

Chapter 20 Rigid Body: Translation and Rotational Motion Kinematics for Fixed Axis Rotation

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Chapter 20 Rigid Body: Translation and Rotational Motion

Kinematics for Fixed Axis Rotation

Hence I feel no shame in asserting that this whole region engirdled by the moon, and the center of the earth, traverse this grand circle amid the rest of the planets in an annual revolution around the sun. Near the sun is the center of the universe. Moreover, since the sun remains stationary, whatever appears as a motion of the sun is really due rather to the motion of the earth.¹

Copernicus

20.1 Introduction

The general motion of a rigid body of mass m consists of a translation of the center of mass with velocity \vec{V}_{cm} and a rotation about the center of mass with all elements of the rigid body rotating with the same angular velocity $\vec{\omega}_{cm}$. We prove this result in Appendix A. Figure 20.1 shows the center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

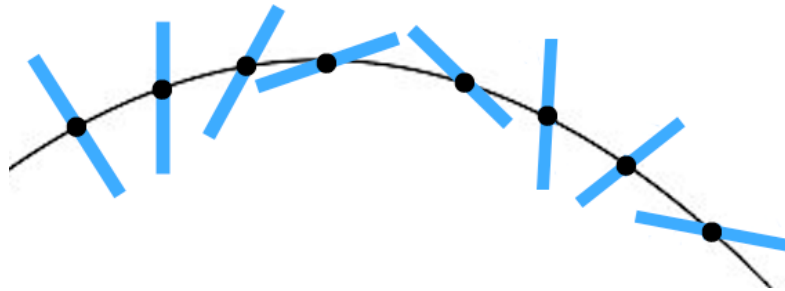


Figure 20.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

20.2 Constrained Motion: Translation and Rotation

We shall encounter many examples of a rolling object whose motion is constrained. For example we will study the motion of an object rolling along a level or inclined surface and the motion of a yo-yo unwinding and winding along a string. We will examine the constraint conditions between the translational quantities that describe the motion of the center of mass, displacement, velocity and acceleration, and the rotational quantities that describe the motion about the center of mass, angular displacement, angular velocity and angular acceleration. We begin with a discussion about the rotation and translation of a rolling wheel.

¹Nicolaus Copernicus, *De revolutionibus orbium coelestium* (*On the Revolutions of the Celestial Spheres*), Book 1 Chapter 10.

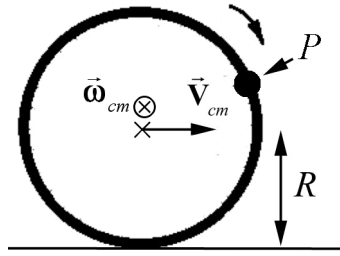


Figure 20.2 Rolling Wheel

Consider a wheel of radius R is rolling in a straight line (Figure 20.2). The center of mass of the wheel is moving in a straight line at a constant velocity \vec{V}_{cm} . Let's analyze the motion of a point P on the rim of the wheel.

Let \vec{v}_p denote the velocity of a point P on the rim of the wheel with respect to reference frame O at rest with respect to the ground (Figure 20.3a). Let \vec{v}'_p denote the velocity of the point P on the rim with respect to the center of mass reference frame O_{cm} moving with velocity \vec{V}_{cm} with respect to at O (Figure 20.3b). (You should review the definition of the center of mass reference frame in Chapter 15.2.1.) We can use the law of addition of velocities (Eq.15.2.4) to relate these three velocities,

$$\vec{v}_p = \vec{v}'_p + \vec{V}_{cm}. \quad (20.2.1)$$

Let's choose Cartesian coordinates for the translation motion and polar coordinates for the motion about the center of mass as shown in Figure 20.3.

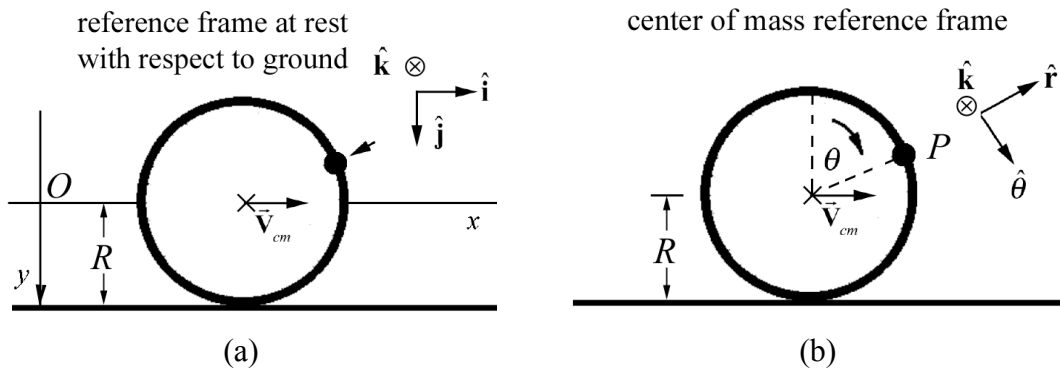


Figure 20.3 (a) reference frame fixed to ground, (b) center of mass reference frame

The center of mass velocity in the reference frame fixed to the ground is given by

$$\vec{V}_{cm} = V_{cm} \hat{i}. \quad (20.2.2)$$

where V_{cm} is the speed of the center of mass. The position of the center of mass in the reference frame fixed to the ground is given by

$$\vec{\mathbf{R}}_{\text{cm}}(t) = (X_{\text{cm},0} + V_{\text{cm}}t)\hat{\mathbf{i}}, \quad (20.2.3)$$

where $X_{\text{cm},0}$ is the initial x -component of the center of mass at $t=0$. The angular velocity of the wheel in the center of mass reference frame is given by

$$\vec{\omega}_{\text{cm}} = \omega_{\text{cm}} \hat{\mathbf{k}}. \quad (20.2.4)$$

where ω_{cm} is the angular speed. The point P on the rim is undergoing uniform circular motion with the velocity in the center of mass reference frame given by

$$\vec{\mathbf{v}}'_P = R\omega_{\text{cm}} \hat{\boldsymbol{\theta}}. \quad (20.2.5)$$

If we want to use the law of addition of velocities then we should express $\vec{\mathbf{v}}'_P = R\omega_{\text{cm}} \hat{\boldsymbol{\theta}}$ in Cartesian coordinates. Assume that at $t=0$, $\theta(t=0)=0$ i.e. the point P is at the top of the wheel at $t=0$. Then the unit vectors in polar coordinates satisfy (Figure 20.4)

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta \hat{\mathbf{i}} - \cos\theta \hat{\mathbf{j}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta \hat{\mathbf{i}} + \sin\theta \hat{\mathbf{j}} \end{aligned} \quad (20.2.6)$$

Therefore the velocity of the point P on the rim in the center of mass reference frame is given by

$$\vec{\mathbf{v}}'_P = R\omega_{\text{cm}} \hat{\boldsymbol{\theta}} = R\omega_{\text{cm}} (\cos\theta \hat{\mathbf{i}} + \sin\theta \hat{\mathbf{j}}). \quad (20.2.7)$$

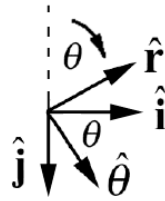


Figure 20.4 Unit vectors

Now substitute Eqs. (20.2.2) and (20.2.7) into Eq. (20.2.1) for the velocity of a point P on the rim in the reference frame fixed to the ground

$$\begin{aligned} \vec{\mathbf{v}}_P &= R\omega_{\text{cm}} (\cos\theta \hat{\mathbf{i}} + \sin\theta \hat{\mathbf{j}}) + V_{\text{cm}} \hat{\mathbf{i}} \\ &= (V_{\text{cm}} + R\omega_{\text{cm}} \cos\theta) \hat{\mathbf{i}} + R\omega_{\text{cm}} \sin\theta \hat{\mathbf{j}} \end{aligned} \quad (20.2.8)$$

The point P is in contact with the ground when $\theta = \pi$. At that instant the velocity of a point P on the rim in the reference frame fixed to the ground is

$$\vec{v}_P(\theta = \pi) = (V_{cm} - R\omega_{cm})\hat{i}. \quad (20.2.9)$$

What velocity does the observer at rest on the ground measure for the point on the rim when that point is in contact with the ground? In order to understand the relationship between V_{cm} and ω_{cm} , we consider the displacement of the center of mass for a small time interval Δt (Figure 20.5).

