

## Chapter 19 Angular Momentum

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## Chapter 19 Angular Momentum

*The situation, in brief, is that newtonian physics is incapable of predicting conservation of angular momentum, but no isolated system has yet been encountered experimentally for which angular momentum is not conserved. We conclude that conservation of angular momentum is an independent physical law, and until a contradiction is observed, our physical understanding must be guided by it.*<sup>1</sup>

Dan Kleppner

### 19.1 Introduction

When we consider a system of objects, we have shown that the external force, acting at the center of mass of the system, is equal to the time derivative of the total momentum of the system,

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}. \quad (19.1.1)$$

We now introduce the rotational analog of Equation (19.1.1). We will first introduce the concept of angular momentum for a point-like particle of mass  $m$  with linear momentum  $\vec{\mathbf{p}}$  about a point  $S$ , defined by the equation

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}, \quad (19.1.2)$$

where  $\vec{\mathbf{r}}_S$  is the vector from the point  $S$  to the particle. We will show in this chapter that the torque about the point  $S$  acting on the particle is equal to the rate of change of the angular momentum about the point  $S$  of the particle,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \quad (19.1.3)$$

Equation (19.1.3) generalizes to any body undergoing rotation.

We shall concern ourselves first with the special case of rigid body undergoing fixed axis rotation about the  $z$ -axis with angular velocity  $\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}}$ . We divide up the rigid body into  $N$  elements labeled by the index  $i$ ,  $i = 1, 2, \dots, N$ , the  $i^{\text{th}}$  element having mass  $m_i$  and position vector  $\vec{\mathbf{r}}_{S,i}$ . The rigid body has a moment of inertia  $I_S$  about some point  $S$  on the fixed axis, (often taken to be the  $z$ -axis, but not always) which rotates with angular velocity  $\vec{\boldsymbol{\omega}}$  about this axis. The angular momentum is then the vector sum of the individual angular momenta,

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<sup>1</sup> Kleppner, Daniel, An Introduction to Mechanics (1973), p. 307.

$$\vec{\mathbf{L}}_S = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i \quad (19.1.4)$$

When the rotation axis is the  $z$ -axis the  $z$ -component of the angular momentum,  $L_{S,z}$ , about the point  $S$  is then given by

$$L_{S,z} = I_S \omega_z. \quad (19.1.5)$$

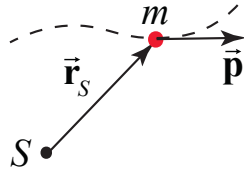
We shall show that the  $z$ -component of the torque about the point  $S$ ,  $\tau_{S,z}$ , is then the time derivative of the  $z$ -component of angular momentum about the point  $S$ ,

$$\tau_{S,z} = \frac{dL_{S,z}}{dt} = I_S \frac{d\omega_z}{dt} = I_S \alpha_z. \quad (19.1.6)$$

## 19.2 Angular Momentum about a Point for a Particle

### 19.2.1 Angular Momentum for a Point Particle

Consider a point-like particle of mass  $m$  moving with a velocity  $\vec{\mathbf{v}}$  (Figure 19.1) with momentum  $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ .



**Figure 19.1** A point-like particle and its angular momentum about  $S$ .

Consider a point  $S$  located anywhere in space. Let  $\vec{\mathbf{r}}_S$  denote the vector from the point  $S$  to the location of the object.

*Define the angular momentum  $\vec{\mathbf{L}}_S$  about the point  $S$  of a point-like particle as the vector product of the vector from the point  $S$  to the location of the object with the momentum of the particle,*

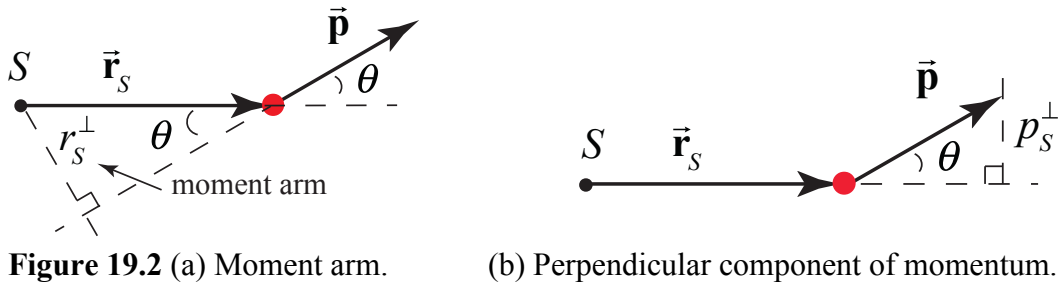
$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}. \quad (19.2.1)$$

The derived SI units for angular momentum are  $[\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}] = [\text{N} \cdot \text{m} \cdot \text{s}] = [\text{J} \cdot \text{s}]$ . There is no special name for this set of units.

Because angular momentum is defined as a vector, we begin by studying its magnitude and direction. The magnitude of the angular momentum about  $S$  is given by

$$|\vec{L}_S| = |\vec{r}_S| |\vec{p}| \sin \theta, \quad (19.2.2)$$

where  $\theta$  is the angle between the vectors  $\vec{r}_S$  and  $\vec{p}$ , and lies within the range  $[0 \leq \theta \leq \pi]$ . Analogous to the magnitude of torque, there are two ways to determine the magnitude of the angular momentum about  $S$ .



Define the *moment arm*,  $r_S^\perp$ , (Figure 19.2 (a)), as the perpendicular distance from the point  $S$  to the line defined by the direction of the momentum. Then

$$r_S^\perp = |\vec{r}_S| \sin \theta. \quad (19.2.3)$$

Hence the magnitude of the angular momentum is the product of the moment arm with the magnitude of the momentum,

$$|\vec{L}_S| = r_S^\perp |\vec{p}|. \quad (19.2.4)$$

Alternatively, let **Error! Objects cannot be created from editing field codes.** denote the magnitude of the component of the momentum perpendicular to the line defined by the direction of the vector  $\vec{r}_S$ . From the geometry shown in Figure 19.2 (b),

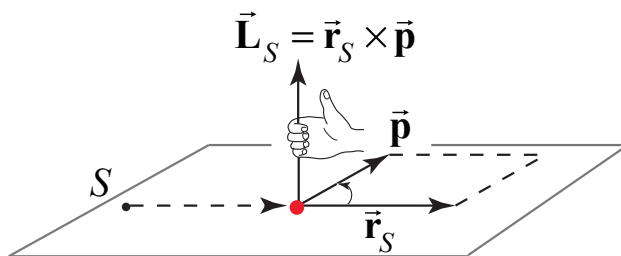
$$p_S^\perp = |\vec{p}| \sin \theta. \quad (19.2.5)$$

Thus the magnitude of the angular momentum is the product of the distance from  $S$  to the particle with  $p_S^\perp$ ,

$$|\vec{L}_S| = |\vec{r}_S| p_S^\perp. \quad (19.2.6)$$

### 19.2.2 Right-Hand-Rule for the Direction of the Angular Momentum

We shall define the direction of the angular momentum about the point  $S$  by a right hand rule. Draw the vectors  $\vec{r}_S$  and  $\vec{p}$  so their tails are touching. Then draw an arc starting from the vector  $\vec{r}_S$  and finishing on the vector  $\vec{p}$ . (There are two such arcs; choose the shorter one.) This arc is either in the clockwise or counterclockwise direction. Curl the fingers of your right hand in the same direction as the arc. Your right thumb points in the direction of the angular momentum.

