{k}}_b - \cos\phi\, \hat{\mathbf{i}}_b \\ \hat{\mathbf{n}}_b &= \cos\phi\, \hat{\mathbf{k}}_b + \sin\phi\, \hat{\mathbf{i}}_b \\ \hat{\mathbf{u}}_b &= \hat{\mathbf{j}}_b. \end{aligned} \tag{43}$$

At $t > 0$, as the body rotates, the body and space z-axes always remain aligned so $\hat{\mathbf{k}}_b = \hat{\mathbf{k}}_s$ however the body unit vectors $(\hat{\mathbf{i}}_b, \hat{\mathbf{j}}_b)$ no longer coincide with the fixed space unit vectors $(\hat{\mathbf{i}}_s, \hat{\mathbf{j}}_s)$. As the body rotates, the components of the unit vectors for the principal axes lying in the x-y plane change according to

$$\hat{\mathbf{i}}_b = \cos\omega t\,\hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s$$
$$\hat{\mathbf{j}}_b = -\sin\omega t\,\hat{\mathbf{i}}_s + \cos\omega t\hat{\mathbf{j}}_s. \tag{44}$$



So the principal axes unit vectors are given in terms of the fixed space unit vectors by

$$\hat{\mathbf{e}}_b = \sin\phi\,\hat{\mathbf{k}}_s - \cos\phi\,(\cos\omega t\,\hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s)$$
$$\hat{\mathbf{n}}_b = \cos\phi\,\hat{\mathbf{k}}_s + \sin\phi\,(\cos\omega t\,\hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s) \tag{45}$$
$$\hat{\mathbf{u}}_b = -\sin\omega t\,\hat{\mathbf{i}}_s + \cos\omega t\hat{\mathbf{j}}_s.$$

b) The angular velocity $\vec{\omega} = \omega\,\hat{\mathbf{k}}_b$ can be decomposed into components along the body principal axes,

$$\vec{\omega} = \omega_n\,\hat{\mathbf{n}}_b + \omega_e\hat{\mathbf{e}}_b = \omega\cos\phi\hat{\mathbf{n}}_b + \omega\sin\phi\hat{\mathbf{e}}_b \tag{46}$$

We now use the principal axes theorem that states that the angular momentum about the center of mass for a symmetric body can be written as a sum

$$\vec{\mathbf{L}}_{cm} = I_{cm,n}\,\omega_n\hat{\mathbf{n}}_b + I_{cm,e}\,\omega_e\hat{\mathbf{e}}_b. \tag{47}$$

The moment of inertia about the perpendicular to the rod is $I_{cm,n} = 2ml^2$. Also since we are assuming that the rod is massless and that the particles are point-like particles $I_{cm,e} = 0$. Then Eq. (47) for the angular momentum about the center of mass becomes

$$\vec{\mathbf{L}}_{cm} = 2ml^2\omega\cos\phi\,\hat{\mathbf{n}}_b. \tag{48}$$

At time $t = 0$, using the fact that $\hat{\mathbf{n}}_b = (\cos\phi\,\hat{\mathbf{k}}_b + \sin\phi\,\hat{\mathbf{i}}_b)$, the angular momentum about the center of mass is

$$\vec{\mathbf{L}}_{cm}(t=0) = 2ml^2\omega\cos\phi(\cos\phi\,\hat{\mathbf{k}}_b + \sin\phi\,\hat{\mathbf{i}}_b), \qquad (49)$$

in agreement with our earlier result (Eq. (38)). We can use Eq. (44) to write the angular momentum about the center of mass as a function of time in terms of the fixed space unit vectors

$$\vec{\mathbf{L}}_{cm}(t) = 2ml^2\omega\cos\phi\,\hat{\mathbf{n}}_b = 2ml^2\omega\cos\phi\cos\phi\;\hat{\mathbf{k}}_s + 2ml^2\omega\cos\phi\sin\phi(\cos\omega t\,\hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s). \qquad (50)$$