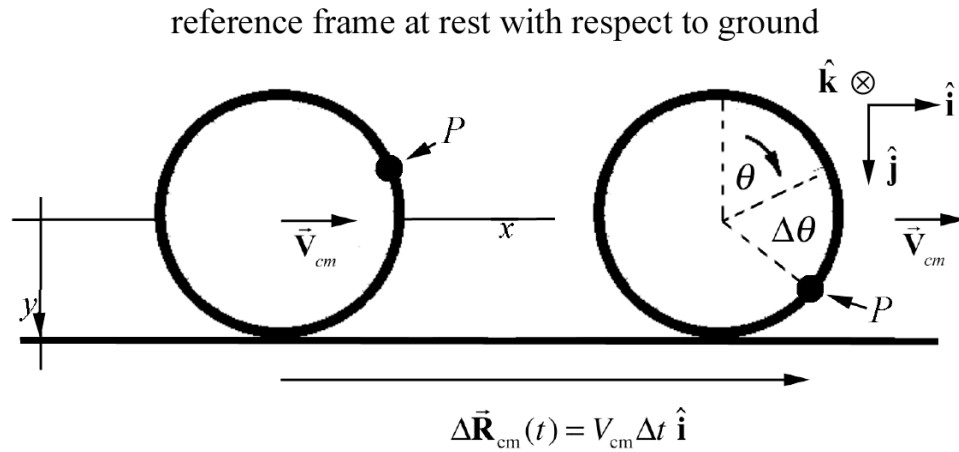


Figure 20.5 Displacement of center of mass in ground reference frame.

From Eq. (20.2.3) the x -component of the displacement of the center of mass is

$$\Delta X_{cm} = V_{cm} \Delta t. \quad (20.2.10)$$

The point P on the rim in the center of mass reference frame is undergoing circular motion (Figure 20.6).

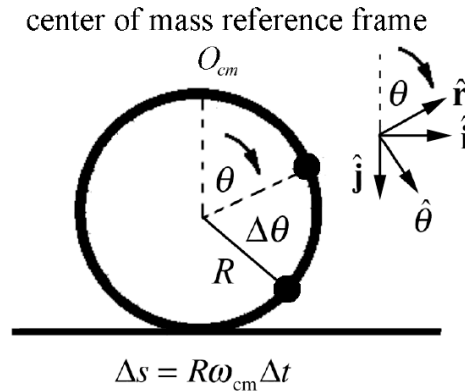


Figure 20.6: Small displacement of point on rim in center of mass reference frame.

In the center of mass reference frame, the magnitude of the tangential displacement is given by the arc length subtended by the angular displacement $\Delta\theta = \omega_{\text{cm}}\Delta t$,

$$\Delta s = R\Delta\theta = R\omega_{\text{cm}}\Delta t. \quad (20.2.11)$$

Case 1: if the x -component of the displacement of the center of mass is equal to the arc length subtended by $\Delta\theta$, then the wheel is rolling without slipping or skidding, **rolling without slipping** for short, along the surface with

$$\Delta X_{\text{cm}} = \Delta s. \quad (20.2.12)$$

Substitute Eq. (20.2.10) and Eq. (20.2.11) into Eq. (20.2.12) and divide through by Δt . Then the rolling without slipping condition becomes

$$V_{\text{cm}} = R\omega_{\text{cm}}, \quad (\text{rolling without slipping}). \quad (20.2.13)$$

Case 2: if the x -component of the displacement of the center of mass is greater than the arc length subtended by $\Delta\theta$, then the wheel is **skidding** along the surface with

$$\Delta X_{\text{cm}} > \Delta s. \quad (20.2.14)$$

Substitute Eqs. (20.2.10) and (20.2.11) into Eq. (20.2.14) and divide through by Δt , then

$$V_{\text{cm}} > R\omega_{\text{cm}}, \quad (\text{skidding}). \quad (20.2.15)$$

Case 3: if the x -component of the displacement of the center of mass is less than the arc length subtended by $\Delta\theta$, then the wheel is slipping along the surface with

$$\Delta X_{\text{cm}} < \Delta s. \quad (20.2.16)$$

Arguing as above the slipping condition becomes

$$V_{\text{cm}} < R\omega_{\text{cm}}, \quad (\text{slipping}). \quad (20.2.17)$$

20.2.1 Rolling without slipping

When a wheel is rolling without slipping, the velocity of a point P on the rim is zero when it is in contact with the ground. In Eq. (20.2.9) set $\theta = \pi$,

$$\vec{v}_P(\theta = \pi) = (V_{\text{cm}} - R\omega_{\text{cm}})\hat{\mathbf{i}} = (R\omega_{\text{cm}} - R\omega_{\text{cm}})\hat{\mathbf{i}} = \vec{0}. \quad (20.2.18)$$

This makes sense because the velocity of the point P on the rim in the center of mass reference frame when it is in contact with the ground points in the opposite direction as the translational motion of the center of mass of the wheel. The two velocities have the

same magnitude so the vector sum is zero. The observer at rest on the ground sees the contact point on the rim at rest relative to the ground.

Thus any frictional force acting between the tire and the ground on the wheel is static friction because the two surfaces are instantaneously at rest with respect to each other. Recall that the direction of the static frictional force depends on the other forces acting on the wheel.

Example 20.1 Bicycle Wheel Rolling Without Slipping

Consider a bicycle wheel of radius R that is rolling in a straight line without slipping. The velocity of the center of mass in a reference frame fixed to the ground is given by velocity \vec{V}_{cm} . A bead is fixed to a spoke a distance b from the center of the wheel (Figure 20.7). (a) Find the position, velocity, and acceleration of the bead as a function of time in the center of mass reference frame. (b) Find the position, velocity, and acceleration of the bead as a function of time as seen in a reference frame fixed to the ground.