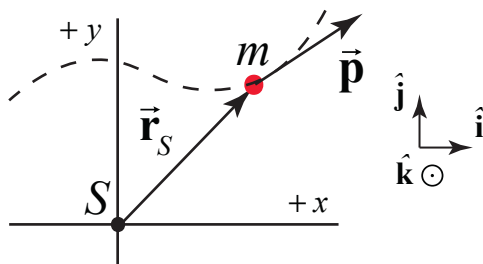


**Figure 19.3** The right hand rule for determining the direction of angular momentum about  $S$ .

Remember that, as in all vector products, the direction of the angular momentum about  $S$  is perpendicular to the plane formed by  $\vec{r}_S$  and  $\vec{p}$ .

### Example 19.1 Angular Momentum: Constant Velocity

A particle of mass  $m = 2.0 \text{ kg}$  moves as shown in Figure 19.4 with a uniform velocity  $\vec{v} = 3.0 \text{ m} \cdot \text{s}^{-1} \hat{i} + 3.0 \text{ m} \cdot \text{s}^{-1} \hat{j}$ . At time  $t$ , the particle passes through the point  $(2.0 \text{ m}, 3.0 \text{ m})$ . Find the direction and the magnitude of the angular momentum about the point  $S$  (the origin) at time  $t$ .



**Figure 19.4** Example 19.4

**Solution:** Choose Cartesian coordinates with unit vectors shown in the figure above. The vector from the point  $S$  to the location of the particle is  $\vec{r}_S = 2.0 \text{ m} \hat{i} + 3.0 \text{ m} \hat{j}$ . The angular momentum vector  $\vec{L}_O$  of the particle about the origin  $S$  is given by:

$$\begin{aligned} \vec{L}_S &= \vec{r}_S \times \vec{p} = \vec{r}_S \times m \vec{v} \\ &= (2.0 \text{ m} \hat{i} + 3.0 \text{ m} \hat{j}) \times (2 \text{ kg})(3.0 \text{ m} \cdot \text{s}^{-1} \hat{i} + 3.0 \text{ m} \cdot \text{s}^{-1} \hat{j}) \\ &= 0 + 12 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \hat{k} - 18 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} (-\hat{k}) + \vec{0} \\ &= -6 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \hat{k}. \end{aligned}$$

In the above, the relations  $\hat{i} \times \hat{j} = \hat{k}$ ,  $\hat{j} \times \hat{i} = -\hat{k}$ ,  $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \vec{0}$  were used.

### Example 19.2 Angular Momentum and Circular Motion

A particle of mass  $m$  moves in a circle of radius  $R$  about the  $z$ -axis in the  $x$ - $y$  plane defined by  $z = 0$  with angular velocity  $\vec{\omega} = \omega_z \hat{\mathbf{k}}$ ,  $\omega_z > 0$ , (Figure 19.5). Find the magnitude and the direction of the angular momentum  $\vec{\mathbf{L}}_S$  relative to the point  $S$  lying at the center of the circular orbit, (the origin).

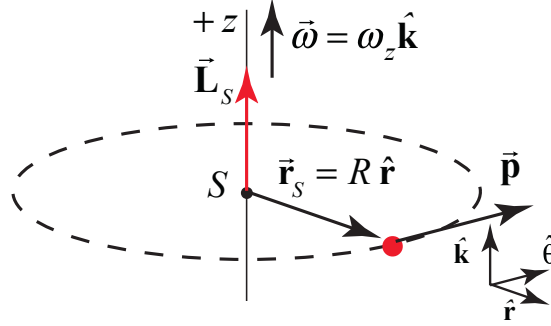


Figure 19.5 Example 19.2

**Solution:** The velocity of the particle is given by  $\vec{\mathbf{v}} = R\omega_z \hat{\boldsymbol{\theta}}$ . The vector from the center of the circle (the point  $S$ ) to the object is given by  $\vec{\mathbf{r}}_S = R \hat{\mathbf{r}}$ . The angular momentum about the center of the circle is the vector product

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}} = Rm\mathbf{v} \hat{\mathbf{k}} = RmR\omega_z \hat{\mathbf{k}} = mR^2\omega_z \hat{\mathbf{k}} = I_S \vec{\boldsymbol{\omega}}.$$

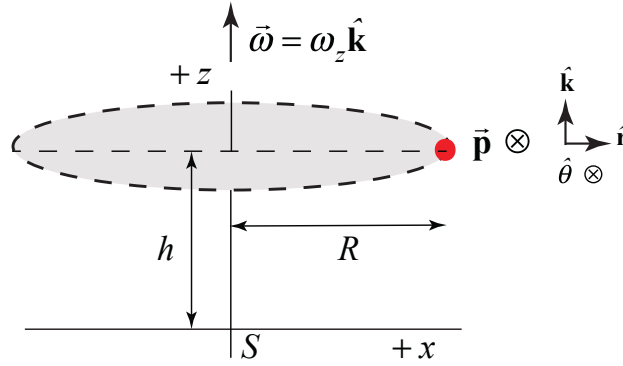
The magnitude is  $|\vec{\mathbf{L}}_S| = mR^2\omega_z$ , and the direction is in the  $+\hat{\mathbf{k}}$ -direction. For the particle, the moment of inertia about the  $z$ -axis is  $I_S = mR^2$ , therefore the angular momentum about  $S$  is

$$\vec{\mathbf{L}}_S = I_S \vec{\boldsymbol{\omega}}.$$

The fact that  $\vec{\mathbf{L}}_S$  is in the same direction as the angular velocity is due to the fact that the point  $S$  lies on the plane of motion.

### Example 19.3 Angular Momentum About a Point along Central Axis for Circular Motion

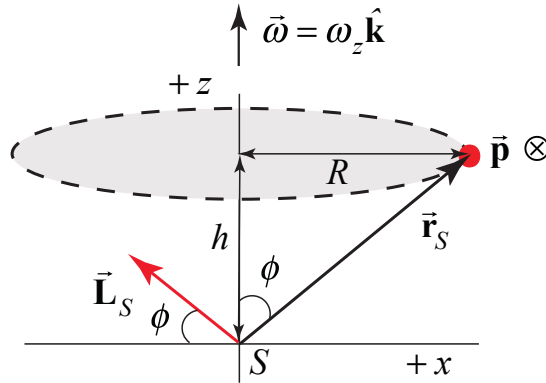
A particle of mass  $m$  moves in a circle of radius  $R$  with angular velocity  $\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}}$ ,  $\omega_z > 0$ , about the  $z$ -axis in a plane parallel to but a distance  $h$  above the  $x$ - $y$  plane (Figure 19.6). Find the magnitude and the direction of the angular momentum  $\vec{\mathbf{L}}_S$  relative to the point  $S$  (the origin).



**Figure 19.6** Example 19.3

**Solution:** The easiest way to calculate  $\vec{L}_S$  is to use cylindrical coordinates. We begin by writing the two vectors  $\vec{r}_S$  and  $\vec{p}$  in polar coordinates. We start with the vector from point  $S$  (the origin) to the location of the moving object,  $\vec{r}_S = R\hat{r} + h\hat{k}$ . The momentum vector is tangent to the circular orbit so  $\vec{p} = m\vec{v} = mR\omega_z\hat{\theta}$ . Using the fact that  $\hat{r} \times \hat{\theta} = \hat{k}$  and  $\hat{k} \times \hat{\theta} = -\hat{r}$ , the angular momentum about point  $S$  is

$$\vec{L}_S = \vec{r}_S \times \vec{p} = (R\hat{r} + h\hat{k}) \times mR\omega_z\hat{\theta} = mR^2\omega_z\hat{k} - hmR\omega_z\hat{r}$$



**Figure 19.7** Angular momentum about the point  $S$

The magnitude of  $\vec{L}_S$  is given by

$$|\vec{L}_S| = ((mR^2\omega_z)^2 + (hmR\omega_z)^2)^{1/2} = mR\omega_z(h^2 + R^2)^{1/2}$$

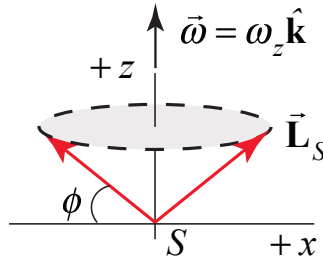
The direction of  $\vec{L}_S$  is given by (Figure 19.7)

$$-\frac{L_{S,z}}{L_{S,r}} = \frac{R}{h} = \tan \phi$$

We also present a geometric argument. Suppose the particle has coordinates  $(x, y, h)$ . The angular momentum about the origin is given by  $\vec{L}_S = \vec{r}_S \times \vec{p}$ . The vectors  $\vec{r}_S$  and  $\vec{p}$  are perpendicular to each other so the angular momentum is perpendicular to the plane formed by those two vectors. Recall that the speed  $v = R\omega_z$ . Suppose the vector  $\vec{r}_S$  forms an angle  $\phi$  with the  $z$ -axis. Then  $\vec{L}_S$  forms an angle  $\phi$  with respect to the  $x - y$  plane as shown in the figure above. The magnitude of  $\vec{L}_S$  is

$$|\vec{L}_S| = |\vec{r}_S| m |\vec{v}| = (h^2 + R^2)^{1/2} m R \omega_z$$

The magnitude of  $\vec{L}_S$  is constant, but its direction is changing as the particle moves in a circular orbit about the  $z$ -axis, sweeping out a cone as shown in Figure 19.8. We draw the vector  $\vec{L}_S$  at the origin because it is defined at that point.



**Figure 19.8** Direction of angular momentum about the point  $S$  sweeps out a cone

The important point to keep in mind regarding this calculation is that for any point along the  $z$ -axis not at the center of the circular orbit of a single particle, the angular momentum about that point does not point along the  $z$ -axis but it has a non-zero component in the  $x - y$  plane (or in the  $-\hat{r}$  direction if you use polar coordinates). The  $z$ -component of the angular momentum about any point along the  $z$ -axis is independent of the location of that point along the axis.

### 19.3 Torque and the Time Derivative of Angular Momentum about a Point for a Particle

We will now show that the torque about a point  $S$  is equal to the time derivative of the angular momentum about  $S$ ,

$$\vec{\tau}_S = \frac{d\vec{L}_S}{dt}. \quad (19.3.1)$$

Take the time derivative of the angular momentum about  $S$ ,

$$\frac{d\vec{L}_S}{dt} = \frac{d}{dt}(\vec{r}_S \times \vec{p}). \quad (19.3.2)$$



In this equation we are taking the time derivative of a vector product of two vectors. There are two important facts that will help us simplify this expression. First, the time derivative of the vector product of two vectors satisfies the product rule,

$$\frac{d\vec{L}_S}{dt} = \frac{d}{dt}(\vec{r}_S \times \vec{p}) = \left( \left( \frac{d\vec{r}_S}{dt} \right) \times \vec{p} \right) + \left( \vec{r}_S \times \left( \frac{d\vec{p}}{dt} \right) \right). \quad (19.3.3)$$

Second, the first term on the right hand side vanishes,

$$\frac{d\vec{r}_S}{dt} \times \vec{p} = \vec{v} \times m \vec{v} = \vec{0}. \quad (19.3.4)$$

The rate of angular momentum change about the point  $S$  is then

$$\frac{d\vec{L}_S}{dt} = \vec{r}_S \times \frac{d\vec{p}}{dt}. \quad (19.3.5)$$

From Newton's Second Law, the force on the particle is equal to the derivative of the linear momentum,

$$\vec{F} = \frac{d\vec{p}}{dt}. \quad (19.3.6)$$

Therefore the rate of change in time of angular momentum about the point  $S$  is

$$\frac{d\vec{L}_S}{dt} = \vec{r}_S \times \vec{F}. \quad (19.3.7)$$

Recall that the torque about the point  $S$  due to the force  $\vec{F}$  acting on the particle is

$$\vec{\tau}_S = \vec{r}_S \times \vec{F}. \quad (19.3.8)$$

Combining the expressions in (19.3.7) and (19.3.8), it is readily seen that the torque about the point  $S$  is equal to the rate of change of angular momentum about the point  $S$ ,

$$\vec{\tau}_S = \frac{d\vec{L}_S}{dt}. \quad (19.3.9)$$

## 19.4 Conservation of Angular Momentum about a Point

So far we have introduced two conservation principles, showing that energy is constant for closed systems (no change in energy in the surroundings) and linear momentum is constant isolated system. The change in mechanical energy of a closed system is

$$W_{nc} = \Delta E_m = \Delta K + \Delta U, \quad (\text{closed system}). \quad (19.3.10)$$

If the non-conservative work done in the system is zero, then the mechanical energy is constant,

$$0 = W_{nc} = \Delta E_{\text{mechanical}} = \Delta K + \Delta U, \quad (\text{closed system}). \quad (19.3.11)$$

The conservation of linear momentum arises from Newton's Second Law applied to systems,

$$\vec{F}^{\text{ext}} = \sum_{i=1}^N \frac{d}{dt} \vec{p}_i = \frac{d}{dt} \vec{p}_{\text{sys}} \quad (19.3.12)$$

Thus if the external force in any direction is zero, then the component of the momentum of the system in that direction is a constant. For example, if there are no external forces in the  $x$ - and  $y$ -directions then

$$\begin{aligned} \vec{0} &= (\vec{F}^{\text{ext}})_x = \frac{d}{dt} (\vec{p}_{\text{sys}})_x \\ \vec{0} &= (\vec{F}^{\text{ext}})_y = \frac{d}{dt} (\vec{p}_{\text{sys}})_y. \end{aligned} \quad (19.3.13)$$

We can now use our relation between torque about a point  $S$  and the change of the angular momentum about  $S$ , Eq. (19.3.9), to introduce a new conservation law. Suppose we can find a point  $S$  such that torque about the point  $S$  is zero,

$$\vec{0} = \vec{\tau}_S = \frac{d\vec{L}_S}{dt}, \quad (19.3.14)$$

then the angular momentum about the point  $S$  is a constant vector, and so the change in angular momentum is zero,

$$\Delta \vec{L}_S \equiv \vec{L}_{S,f} - \vec{L}_{S,i} = \vec{0}. \quad (19.3.15)$$

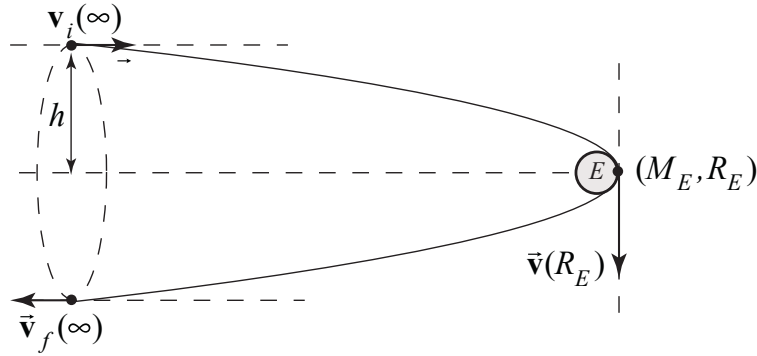
Thus when the torque about a point  $S$  is zero, the final angular momentum about  $S$  is equal to the initial angular momentum,

$$\vec{L}_{S,f} = \vec{L}_{S,i}. \quad (19.3.16)$$

### Example 19.4 Meteor Flyby of Earth

A meteor of mass  $m$  is approaching Earth as shown in the figure. The radius of Earth is  $R_E$ . The mass of Earth is  $M_E$ . Assume that the meteor started very far away from Earth with

speed  $v_i$  and at a perpendicular distance  $h$  from the axis of symmetry of the orbit. At some later time the meteor just grazes Earth (Figure 19.9). You may ignore all other gravitational forces except due to Earth. Find the distance  $h$ . Hint: What quantities are constant for this orbit?