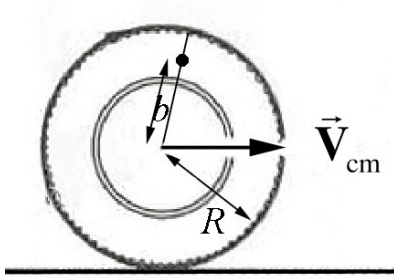


Figure 20.7 Example 20.1

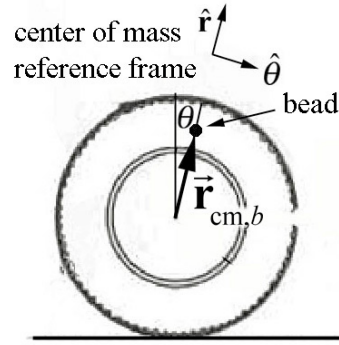


Figure 20.8 Coordinate system for bead in center of mass reference frame

Solution: a) Choose the center of mass reference frame with an origin at the center of the wheel, and moving with the wheel. Choose polar coordinates (Figure 20.8). The z -component of the angular velocity $\omega_{\text{cm}} = d\theta / dt > 0$. Then the bead is moving uniformly in a circle of radius $r = b$ with the position, velocity, and acceleration given by

$$\vec{r}'_b = b \hat{r}, \quad \vec{v}'_b = b\omega_{\text{cm}} \hat{\theta}, \quad \vec{a}'_b = -b\omega_{\text{cm}}^2 \hat{r}. \quad (20.2.19)$$

Because the wheel is rolling without slipping, the velocity of a point on the rim of the wheel has speed $v'_p = R\omega_{\text{cm}}$. This is equal to the speed of the center of mass of the wheel V_{cm} , thus

$$V_{\text{cm}} = R\omega_{\text{cm}}. \quad (20.2.20)$$

Note that at $t = 0$, the angle $\theta = \theta_0 = 0$. So the angle grows in time as

$$\theta(t) = \omega_{cm} t = (V_{cm} / R)t . \quad (20.2.21)$$

The velocity and acceleration of the bead with respect to the center of the wheel are then

$$\vec{v}'_b = \frac{bV_{cm}}{R} \hat{\theta}, \quad \vec{a}'_b = -\frac{bV_{cm}^2}{R^2} \hat{r} . \quad (20.2.22)$$

b) Define a second reference frame fixed to the ground with choice of origin, Cartesian coordinates and unit vectors as shown in Figure 20.9.

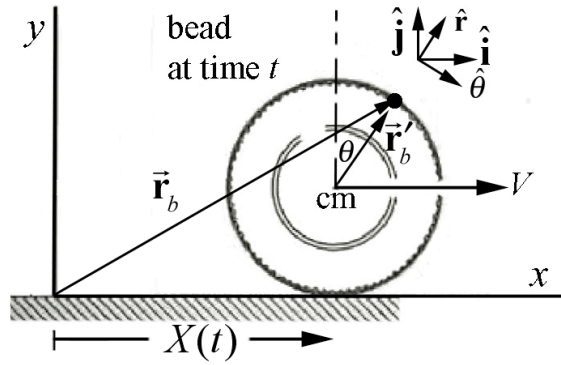


Figure 20.9 Coordinates of bead in reference frame fixed to ground

Then the position vector of the center of mass in the reference frame fixed to the ground is given by

$$\vec{R}_{cm}(t) = X_{cm} \hat{i} + R \hat{j} = V_{cm} t \hat{i} + R \hat{j} . \quad (20.2.23)$$

The relative velocity of the two frames is the derivative

$$\vec{V}_{cm} = \frac{d\vec{R}_{cm}}{dt} = \frac{dX_{cm}}{dt} \hat{i} = V_{cm} \hat{i} . \quad (20.2.24)$$

Because the center of the wheel is moving at a uniform speed the relative acceleration of the two frames is zero,

$$\vec{A}_{cm} = \frac{d\vec{V}_{cm}}{dt} = \vec{0} . \quad (20.2.25)$$

Define the position, velocity, and acceleration in this frame (with respect to the ground) by

$$\vec{r}_b(t) = x_b(t) \hat{i} + y_b(t) \hat{j}, \quad \vec{v}_b(t) = v_{b,x}(t) \hat{i} + v_{b,y}(t) \hat{j}, \quad \vec{a}(t) = a_{b,x}(t) \hat{i} + a_{b,y}(t) \hat{j} . \quad (20.2.26)$$

Then the position vectors are related by

$$\vec{\mathbf{r}}_b(t) = \vec{\mathbf{R}}_{\text{cm}}(t) + \vec{\mathbf{r}}'_b(t). \quad (20.2.27)$$

In order to add these vectors we need to decompose the position vector in the center of mass reference frame into Cartesian components,

$$\vec{\mathbf{r}}'_b(t) = b \hat{\mathbf{r}}(t) = b \sin \theta(t) \hat{\mathbf{i}} + b \cos \theta(t) \hat{\mathbf{j}}. \quad (20.2.28)$$

Then using the relation $\theta(t) = (V_{\text{cm}} / R)t$, Eq. (20.2.28) becomes

$$\begin{aligned} \vec{\mathbf{r}}_b(t) &= \vec{\mathbf{R}}_{\text{cm}}(t) + \vec{\mathbf{r}}'_b(t) = (V_{\text{cm}} t \hat{\mathbf{i}} + R \hat{\mathbf{j}}) + (b \sin \theta(t) \hat{\mathbf{i}} + b \cos \theta(t) \hat{\mathbf{j}}) \\ &= (V_{\text{cm}} t + b \sin((V_{\text{cm}} / R)t)) \hat{\mathbf{i}} + (R + b \cos((V_{\text{cm}} / R)t)) \hat{\mathbf{j}} \end{aligned} \quad (20.2.29)$$

Thus the position components of the bead with respect to the reference frame fixed to the ground are given by

$$x_b(t) = V_{\text{cm}} t + b \sin((V_{\text{cm}} / R)t) \quad (20.2.30)$$

$$y_b(t) = R + b \cos((V_{\text{cm}} / R)t) \quad (20.2.31)$$

A plot of the y -component vs. the x -component of the position of the bead in the reference frame fixed to the ground is shown in Figure 20.10 below using the values $V_{\text{cm}} = 5 \text{ m} \cdot \text{s}^{-1}$, $R = 0.25 \text{ m}$, and $b = 0.125 \text{ m}$. We can differentiate the position vector in the reference frame fixed to the ground to find the velocity of the bead

$$\vec{\mathbf{v}}_b(t) = \frac{d\vec{\mathbf{r}}_b}{dt}(t) = \frac{d}{dt}(V_{\text{cm}} t + b \sin((V_{\text{cm}} / R)t)) \hat{\mathbf{i}} + \frac{d}{dt}(R + b \cos((V_{\text{cm}} / R)t)) \hat{\mathbf{j}}, \quad (20.2.32)$$

$$\vec{\mathbf{v}}_b(t) = (V_{\text{cm}} + (b / R)V_{\text{cm}} \cos((V_{\text{cm}} / R)t)) \hat{\mathbf{i}} - ((b / R)V_{\text{cm}} \sin((V_{\text{cm}} / R)t)) \hat{\mathbf{j}}. \quad (20.2.33)$$