**Figure 19.9** Meteor flyby of earth

**Solution:** In this problem both energy and angular momentum about the center of Earth are constant (see below for justification).

The meteor's mass is so much small than the mass of Earth that we will assume that the earth's motion is not affected by the meteor. We'll also need to neglect any air resistance when the meteor approaches Earth. Choose the center of Earth, (point  $S$ ) to calculate the torque and angular momentum. The force on the meteor is

$$\vec{\mathbf{F}}_{E,m}^G = -\frac{GmM_E}{r^2} \hat{\mathbf{r}}$$

The vector from the center of Earth to the meteor is  $\vec{\mathbf{r}}_S = r\hat{\mathbf{r}}$ . The torque about  $S$  is zero because they two vectors are anti-parallel

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}}_{E,m}^G = r\hat{\mathbf{r}} \times -\frac{GmM_E}{r^2} \hat{\mathbf{r}} = \vec{\mathbf{0}}$$

Therefore the angular momentum about the center of Earth is a constant.

The initial angular momentum is

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times m\vec{\mathbf{v}}_i = (x_i\hat{\mathbf{i}} + h\hat{\mathbf{j}}) \times mv_i\hat{\mathbf{i}} = -hmv_i\hat{\mathbf{k}}$$

When the meteor just grazes Earth, the angular momentum is

$$\vec{\mathbf{L}}_{S,E} = \vec{\mathbf{r}}_{S,E} \times m\vec{\mathbf{v}}_p = R_E\hat{\mathbf{i}} \times mv_p(-\hat{\mathbf{j}}) = -R_Emv_p\hat{\mathbf{k}}$$

where we have used  $v_p$  for the speed of the meteor at its nearest approach to Earth. The constancy of angular momentum requires that

$$mv_i h = mv_p R_E$$

In order to solve for  $h$ , we need to find  $v_p$ . Because we are neglecting all forces on the meteor other than Earth's gravity, mechanical energy is constant, and

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_p^2 - \frac{GmM_E}{R_E},$$

where we have taken the meteor to have speed  $v_i$  at a distance “very far away from Earth” to mean that we neglect any gravitational potential energy in the meteor-Earth system, when  $r \rightarrow \infty$ ,  $U(r) = -GmM_E / r \rightarrow 0$ . From the angular momentum condition,  $v_p = v_i h / R_E$  and therefore the energy condition can be rewritten as

$$v_i^2 = v_p^2 \left( \frac{h}{R_E} \right)^2 - \frac{2GM_E}{R_E}$$

which we solve for the impact parameter  $h$

$$h = \sqrt{R_E^2 + \frac{2GM_E R_E}{v_i^2}}.$$

## 19.5 Angular Impulse and Change in Angular Momentum

If there is a total applied torque  $\vec{\tau}_S$  about a point  $S$  over an interval of time  $\Delta t = t_f - t_i$ , then the torque applies an **angular impulse** about a point  $S$ , given by

$$\vec{J}_S = \int_{t_i}^{t_f} \vec{\tau}_S dt. \quad (19.4.1)$$

Because  $\vec{\tau}_S = d\vec{L}_S^{\text{total}} / dt$ , the angular impulse about  $S$  is equal to the change in angular momentum about  $S$ ,

$$\vec{J}_S = \int_{t_i}^{t_f} \vec{\tau}_S dt = \int_{t_i}^{t_f} \frac{d\vec{L}_S}{dt} dt = \Delta \vec{L}_S = \vec{L}_{S,f} - \vec{L}_{S,i}. \quad (19.4.2)$$

This result is the rotational analog to linear impulse, which is equal to the change in momentum,

$$\vec{\mathbf{I}} = \int_{t_i}^{t_f} \vec{\mathbf{F}} dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{p}}}{dt} dt = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i. \quad (19.4.3)$$

## 19.6 Angular Momentum of a System of Particles

We now calculate the angular momentum about the point  $S$  associated with a system of  $N$  point particles. Label each individual particle by the index  $j$ ,  $j=1,2,\dots,N$ . Let the  $j^{\text{th}}$  particle have mass  $m_j$  and velocity  $\vec{\mathbf{v}}_j$ . The momentum of an individual particle is then  $\vec{\mathbf{p}}_j = m_j \vec{\mathbf{v}}_j$ . Let  $\vec{\mathbf{r}}_{S,j}$  be the vector from the point  $S$  to the  $j^{\text{th}}$  particle, and let  $\theta_j$  be the angle between the vectors  $\vec{\mathbf{r}}_{S,j}$  and  $\vec{\mathbf{p}}_j$  (Figure 19.10).

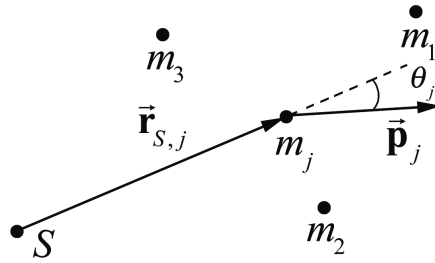


Figure 19.10 System of particles

The angular momentum  $\vec{\mathbf{L}}_{S,j}$  of the  $j^{\text{th}}$  particle is

$$\vec{\mathbf{L}}_{S,j} = \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j. \quad (19.5.1)$$

The angular momentum for the system of particles is the vector sum of the individual angular momenta,

$$\vec{\mathbf{L}}_S^{\text{sys}} = \sum_{j=1}^{j=N} \vec{\mathbf{L}}_{S,j} = \sum_{j=1}^{j=N} \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j. \quad (19.5.2)$$

The change in the angular momentum of the system of particles about a point  $S$  is given by

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \frac{d}{dt} \sum_{j=1}^{j=N} \vec{\mathbf{L}}_{S,j} = \sum_{j=1}^{j=N} \left( \frac{d\vec{\mathbf{r}}_{S,j}}{dt} \times \vec{\mathbf{p}}_j + \vec{\mathbf{r}}_{S,j} \times \frac{d\vec{\mathbf{p}}_j}{dt} \right). \quad (19.5.3)$$

Because the velocity of the  $j^{\text{th}}$  particle is  $\vec{\mathbf{v}}_{S,j} = d\vec{\mathbf{r}}_{S,j} / dt$ , the first term in the parentheses vanishes (the cross product of a vector with itself is zero because they are parallel to each other)

$$\frac{d\vec{r}_{S,j}}{dt} \times \vec{p}_j = \vec{v}_{S,j} \times m_j \vec{v}_{S,j} = 0. \quad (19.5.4)$$

Substitute Eq. (19.5.4) and  $\vec{F}_j = d\vec{p}_j / dt$  into Eq. (19.5.3) yielding

$$\frac{d\vec{L}_S^{\text{sys}}}{dt} = \sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \frac{d\vec{p}_j}{dt} \right) = \sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \vec{F}_j \right). \quad (19.5.5)$$

Because

$$\sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \vec{F}_j \right) = \sum_{j=1}^{j=N} \vec{\tau}_{S,j} = \vec{\tau}_S^{\text{ext}} + \vec{\tau}_S^{\text{int}} \quad (19.5.6)$$

We have already shown in Chapter 17.4 that when we assume all internal forces are directed along the line connecting the two interacting objects then the internal torque about the point  $S$  is zero,

$$\vec{\tau}_S^{\text{int}} = \vec{0}. \quad (19.5.7)$$

Eq. (19.5.6) simplifies to

$$\sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \vec{F}_j \right) = \sum_{j=1}^{j=N} \vec{\tau}_{S,j} = \vec{\tau}_S^{\text{ext}}. \quad (19.5.8)$$

Therefore Eq. (19.5.5) becomes

$$\vec{\tau}_S^{\text{ext}} = \frac{d\vec{L}_S^{\text{sys}}}{dt}. \quad (19.5.9)$$

The external torque about the point  $S$  is equal to the time derivative of the angular momentum of the system about that point.

### Example 19.5 Angular Momentum of Two Particles undergoing Circular Motion

Two identical particles of mass  $m$  move in a circle of radius  $R$ , with angular velocity  $\vec{\omega} = \omega_z \hat{k}$ ,  $\omega_z > 0$ ,  $\omega$  about the  $z$ -axis in a plane parallel to but a distance  $h$  above the  $x$ - $y$  plane. The particles are located on opposite sides of the circle (Figure 19.11). Find the magnitude and the direction of the angular momentum about the point  $S$  (the origin).

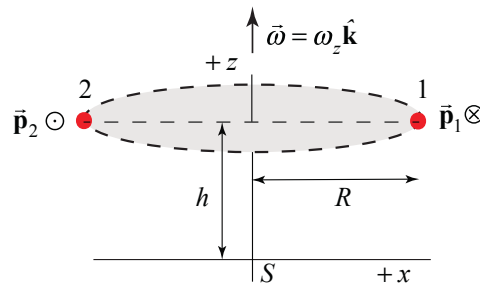
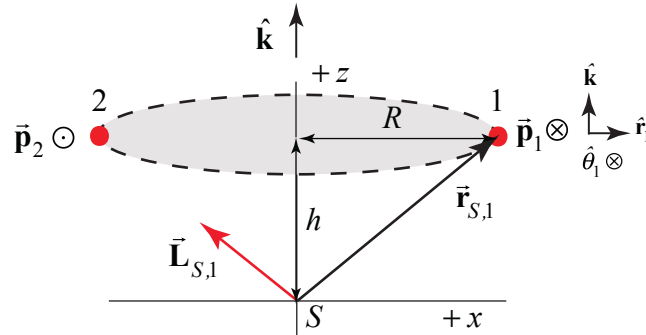
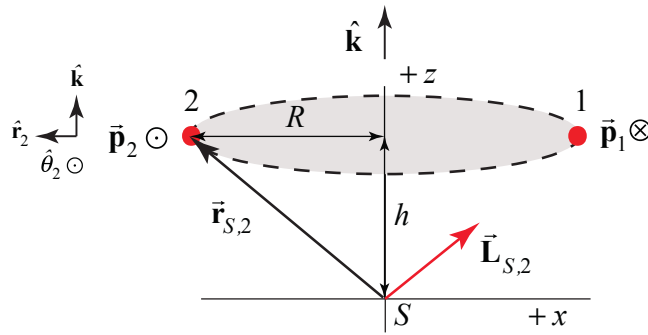


Figure 19.11 Example 19.5

**Solution:** The angular momentum about the origin is the sum of the contributions from each object. The calculation of each contribution will be identical to the calculation in Example 19.3



**Figure 19.12** Angular momentum of particle 1 about origin



**Figure 19.13** Angular momentum of particle 2 about origin

For particle 1 (Figure 19.12), the angular momentum about the point  $S$  is

$$\vec{L}_{S,1} = \vec{r}_{S,1} \times \vec{p}_1 = (R\hat{r}_1 + h\hat{k}) \times mR\omega_z\hat{\theta}_1 = mR^2\omega_z\hat{k} - hmR\omega_z\hat{r}_1.$$

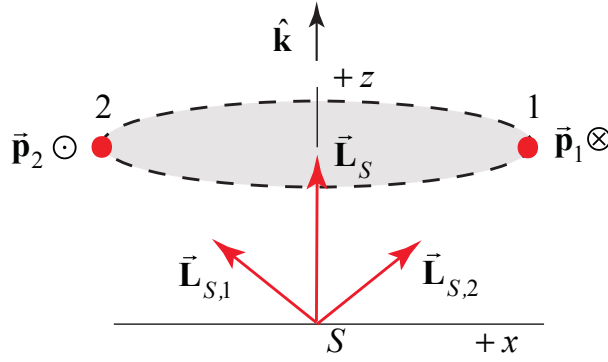
For particle 2, (Figure 19.13), the angular momentum about the point  $S$  is

$$\vec{L}_{S,2} = \vec{r}_{S,2} \times \vec{p}_2 = (R\hat{r}_2 + h\hat{k}) \times mR\omega_z\hat{\theta}_2 = mR^2\omega_z\hat{k} - hmR\omega_z\hat{r}_2.$$

Because the particles are located on opposite sides of the circle,  $\hat{r}_1 = -\hat{r}_2$ . The vector sum only points along the  $z$ -axis and is equal to

$$\vec{L}_S = \vec{L}_{S,1} + \vec{L}_{S,2} = 2mR^2\omega_z\hat{k}. \quad (19.5.10)$$

The two angular momentum vectors are shown in Figure 19.14.



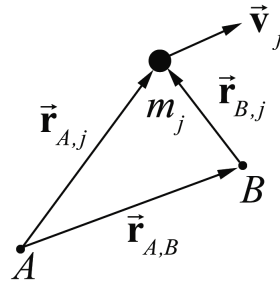
**Figure 19.14** Angular momentum about the point  $S$  of both particles and their sum

The moment of inertia of the two particles about the  $z$ -axis is given by  $I_S = 2mR^2$ . Therefore  $\vec{L}_S = I_S \vec{\omega}$ . The important point about this example is that the two objects are symmetrically distributed with respect to the  $z$ -axis (opposite sides of the circular orbit). Therefore the angular momentum about any point  $S$  along the  $z$ -axis has the same value  $\vec{L}_S = 2mr^2\omega\hat{k}$ , which is constant in magnitude and points in the  $+z$ -direction. This result generalizes to any rigid body in which the mass is distributed symmetrically about the axis of rotation.

### Example 19.6 Angular Momentum of a System of Particles about Different Points

Consider a system of  $N$  particles, and two points  $A$  and  $B$  (Figure 19.15). The angular momentum of the  $j^{\text{th}}$  particle about the point  $A$  is given by

$$\vec{L}_{A,j} = \vec{r}_{A,j} \times m_j \vec{v}_j. \quad (19.5.11)$$



**Figure 19.15** Vector triangle relating position of object and points  $A$  and  $B$

The angular momentum of the system of particles about the point  $A$  is given by the sum

$$\vec{L}_A = \sum_{j=1}^N \vec{L}_{A,j} = \sum_{j=1}^N \vec{r}_{A,j} \times m_j \vec{v}_j \quad (19.5.12)$$

The angular momentum about the point  $B$  can be calculated in a similar way and is given by



$$\vec{\mathbf{L}}_B = \sum_{j=1}^N \vec{\mathbf{L}}_{B,j} = \sum_{j=1}^N \vec{\mathbf{r}}_{B,j} \times m_j \vec{\mathbf{v}}_j . \quad (19.5.13)$$

From Figure 19.15, the vectors

$$\vec{\mathbf{r}}_{A,j} = \vec{\mathbf{r}}_{B,j} + \vec{\mathbf{r}}_{A,B} . \quad (19.5.14)$$

We can substitute Eq. (19.5.14) into Eq. (19.5.12) yielding

$$\vec{\mathbf{L}}_A = \sum_{j=1}^N (\vec{\mathbf{r}}_{B,j} + \vec{\mathbf{r}}_{A,B}) \times m_j \vec{\mathbf{v}}_j = \sum_{j=1}^N \vec{\mathbf{r}}_{B,j} \times m_j \vec{\mathbf{v}}_j + \sum_{j=1}^N \vec{\mathbf{r}}_{A,B} \times m_j \vec{\mathbf{v}}_j . \quad (19.5.15)$$

The first term in Eq. (19.5.15) is the angular momentum about the point  $B$ . The vector  $\vec{\mathbf{r}}_{A,B}$  is a constant and so can be pulled out of the sum in the second term, and Eq. (19.5.15) becomes

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \sum_{j=1}^N m_j \vec{\mathbf{v}}_j \quad (19.5.16)$$

The sum in the second term is the momentum of the system

$$\vec{\mathbf{p}}_{\text{sys}} = \sum_{j=1}^N m_j \vec{\mathbf{v}}_j . \quad (19.5.17)$$

Therefore the angular momentum about the points  $A$  and  $B$  are related by

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{p}}_{\text{sys}} \quad (19.5.18)$$

Thus if the momentum of the system is zero, the angular momentum is the same about any point.