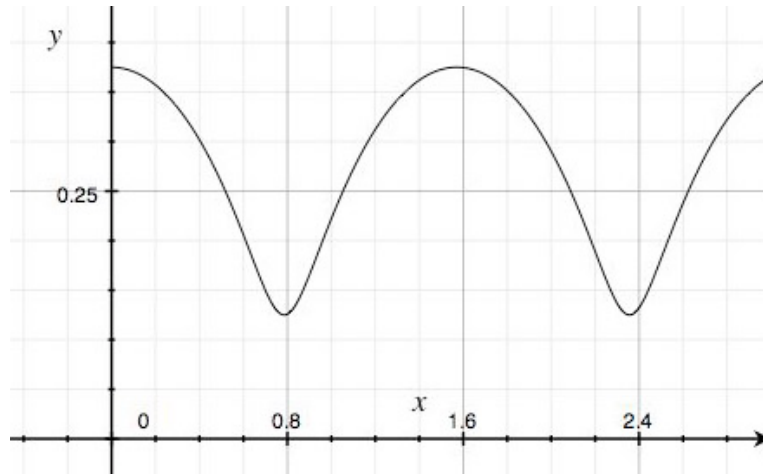


Figure 20.10 Plot of the y -component vs. the x -component of the position of the bead

Alternatively, we can decompose the velocity of the bead in the center of mass reference frame into Cartesian coordinates

$$\vec{v}'_b(t) = (b/R)V_{cm} (\cos((V_{cm}/R)t) \hat{i} - \sin((V_{cm}/R)t) \hat{j}). \quad (20.2.34)$$

The law of addition of velocities is then

$$\vec{v}_b(t) = \vec{V}_{cm} + \vec{v}'_b(t), \quad (20.2.35)$$

$$\vec{v}_b(t) = V_{cm} \hat{i} + (b/R)V_{cm} (\cos((V_{cm}/R)t) \hat{i} - \sin((V_{cm}/R)t) \hat{j}), \quad (20.2.36)$$

$$\vec{v}_b(t) = (V_{cm} + (b/R)V_{cm} \cos((V_{cm}/R)t)) \hat{i} - (b/R)V_{cm} \sin((V_{cm}/R)t) \hat{j}, \quad (20.2.37)$$

in agreement with our previous result. The acceleration is the same in either frame so

$$\vec{a}_b(t) = \vec{a}'_b = -(b/R^2)V_{cm}^2 \hat{r} = -(b/R^2)V_{cm}^2 (\sin((V_{cm}/R)t) \hat{i} + \cos((V_{cm}/R)t) \hat{j}). \quad (20.2.38)$$

When the bead is at the rim of the wheel, $b = R$, then the position of the bead in the reference frame fixed to the ground is given by

$$\vec{r}_b(t) = (V_{cm}t + R \sin((V_{cm}/R)t)) \hat{i} + R(1 + \cos((V_{cm}/R)t)) \hat{j}. \quad (20.2.39)$$

The path traced out by the bead in the reference frame fixed to the ground is called a *cycloid*.

Example 20.2 Cylinder Rolling Without Slipping Down an Inclined Plane

A uniform cylinder of outer radius R and mass M with moment of inertia about the center of mass $I_{cm} = (1/2)MR^2$ starts from rest and rolls without slipping down an incline tilted at an angle β from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. What is the relation between the component of the acceleration of the center of mass in the direction down the inclined plane and the component of the angular acceleration into the page of Figure 20.11?

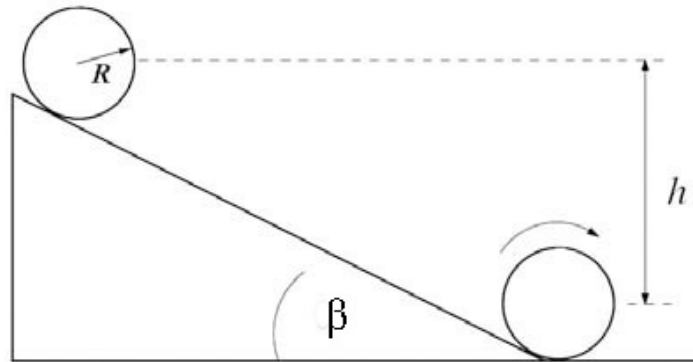


Figure 20.11 Example 20.2

Solution: We begin by choosing a coordinate system for the translational and rotational motion as shown in Figure 20.12.

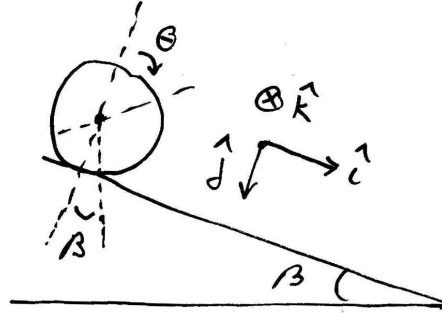


Figure 20.12 Coordinate system for rolling cylinder

For a time interval Δt , the displacement of the center of mass is given by $\Delta \vec{R}_{cm}(t) = \Delta X_{cm} \hat{i}$. The arc length due to the angular displacement of a point on the rim during the time interval Δt is given by $\Delta s = R\Delta\theta$. The rolling without slipping condition is

$$\Delta X_{cm} = R\Delta\theta .$$

If we divide both sides by Δt and take the limit as $\Delta t \rightarrow 0$ then the rolling without slipping condition show that the x -component of the center of mass velocity is equal to the magnitude of the tangential component of the velocity of a point on the rim

$$V_{cm} = \lim_{\Delta t \rightarrow 0} \frac{\Delta X_{cm}}{\Delta t} = \lim_{\Delta t \rightarrow 0} R \frac{\Delta\theta}{\Delta t} = R\omega_{cm} .$$

Similarly if we differentiate both sides of the above equation, we find a relation between the x -component of the center of mass acceleration is equal to the magnitude of the tangential component of the acceleration of a point on the rim

$$A_{cm} = \frac{dV_{cm}}{dt} = R \frac{d\omega_{cm}}{dt} = R\alpha_{cm} .$$

Example 20.3 Falling Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R (Figure 20.13a). Its moment of inertia about the center of mass can be taken to be $I = (1/2)mR^2$ (the thickness of the string can be neglected). The Yo-Yo is released from rest. What is the relation between the angular acceleration about the center of mass and the linear acceleration of the center of mass?

Solution: Choose coordinates as shown in Figure 20.13b.

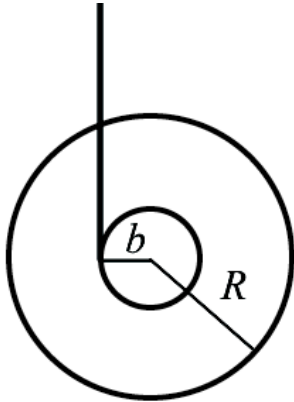


Figure 20.13a Example 20.3

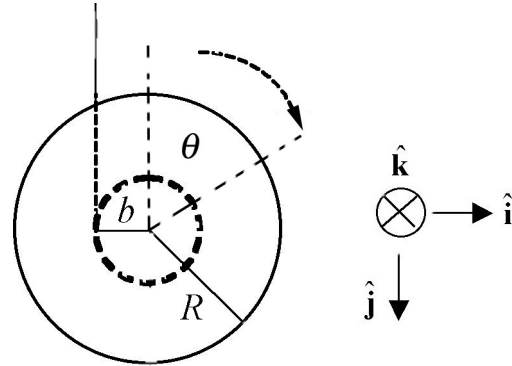


Figure 20.13b Coordinate system for Yo-Yo

Consider a point on the rim of the axle at a distance $r = b$ from the center of mass. As the yo-yo falls, the arc length $\Delta s = b\Delta\theta$ subtended by the rotation of this point is equal to length of string that has unraveled, an amount Δl . In a time interval Δt , $b\Delta\theta = \Delta l$.