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B, \quad (\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}}) . \quad (19.5.19)$$

In particular, the momentum of a system of particles is zero by definition in the center of mass reference frame because in that reference frame  $\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}}$ . Hence the angular momentum is the same about any point in the center of mass reference frame.

## 19.7 Angular Momentum and Torque for Fixed Axis Rotation

We have shown that, for fixed axis rotation, the component of torque that causes the angular velocity to change is the rotational analog of Newton's Second Law,

$$\vec{\tau}_S^{\text{ext}} = I_S \vec{\alpha}. \quad (19.5.20)$$

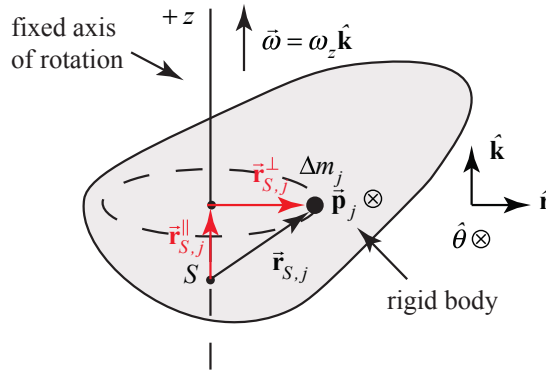
We shall now see that this is a special case of the more general result

$$\vec{\tau}_S^{\text{ext}} = \frac{d}{dt} \vec{L}_S^{\text{sys}}. \quad (19.5.21)$$

Consider a rigid body rotating about a fixed axis passing through the point  $S$  and take the fixed axis of rotation to be the  $z$ -axis. Recall that all the points in the rigid body rotate about the  $z$ -axis with the same angular velocity  $\vec{\omega} = (d\theta/dt)\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}}$ . In a similar fashion, all points in the rigid body have the same angular acceleration,  $\vec{\alpha} = (d^2\theta/dt^2)\hat{\mathbf{k}} = \alpha_z \hat{\mathbf{k}}$ . Let the point  $S$  lie somewhere along the  $z$ -axis.

As before, the body is divided into individual elements. We calculate the contribution of each element to the angular momentum about the point  $S$ , and then sum over all the elements. The summation will become an integral for a continuous body.

Each individual element has a mass  $\Delta m_j$  and is moving in a circle of radius  $r_{S,j}^\perp$  about the axis of rotation. Let  $\vec{r}_{S,j}$  be the vector from the point  $S$  to the element. The momentum of the element,  $\vec{p}_j$ , is tangent to this circle (Figure 19.16).



**Figure 19.16** Geometry of instantaneous rotation.

The angular momentum of the  $j^{\text{th}}$  element about the point  $S$  is given by  $\vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j$ . The vector  $\vec{r}_{S,j}$  can be decomposed into parallel and perpendicular components with respect to the axis of rotation  $\vec{r}_{S,j} = \vec{r}_{S,j}^{\parallel} + \vec{r}_{S,j}^{\perp}$  (Figure 19.16), where  $r_{S,j}^{\perp} = |\vec{r}_{S,j}^{\perp}|$  and  $r_{S,j}^{\parallel} = |\vec{r}_{S,j}^{\parallel}|$ . The momentum is given by  $\vec{p}_j = \Delta m_j r_{S,j}^{\perp} \omega_z \hat{\theta}$ . Then the angular momentum about the point  $S$  is

$$\begin{aligned}\vec{L}_{S,j} &= \vec{r}_{S,j} \times \vec{p}_j = (r_{S,j}^\perp \hat{r} + r_{S,j}^\parallel \hat{k}) \times (\Delta m_j r_{S,j}^\perp \omega_z \hat{\theta}) \\ &= \Delta m_j (r_{S,j}^\perp)^2 \omega_z \hat{k} - \Delta m_j r_{S,j}^\parallel r_{S,j}^\perp \omega_z \hat{r}.\end{aligned}\quad (19.5.22)$$

In the last expression in Equation (19.5.22), the second term has a direction that is perpendicular to the  $z$ -axis. Therefore the  $z$ -component of the angular momentum about the point  $S$ ,  $(L_{S,j})_z$ , arises entirely from the second term,  $\vec{r}_{S,j}^\perp \times \vec{p}_j$ . Therefore the  $z$ -component of the angular momentum about  $S$  is

$$(L_{S,j})_z = \Delta m_j (r_{S,j}^\perp)^2 \omega_z. \quad (19.5.23)$$

The  $z$ -component of the angular momentum of the system about  $S$  is the summation over all the elements,

$$L_{S,z}^{\text{sys}} = \sum_j (L_{S,j})_z = \sum_j \Delta m_j (r_{S,j}^\perp)^2 \omega_z. \quad (19.5.24)$$

For a continuous mass distribution the summation becomes an integral over the body,

$$L_{S,z}^{\text{sys}} = \int_{\text{body}} dm (r_{dm})^2 \omega_z, \quad (19.5.25)$$

where  $r_{dm}$  is the distance from the fixed  $z$ -axis to the infinitesimal element of mass  $dm$ . The moment of inertia of a rigid body about a fixed  $z$ -axis passing through a point  $S$  is given by an integral over the body

$$I_S = \int_{\text{body}} dm (r_{dm})^2. \quad (19.5.26)$$

Thus the  $z$ -component of the angular momentum about  $S$  for a fixed axis that passes through  $S$  in the  $z$ -direction is proportional to the  $z$ -component of the angular velocity,  $\omega_z$ ,

$$L_{S,z}^{\text{sys}} = I_S \omega_z. \quad (19.5.27)$$

For fixed axis rotation, our result that torque about a point is equal to the time derivative of the angular momentum about that point,

$$\vec{\tau}_S^{\text{ext}} = \frac{d}{dt} \vec{L}_S^{\text{sys}}, \quad (19.5.28)$$

can now be resolved in the  $z$ -direction,

$$\tau_{S,z}^{\text{ext}} = \frac{dL_{S,z}^{\text{sys}}}{dt} = \frac{d}{dt} (I_S \omega_z) = I_S \frac{d\omega_z}{dt} = I_S \frac{d^2\theta}{dt^2} = I_S \alpha_z, \quad (19.5.29)$$

in agreement with our earlier result that the  $z$ -component of torque about the point  $S$  is equal to the product of moment of inertia about  $I_S$ , and the  $z$ -component of the angular acceleration,  $\alpha_z$ .

### Example 19.6 Circular Ring

A circular ring of radius  $R$ , and mass  $M$  is rotating about the  $z$ -axis in a plane parallel to but a distance  $h$  above the  $x$ - $y$  plane. The  $z$ -component of the angular velocity is  $\omega_z$  (Figure 19.17). Find the magnitude and the direction of the angular momentum  $\vec{L}_S$  along at any point  $S$  on the central  $z$ -axis.