Therefore $b\Delta\theta / \Delta t = \Delta l / \Delta t$. Taking limits, noting that, $V_{\text{cm},y} = dl / dt$, we have that

$b\omega_{\text{cm}} = V_{\text{cm},y}$. Differentiating a second time yields $b\alpha_{\text{cm}} = A_{\text{cm},y}$.

Example 20.4 Unwinding Drum

Drum A of mass m and radius R is suspended from a drum B also of mass m and radius R , which is free to rotate about its axis. The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind (Figure 20.14). Gravity acts with acceleration g downwards. Both drums are initially at rest. Find the initial acceleration of drum A , assuming that it moves straight down.

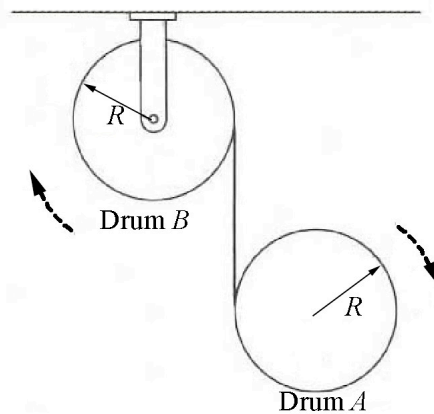


Figure 20.14 Example 20.4

Solution: The key to solving this problem is to determine the relation between the three kinematic quantities α_A , α_B , and a_A , the angular accelerations of the two drums and the linear acceleration of drum A . Choose the positive y -axis pointing downward with the origin at the center of drum B . After a time interval Δt , the center of drum A has undergone a displacement Δy . An amount of tape $\Delta l_A = R\Delta\theta_A$ has unraveled from drum A , and an amount of tape $\Delta l_B = R\Delta\theta_B$ has unraveled from drum B . Therefore the displacement of the center of drum A is equal to the total amount of tape that has unwound from the two drums, $\Delta y = \Delta l_A + \Delta l_B = R\Delta\theta_A + R\Delta\theta_B$. Dividing through by Δt and taking the limit as $\Delta t \rightarrow 0$ yields

$$\frac{dy}{dt} = R \frac{d\theta_A}{dt} + R \frac{d\theta_B}{dt}.$$

Differentiating a second time yields the desired relation between the angular accelerations of the two drums and the linear acceleration of drum A ,

$$\begin{aligned} \frac{d^2 y}{dt^2} &= R \frac{d^2 \theta_A}{dt^2} + R \frac{d^2 \theta_B}{dt^2} \\ a_{A,y} &= R\alpha_A + R\alpha_B. \end{aligned}$$

20.3 Angular Momentum for a System of Particles Undergoing Translational and Rotational

We shall now show that the angular momentum of a body about a point S can be decomposed into two vector parts, the angular momentum of the center of mass (treated as a point particle) about the point S , and the angular momentum of the rotational motion about the center of mass.

Consider a system of N particles located at the points labeled $i = 1, 2, \dots, N$. The angular momentum about the point S is the sum

$$\vec{L}_S^{\text{total}} = \sum_{i=1}^N \vec{L}_{S,i} = \left(\sum_{i=1}^N \vec{r}_{S,i} \times m_i \vec{v}_i \right), \quad (20.3.1)$$

where $\vec{r}_{S,i}$ is the vector from the point S to the i^{th} particle (Figure 20.15) satisfying

$$\vec{r}_{S,i} = \vec{r}_{S,cm} + \vec{r}_{cm,i}, \quad (20.3.2)$$

$$\vec{v}_{S,i} = \vec{V}_{cm} + \vec{v}_{cm,i}, \quad (20.3.3)$$

where $\vec{v}_{S,cm} = \vec{V}_{cm}$. We can now substitute both Eqs. (20.3.2) and (20.3.3) into Eq. (20.3.1) yielding

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^N (\vec{\mathbf{r}}_{S,cm} + \vec{\mathbf{r}}_{cm,i}) \times m_i (\vec{\mathbf{V}}_{cm} + \vec{\mathbf{v}}_{cm,i}). \quad (20.3.4)$$

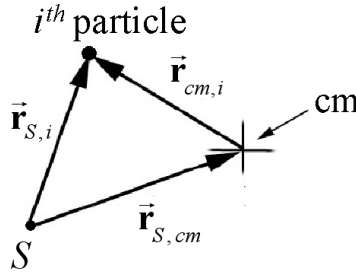


Figure 20.15 Vector Triangle

When we expand the expression in Equation (20.3.4), we have four terms,

$$\begin{aligned} \vec{\mathbf{L}}_S^{\text{total}} &= \sum_{i=1}^N (\vec{\mathbf{r}}_{S,cm} \times m_i \vec{\mathbf{v}}_{cm,i}) + \sum_{i=1}^N (\vec{\mathbf{r}}_{S,cm} \times m_i \vec{\mathbf{V}}_{cm}) \\ &+ \sum_{i=1}^N (\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) + \sum_{i=1}^N (\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{V}}_{cm}). \end{aligned} \quad (20.3.5)$$

The vector $\vec{\mathbf{r}}_{S,cm}$ is a constant vector that depends only on the location of the center of mass and not on the location of the i^{th} particle. Therefore in the first term in the above equation, $\vec{\mathbf{r}}_{S,cm}$ can be taken outside the summation. Similarly, in the second term the velocity of the center of mass $\vec{\mathbf{V}}_{cm}$ is the same for each term in the summation, and may be taken outside the summation,

$$\begin{aligned} \vec{\mathbf{L}}_S^{\text{total}} &= \vec{\mathbf{r}}_{S,cm} \times \left(\sum_{i=1}^N m_i \vec{\mathbf{v}}_{cm,i} \right) + \vec{\mathbf{r}}_{S,cm} \times \left(\sum_{i=1}^N m_i \right) \vec{\mathbf{V}}_{cm} \\ &+ \sum_{i=1}^N (\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) + \left(\sum_{i=1}^N m_i \vec{\mathbf{r}}_{cm,i} \right) \times \vec{\mathbf{V}}_{cm}. \end{aligned} \quad (20.3.6)$$

The first and third terms in Eq. (20.3.6) are both zero due to the fact that

$$\begin{aligned} \sum_{i=1}^N m_i \vec{\mathbf{r}}_{cm,i} &= 0 \\ \sum_{i=1}^N m_i \vec{\mathbf{v}}_{cm,i} &= 0. \end{aligned} \quad (20.3.7)$$

We first show that $\sum_{i=1}^N m_i \vec{\mathbf{r}}_{cm,i}$ is zero. We begin by using Eq. (20.3.2),

$$\begin{aligned}
\sum_{i=1}^N (m_i \vec{r}_{cm,i}) &= \sum_{i=1}^N (m_i (\vec{r}_i - \vec{r}_{S,cm})) \\
&= \sum_{i=1}^N m_i \vec{r}_i - \left(\sum_{i=1}^N m_i \right) \vec{r}_{S,cm} = \sum_{i=1}^N m_i \vec{r}_i - m^{total} \vec{r}_{S,cm}.
\end{aligned} \tag{20.3.8}$$

Substitute the definition of the center of mass (Eq. 10.5.3) into Eq. (20.3.8) yielding

$$\sum_{i=1}^N (m_i \vec{r}_{cm,i}) = \sum_{i=1}^N m_i \vec{r}_i - m^{total} \frac{1}{m^{total}} \sum_{i=1}^N m_i \vec{r}_i = \vec{0}. \tag{20.3.9}$$

The vanishing of $\sum_{i=1}^N m_i \vec{v}_{cm,i} = 0$ follows directly from the definition of the center of mass frame, that the momentum in the center of mass is zero. Equivalently the derivative of Eq. (20.3.9) is zero. We could also simply calculate and find that

$$\begin{aligned}
\sum_i m_i \vec{v}_{cm,i} &= \sum_i m_i (\vec{v}_i - \vec{v}_{cm}) \\
&= \sum_i m_i \vec{v}_i - \vec{v}_{cm} \sum_i m_i \\
&= m^{total} \vec{v}_{cm} - \vec{v}_{cm} m^{total} \\
&= \vec{0}.
\end{aligned} \tag{20.3.10}$$

We can now simplify Eq. (20.3.6) for the angular momentum about the point S using the fact that, $m_T = \sum_{i=1}^N m_i$, and $\vec{p}_{sys} = m_T \vec{V}_{cm}$ (in reference frame O):

$$\vec{L}_S^{total} = \vec{r}_{S,cm} \times \vec{p}_{sys} + \sum_{i=1}^N (\vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}). \tag{20.3.11}$$

Consider the first term in Equation (20.3.11), $\vec{r}_{S,cm} \times \vec{p}_{sys}$; the vector $\vec{r}_{S,cm}$ is the vector from the point S to the center of mass. If we treat the system as a point-like particle of mass m_T located at the center of mass, then the momentum of this point-like particle is $\vec{p}_{sys} = m_T \vec{V}_{cm}$. Thus the first term is the angular momentum about the point S of this “point-like particle”, which is called the **orbital angular momentum** about S ,

$$\vec{L}_S^{orbital} = \vec{r}_{S,cm} \times \vec{p}_{sys}. \tag{20.3.12}$$

for the system of particles.

Consider the second term in Equation (20.3.11), $\sum_{i=1}^N (\vec{r}_{cm,i} \times m_i \vec{v}_{cm,i})$; the quantity inside the summation is the angular momentum of the i^{th} particle with respect to the origin in the center of mass reference frame O_{cm} (recall the origin in the center of mass reference frame is the center of mass of the system),

$$\vec{L}_{cm,i} = \vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}. \quad (20.3.13)$$

Hence the total angular momentum of the system with respect to the center of mass in the center of mass reference frame is given by

$$\vec{L}_{cm}^{spin} = \sum_{i=1}^N \vec{L}_{cm,i} = \sum_{i=1}^N (\vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}). \quad (20.3.14)$$

a vector quantity we call the **spin angular momentum**. Thus we see that the total angular momentum about the point S is the sum of these two terms,

$$\vec{L}_S^{total} = \vec{L}_S^{orbital} + \vec{L}_{cm}^{spin}. \quad (20.3.15)$$

This decomposition of angular momentum into a piece associated with the translational motion of the center of mass and a second piece associated with the rotational motion about the center of mass in the center of mass reference frame is the key conceptual foundation for what follows.

Example 20.5 Earth's Motion Around the Sun

The earth, of mass $m_e = 5.97 \times 10^{24}$ kg and (mean) radius $R_e = 6.38 \times 10^6$ m, moves in a nearly circular orbit of radius $r_{s,e} = 1.50 \times 10^{11}$ m around the sun with a period $T_{orbit} = 365.25$ days, and spins about its axis in a period $T_{spin} = 23$ hr 56 min, the axis inclined to the normal to the plane of its orbit around the sun by 23.5° (in Figure 20.16, the relative size of the earth and sun, and the radius and shape of the orbit are not representative of the actual quantities).

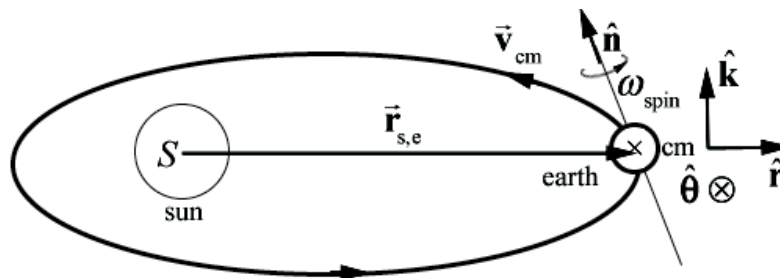


Figure 20.16 Example 20.5

If we approximate the earth as a uniform sphere, then the moment of inertia of the earth about its center of mass is

$$I_{\text{cm}} = \frac{2}{5} m_e R_e^2. \quad (20.3.16)$$

If we choose the point S to be at the center of the sun, and assume the orbit is circular, then the orbital angular momentum is

$$\vec{\mathbf{L}}_S^{\text{orbital}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}_{\text{sys}} = r_{s,e} \hat{\mathbf{r}} \times m_e v_{\text{cm}} \hat{\boldsymbol{\theta}} = r_{s,e} m_e v_{\text{cm}} \hat{\mathbf{k}}. \quad (20.3.17)$$

The velocity of the center of mass of the earth about the sun is related to the orbital angular velocity by

$$v_{\text{cm}} = r_{s,e} \omega_{\text{orbit}}, \quad (20.3.18)$$

where the orbital angular speed is

$$\begin{aligned} \omega_{\text{orbit}} &= \frac{2\pi}{T_{\text{orbit}}} = \frac{2\pi}{(365.25 \text{ d})(8.640 \times 10^4 \text{ s} \cdot \text{d}^{-1})} \\ &= 1.991 \times 10^{-7} \text{ rad} \cdot \text{s}^{-1}. \end{aligned} \quad (20.3.19)$$

The orbital angular momentum about S is then

$$\begin{aligned} \vec{\mathbf{L}}_S^{\text{orbital}} &= m_e r_{s,e}^2 \omega_{\text{orbit}} \hat{\mathbf{k}} \\ &= (5.97 \times 10^{24} \text{ kg})(1.50 \times 10^{11} \text{ m})^2 (1.991 \times 10^{-7} \text{ rad} \cdot \text{s}^{-1}) \hat{\mathbf{k}} \\ &= (2.68 \times 10^{40} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}) \hat{\mathbf{k}}. \end{aligned} \quad (20.3.20)$$

The spin angular momentum is given by

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}} \vec{\boldsymbol{\omega}}_{\text{spin}} = \frac{2}{5} m_e R_e^2 \omega_{\text{spin}} \hat{\mathbf{n}}, \quad (20.3.21)$$

where $\hat{\mathbf{n}}$ is a unit normal pointing along the axis of rotation of the earth and

$$\omega_{\text{spin}} = \frac{2\pi}{T_{\text{spin}}} = \frac{2\pi}{8.616 \times 10^4 \text{ s}} = 7.293 \times 10^{-5} \text{ rad} \cdot \text{s}^{-1}. \quad (20.3.22)$$

The spin angular momentum is then

$$\begin{aligned} \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} &= \frac{2}{5} (5.97 \times 10^{24} \text{ kg})(6.38 \times 10^6 \text{ m})^2 (7.293 \times 10^{-5} \text{ rad} \cdot \text{s}^{-1}) \hat{\mathbf{n}} \\ &= (7.10 \times 10^{33} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}) \hat{\mathbf{n}}. \end{aligned} \quad (20.3.23)$$