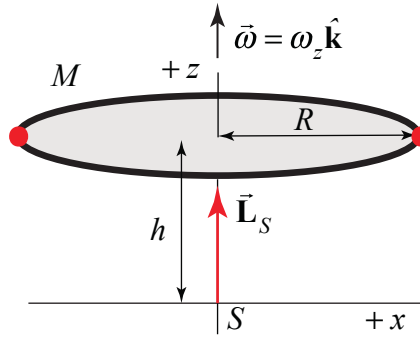


Figure 19.17 Example 19.6

**Solution:** Use the same symmetry argument as we did in Example 19.5. The ring can be thought of as made up of pairs of point like objects on opposite sides of the ring each of mass  $m$  (Figure 19.18).

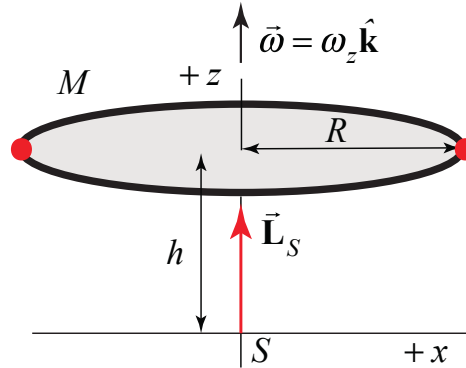


Figure 19.18 Ring as a sum of pairs of symmetrically distributed particles

Each pair has a non-zero  $z$ -component of the angular momentum taken about any point  $S$  along the  $z$ -axis,  $\vec{L}_S^{pair} = \vec{L}_{S,1} + \vec{L}_{S,2} = 2mR^2\omega_z\hat{k} = m^{pair}R^2\omega_z\hat{k}$ . The angular momentum of the ring about the point  $S$  is then the sum over all the pairs

$$\vec{L}_S = \sum_{pairs} m^{pair} R^2 \omega_z \hat{k} = MR^2 \omega_z \hat{k}. \quad (19.5.30)$$

Recall that the moment of inertia of a ring is given by

$$I_S = \int_{\text{body}} dm (r_{dm})^2 = MR^2. \quad (19.5.31)$$

For the symmetric ring, the angular momentum about  $S$  points in the direction of the angular velocity and is equal to

$$\vec{L}_S = I_S \omega \hat{k}. \quad (19.5.32)$$

## 19.8 Principle of Conservation of Angular Momentum

Consider a system of particles. We begin with the result that we derived in Section 19.7 that the torque about a point  $S$  is equal to the time derivative of the angular momentum about that point  $S$ ,

$$\vec{\tau}_S^{\text{ext}} = \frac{d\vec{L}_S^{\text{sys}}}{dt}. \quad (19.5.33)$$

With this assumption, the torque due to the external forces is equal to the rate of change of the angular momentum

$$\vec{\tau}_S^{\text{ext}} = \frac{d\vec{L}_S^{\text{sys}}}{dt}. \quad (19.5.34)$$

### *Principle of Conservation of Angular Momentum*

*If the external torque acting on a system is zero, then the angular momentum of the system is constant. So for any change of state of the system the change in angular momentum is zero*

$$\Delta \vec{L}_S^{\text{sys}} \equiv (\vec{L}_S^{\text{sys}})_f - (\vec{L}_S^{\text{sys}})_i = \vec{0}. \quad (19.5.35)$$

*Equivalently the angular momentum is constant*

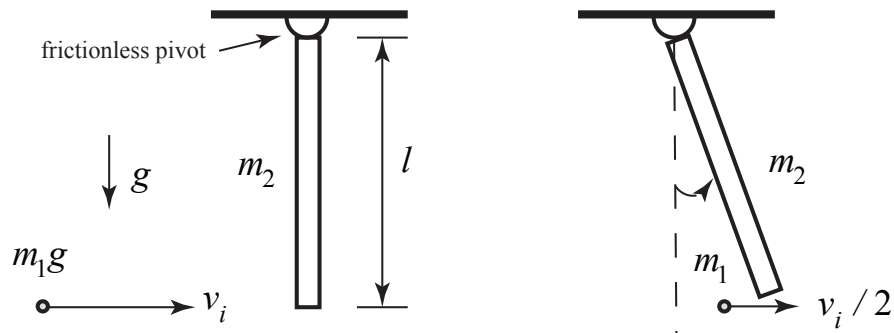
$$(\vec{L}_S^{\text{sys}})_f = (\vec{L}_S^{\text{sys}})_i. \quad (19.5.36)$$

So far no isolated system has been encountered such that the angular momentum is not constant so our assumption that internal torques cancel in pairs can be taken as an experimental observation.

### **Example 19.7 Collision Between Pivoted Rod and Object**

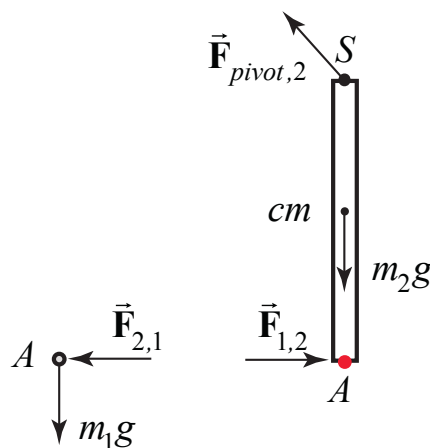
A point-like object of mass  $m_1$  moving with constant speed  $v_i$  strikes a rigid uniform rod of length  $l$  and mass  $m_2$  that is hanging by a frictionless pivot from the ceiling. Immediately

after striking the rod, the object continues forward but its speed decreases to  $v_i / 2$  (Figure 19.19). The moment of inertia of the rod about its center of mass is  $I_{cm} = (1/12)m_2l^2$ . Gravity acts with acceleration  $g$  downward. (a) For what value of  $v_i$  will the rod just touch the ceiling on its first swing? (b) For what ratio  $m_2 / m_1$  will the collision be elastic?



**Figure 19.19** Example 19.7

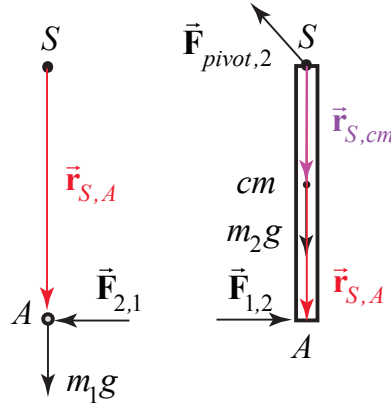
**Solution:** We begin by identifying our system, which consists of the object and the uniform rod. We identify three states; an initial state  $i$ : immediately before the collision, state  $a$ : immediately after the collision, and state  $f$ : the instant the rod touches the ceiling when the final angular speed is zero. We would like to know if any of our fundamental quantities: momentum, energy, and angular momentum, are constant during these state changes, state  $i$  to state  $a$ , state  $a$  to state  $f$ .



**Figure 19.20** Free-body force diagrams on particle and rod

We start with the transition from state  $i$  to state  $a$ . The pivot force holding the rod to the ceiling is an external force acting at the pivot point  $S$ . There is also the gravitational force acting at the center of mass of the rod and on the object. There are also internal forces due to the collision of the rod and the object at point  $A$  (Figure 19.20).

The external force means that momentum is not constant. The point of action of the external pivot force is fixed and so does no work. However, we do not know whether or not the collision is elastic and so we cannot assume that mechanical energy is constant. Choose the pivot point  $S$  as the point about which to calculate torque, then the torque diagrams are shown in Figure 19.21.