The ratio of the magnitudes of the orbital angular momentum about S to the spin angular momentum is greater than a million,

$$\frac{L_S^{\text{orbital}}}{L_{\text{cm}}^{\text{spin}}} = \frac{m_e r_{s,e}^2 \omega_{\text{orbit}}}{(2/5)m_e R_e^2 \omega_{\text{spin}}} = \frac{5}{2} \frac{r_{s,e}^2}{R_e^2} \frac{T_{\text{spin}}}{T_{\text{orbit}}} = 3.77 \times 10^6, \quad (20.3.24)$$

as this ratio is proportional to the square of the ratio of the distance to the sun to the radius of the earth. The angular momentum about S is then

$$\vec{L}_S^{\text{total}} = m_e r_{s,e}^2 \omega_{\text{orbit}} \hat{\mathbf{k}} + \frac{2}{5} m_e R_e^2 \omega_{\text{spin}} \hat{\mathbf{n}}. \quad (20.3.25)$$

The orbit and spin periods are known to far more precision than the average values used for the earth's orbit radius and mean radius. Two different values have been used for one "day;" in converting the orbit period from days to seconds, the value for the *solar day*, $T_{\text{solar}} = 86,400\text{s}$ was used. In converting the earth's spin angular frequency, the *sidereal day*, $T_{\text{sidereal}} = T_{\text{spin}} = 86,160\text{s}$ was used. The two periods, the solar day from noon to noon and the sidereal day from the difference between the times that a fixed star is at the same place in the sky, do differ in the third significant figure.

20.4 Kinetic Energy of a System of Particles

Consider a system of particles. The i^{th} particle has mass m_i and velocity \vec{v}_i with respect to a reference frame O . The kinetic energy of the system of particles is given by

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i \vec{v}_i \cdot \vec{v}_i \\ &= \frac{1}{2} \sum_i m_i (\vec{v}_{\text{cm},i} + \vec{V}_{\text{cm}}) \cdot (\vec{v}_{\text{cm},i} + \vec{V}_{\text{cm}}). \end{aligned} \quad (20.4.1)$$

where Equation 15.2.6 has been used to express \vec{v}_i in terms of $\vec{v}_{\text{cm},i}$ and \vec{V}_{cm} . Expanding the last dot product in Equation (20.4.1),

$$\begin{aligned} K &= \frac{1}{2} \sum_i m_i (\vec{v}_{\text{cm},i} \cdot \vec{v}_{\text{cm},i} + \vec{V}_{\text{cm}} \cdot \vec{V}_{\text{cm}} + 2 \vec{v}_{\text{cm},i} \cdot \vec{V}_{\text{cm}}) \\ &= \frac{1}{2} \sum_i m_i (\vec{v}_{\text{cm},i} \cdot \vec{v}_{\text{cm},i}) + \frac{1}{2} \sum_i m_i (\vec{V}_{\text{cm}} \cdot \vec{V}_{\text{cm}}) + \sum_i m_i \vec{v}_{\text{cm},i} \cdot \vec{V}_{\text{cm}} \\ &= \sum_i \frac{1}{2} m_i v_{\text{cm},i}^2 + \frac{1}{2} \sum_i m_i V_{\text{cm}}^2 + \left(\sum_i m \vec{v}_{\text{cm},i} \right) \cdot \vec{V}_{\text{cm}}. \end{aligned} \quad (20.4.2)$$

The last term in the third equation in (20.4.2) vanishes as we showed in Eq. (20.3.7). Then Equation (20.4.2) reduces to

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i v_{\text{cm},i}^2 + \frac{1}{2} \sum_i m_i V_{\text{cm}}^2 \\ &= \sum_i \frac{1}{2} m_i v_{\text{cm},i}^2 + \frac{1}{2} m^{\text{total}} V_{\text{cm}}^2. \end{aligned} \quad (20.4.3)$$

We interpret the first term as the sum of the individual kinetic energies of the particles of the system in the center of mass reference frame O_{cm} and the second term as the kinetic energy of the center of mass motion in reference frame O .

At this point, it's important to note that no assumption was made regarding the mass elements being constituents of a rigid body. Equation (20.4.3) is valid for a rigid body, a gas, a firecracker (but K is certainly not the same before and after detonation), and the sixteen pool balls after the break, or any collection of objects for which the center of mass can be determined.

20.5 Rotational Kinetic Energy for a Rigid Body Undergoing Fixed Axis Rotation

The rotational kinetic energy for the rigid body, using $\vec{v}_{\text{cm},i} = (r_{\text{cm},i})_{\perp} \omega_{\text{cm}} \hat{\theta}$, simplifies to

$$K_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2. \quad (20.5.1)$$

Therefore the total kinetic energy of a translating and rotating rigid body is

$$K_{\text{total}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} m V_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2. \quad (20.5.2)$$

Appendix 20A Chasles's Theorem: Rotation and Translation of a Rigid Body

We now return to our description of the translating and rotating rod that we first considered when we began our discussion of rigid bodies. We shall now show that the motion of any rigid body consists of a translation of the center of mass and rotation about the center of mass.

We shall demonstrate this for a rigid body by dividing up the rigid body into point-like constituents. Consider two point-like constituents with masses m_1 and m_2 . Choose a coordinate system with a choice of origin such that body 1 has position \vec{r}_1 and body 2 has position \vec{r}_2 (Figure 20A.1). The relative position vector is given by

$$\vec{r}_{1,2} = \vec{r}_1 - \vec{r}_2. \quad (20.A.1)$$

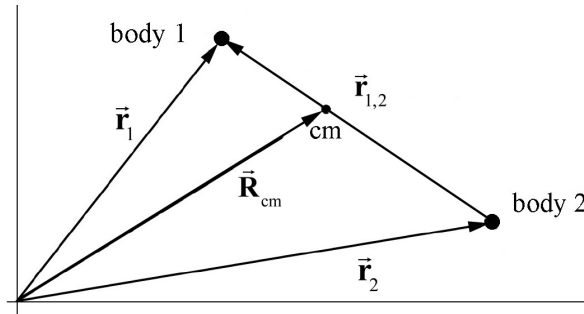


Figure 20A.1 Two-body coordinate system.

Recall we defined the center of mass vector, \vec{R}_{cm} , of the two-body system as

$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}. \quad (20.A.2)$$

In Figure 20A.2 we show the center of mass coordinate system.

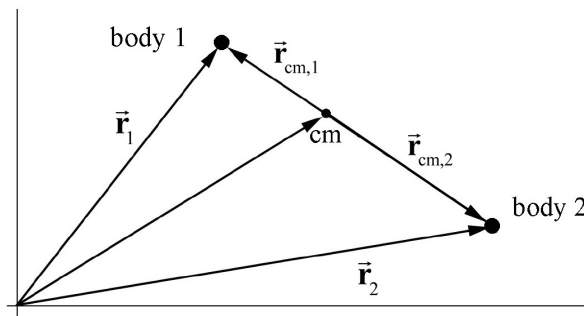


Figure 20A.2 Position coordinates with respect to center of mass

The position vector of the object 1 with respect to the center of mass is given by

$$\vec{r}_{\text{cm},1} = \vec{r}_1 - \vec{R}_{\text{cm}} = \vec{r}_1 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} (\vec{r}_1 - \vec{r}_2) = \frac{\mu}{m_1} \vec{r}_{1,2}, \quad (20.A.3)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (20.A.4)$$

is the reduced mass. In addition, the relative position vector between the two objects is independent of the choice of reference frame,

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = (\vec{r}_{\text{cm},1} + \vec{R}_{\text{cm}}) - (\vec{r}_{\text{cm},2} + \vec{R}_{\text{cm}}) = \vec{r}_{\text{cm},1} - \vec{r}_{\text{cm},2} = \vec{r}_{\text{cm},1,2}. \quad (20.A.5)$$

Because the center of mass is at the origin in the center of mass reference frame,

$$\frac{m_1 \vec{r}_{\text{cm},1} + m_2 \vec{r}_{\text{cm},2}}{m_1 + m_2} = \vec{0}. \quad (20.A.6)$$

Therefore

$$m_1 \vec{r}_{\text{cm},1} = -m_2 \vec{r}_{\text{cm},2} \quad (20.A.7)$$

$$m_1 |\vec{r}_{\text{cm},1}| = m_2 |\vec{r}_{\text{cm},2}|. \quad (20.A.8)$$

The displacement of object 1 about the center of mass is given by taking the derivative of Eq. (20.A.3),

$$d\vec{r}_{\text{cm},1} = \frac{\mu}{m_1} d\vec{r}_{1,2}. \quad (20.A.9)$$

A similar calculation for the position of object 2 with respect to the center of mass yields for the position and displacement with respect to the center of mass

$$\vec{r}_{\text{cm},2} = \vec{r}_2 - \vec{R}_{\text{cm}} = -\frac{\mu}{m_2} \vec{r}_{1,2}, \quad (20.A.10)$$

$$d\vec{r}_{\text{cm},2} = -\frac{\mu}{m_2} d\vec{r}_{1,2}. \quad (20.A.11)$$

Let $i = 1, 2$. An arbitrary displacement of the i^{th} object is given respectively by

$$d\vec{r}_i = d\vec{r}_{\text{cm},i} + d\vec{R}_{\text{cm}}, \quad (20.A.12)$$

which is the sum of a displacement about the center of mass $d\vec{r}_{\text{cm},i}$ and a displacement of the center of mass $d\vec{R}_{\text{cm}}$. The displacement of objects 1 and 2 are constrained by the condition that the distance between the objects must remain constant since the body is rigid. In particular, the distance between objects 1 and 2 is given by

$$|\vec{r}_{1,2}|^2 = (\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2). \quad (20.A.13)$$

Because this distance is constant we can differentiate Eq. (20.A.13), yielding the **rigid body condition** that

$$0 = 2(\vec{r}_1 - \vec{r}_2) \cdot (d\vec{r}_1 - d\vec{r}_2) = 2\vec{r}_{1,2} \cdot d\vec{r}_{1,2} \quad (20.A.14)$$

20A.1. Translation of the Center of Mass

The condition (Eq. (20.A.14)) can be satisfied if the relative displacement vector between the two objects is zero,

$$d\vec{r}_{1,2} = d\vec{r}_1 - d\vec{r}_2 = \vec{0}. \quad (20.A.15)$$

This implies, using, Eq. (20.A.9) and Eq. (20.A.11), that the displacement with respect to the center of mass is zero,

$$d\vec{r}_{\text{cm},1} = d\vec{r}_{\text{cm},2} = \vec{0}. \quad (20.A.16)$$

Thus by Eq. (20.A.12), the displacement of each object is equal to the displacement of the center of mass,

$$d\vec{r}_i = d\vec{R}_{\text{cm}}, \quad (20.A.17)$$

which means that the body is undergoing pure translation.

20A.2 Rotation about the Center of Mass

Now suppose that $d\vec{r}_{1,2} = d\vec{r}_1 - d\vec{r}_2 \neq \vec{0}$. The rigid body condition can be expressed in terms of the center of mass coordinates. Using Eq. (20.A.9), the rigid body condition (Eq. (20.A.14)) becomes

$$0 = 2 \frac{\mu}{m_1} \vec{r}_{1,2} \cdot d\vec{r}_{\text{cm},1}. \quad (20.A.18)$$

Because the relative position vector between the two objects is independent of the choice of reference frame (Eq. (20.A.5)), the rigid body condition Eq. (20.A.14) in the center of mass reference frame is then given by

$$0 = 2\vec{r}_{\text{cm},1,2} \cdot d\vec{r}_{\text{cm},1}. \quad (20.A.19)$$

This condition is satisfied if the relative displacement is perpendicular to the line passing through the center of mass,

$$\vec{r}_{\text{cm},1,2} \perp d\vec{r}_{\text{cm},1}. \quad (20.A.20)$$

By a similar argument, $\vec{r}_{\text{cm},1,2} \perp d\vec{r}_{\text{cm},2}$. In order for these displacements to correspond to a rotation about the center of mass, the displacements must have the same angular displacement.

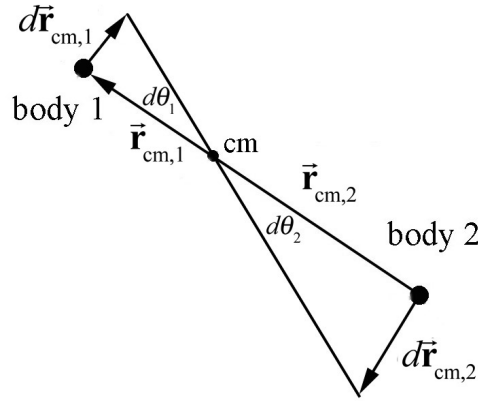


Figure 20A.3 Infinitesimal angular displacements in the center of mass reference frame

In Figure 20A.3, the infinitesimal angular displacement of each object is given by

$$d\theta_1 = \frac{|d\vec{r}_{\text{cm},1}|}{|\vec{r}_{\text{cm},1}|}, \quad (20.A.21)$$

$$d\theta_2 = \frac{|d\vec{r}_{\text{cm},2}|}{|\vec{r}_{\text{cm},2}|}. \quad (20.A.22)$$

From Eq. (20.A.9) and Eq. (20.A.11), we can rewrite Eqs. (20.A.21) and (20.A.22) as

$$d\theta_1 = \frac{\mu}{m_1} \frac{|d\vec{r}_{1,2}|}{|\vec{r}_{\text{cm},1}|}, \quad (20.A.23)$$

$$d\theta_2 = \frac{\mu}{m_2} \frac{|d\vec{r}_{1,2}|}{|\vec{r}_{\text{cm},2}|}. \quad (20.A.24)$$

Recall that in the center of mass reference frame $m_1 |\vec{r}_{\text{cm},1}| = m_2 |\vec{r}_{\text{cm},2}|$ (Eq. (20.A.8))

and hence the angular displacements are equal,

$$d\theta_1 = d\theta_2 = d\theta . \quad (20.A.25)$$

Therefore the displacement of the i^{th} object $d\vec{r}_i$ differs from the displacement of the center of mass $d\vec{R}_{cm}$ by a vector that corresponds to an infinitesimal rotation in the center of mass reference frame

$$d\vec{r}_i = d\vec{R}_{cm} + d\vec{r}_{cm,i} . \quad (20.A.26)$$

We have shown that the displacement of a rigid body is the vector sum of the displacement of the center of mass (translation of the center of mass) and an infinitesimal rotation about the center of mass.