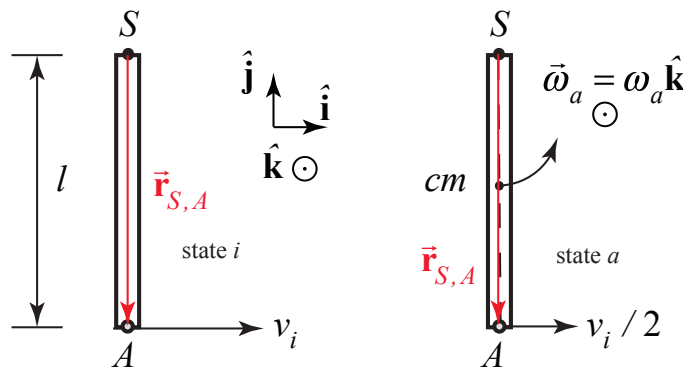
**Figure 19.21** Torque diagrams on particle and rod with torque calculated about pivot point  $S$

The torque on the system about the pivot  $S$  is then the sum of terms

$$\vec{\tau}_S^{\text{sys}} = \vec{r}_{S,S} \times \vec{F}_{\text{pivot},2} + \vec{r}_{S,A} \times \vec{F}_{1,2} + \vec{r}_{S,A} \times \vec{F}_{2,1} + \vec{r}_{S,cm} \times m_2 \vec{g} + \vec{r}_{S,A} \times m_1 \vec{g} . \quad (19.5.37)$$

The external pivot force does not contribute any torque because  $\vec{r}_{S,S} = \vec{0}$ . The internal forces between the rod and the object are equal in magnitude and opposite in direction,  $\vec{F}_{1,2} = -\vec{F}_{2,1}$  (Newton's Third Law), and so their contributions to the torque add to zero. If the collision is instantaneous then the gravitational force is parallel to  $\vec{r}_{S,cm}$  and  $\vec{r}_{S,A}$  so the two gravitational torques are zero. Therefore the torque on the system about the pivot point is zero,  $\vec{\tau}_S^{\text{sys}} = \vec{0}$ . Thus the angular momentum about the pivot point is constant,

$$\vec{L}_{S,i}^{\text{sys}} = \vec{L}_{S,a}^{\text{sys}} . \quad (19.5.38)$$



**Figure 19.22** Angular momentum diagram

In order to calculate the angular momentum we draw a diagram showing the momentum of the object and the angular speed of the rod in (Figure 19.22). The angular momentum about  $S$  immediately before the collision is

$$\vec{L}_{S,i}^{\text{sys}} = \vec{r}_{S,1} \times m_1 \vec{v}_i = l(-\hat{j}) \times m_1 v_i \hat{i} = lm_1 v_i \hat{k}.$$

The angular momentum about  $S$  immediately after the collision is

$$\vec{L}_{S,a}^{\text{sys}} = \vec{r}_{S,1} \times m_1 \vec{v}_i / 2 + I_S \vec{\omega}_a = l(-\hat{j}) \times m_1 (v_i / 2) \hat{i} + I_S \omega_a \hat{k} = (lm_1 v_i / 2) \hat{k} + I_S \omega_a \hat{k}.$$

Therefore the condition that the angular momentum about  $S$  is constant during the collision becomes

$$lm_1 v_i \hat{k} = (lm_1 v_i / 2 + I_S \omega_a) \hat{k}.$$

We can solve for the angular speed immediately after the collision

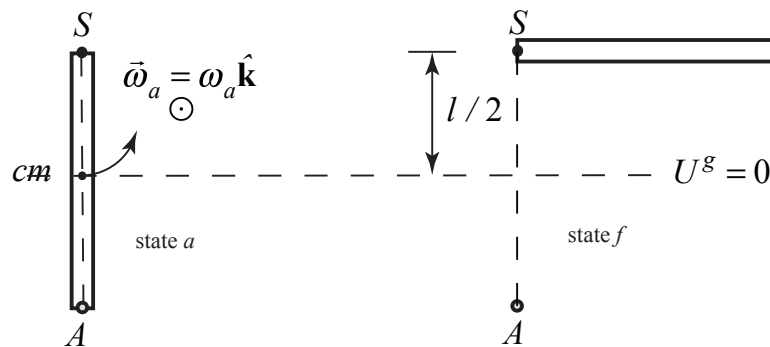
$$\omega_a = \frac{lm_1 v_i}{2I_S}.$$

By the parallel axis theorem the moment of inertial of a uniform rod about the pivot point is

$$I_S = m_2 (l/2)^2 + I_{cm} = (1/4)m_2 l^2 + (1/12)m_2 l^2 = (1/3)m_2 l^2. \quad (19.5.39)$$

Therefore the angular speed immediately after the collision is

$$\omega_2 = \frac{3m_1 v_i}{2m_2 l}. \quad (19.5.40)$$



**Figure 19.23** Energy diagram for transition from state  $a$  to state  $f$ .

For the transition from state  $a$  to state  $f$ , we know that the gravitational force is conservative and the pivot force does no work so mechanical energy is constant.



$$E_a^{mech} = E_f^{mech}$$

We draw an energy diagram only for the rod because the kinetic energy for the particle is not changing between states  $a$  and  $f$ , (Figure 19.23), with a choice of zero for the potential energy at the center of mass. The mechanical energy of the rod and particle immediately after the collision is

$$E_a^{mech} = \frac{1}{2} I_s \omega_a^2 + \frac{1}{2} m_1 (v_i / 2)^2 .$$

Using our results for the moment of inertia  $I_s$  (Eq. (19.5.39)) and  $\omega_2$  (Eq. (19.5.40)), we have that

$$E_a^{mech} = \frac{1}{2} (1/3) m_2 l^2 \left( \frac{3m_1 v_i}{2m_2 l} \right)^2 + \frac{1}{2} m_1 (v_i / 2)^2 = \frac{3m_1^2 v_i^2}{8m_2} + \frac{1}{2} m_1 (v_i / 2)^2 . \quad (19.5.41)$$

The mechanical energy when the rod just reaches the ceiling when the final angular speed is zero is then

$$E_f^{mech} = m_2 g (l / 2) + \frac{1}{2} m_1 (v_i / 2)^2 .$$

Then the condition that the mechanical energy is constant becomes

$$\frac{3m_1^2 v_i^2}{8m_2} + \frac{1}{2} m_1 (v_i / 2)^2 = m_2 g (l / 2) + \frac{1}{2} m_1 (v_i / 2)^2 . \quad (19.5.42)$$

We can now solve Eq. (19.5.42) for the initial speed of the object

$$v_i = \frac{m_2}{m_1} \sqrt{\frac{4gl}{3}} . \quad (19.5.43)$$

We now return to the transition from state  $i$  to state  $a$  . and determine the constraint on the mass ratio in order for the collision to be elastic. The mechanical energy before the collision is

$$E_i^{mech} = \frac{1}{2} m_1 v_i^2 . \quad (19.5.44)$$

If we impose the condition that the collision is elastic then

$$E_i^{mech} = E_a^{mech} . \quad (19.5.45)$$

Substituting Eqs. (19.5.41) and (19.5.44) into Eq. (19.5.45) yields

$$\frac{1}{2} m_1 v_i^2 = \frac{3m_1^2 v_i^2}{8m_2} + \frac{1}{2} m_1 (v_i / 2)^2 .$$

This simplifies to

$$\frac{3}{8} m_1 v_i^2 = \frac{3m_1^2 v_i^2}{8m_2}$$

Hence we can solve for the mass ratio necessary to ensure that the collision is elastic if the final speed of the object is half it's initial speed

$$\frac{m_2}{m_1} = 1 . \quad (19.5.46)$$

Notice that this mass ratio is independent of the initial speed of the object.

## 19.9 External Angular Impulse and Change in Angular Momentum

*Define the **external angular impulse** about a point  $S$  applied as the integral of the external torque about  $S$*

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}} dt . \quad (19.5.47)$$

Then the external angular impulse about  $S$  is equal to the change in angular momentum

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}} dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} dt = \vec{\mathbf{L}}_{S,f}^{\text{sys}} - \vec{\mathbf{L}}_{S,i}^{\text{sys}} . \quad (19.5.48)$$

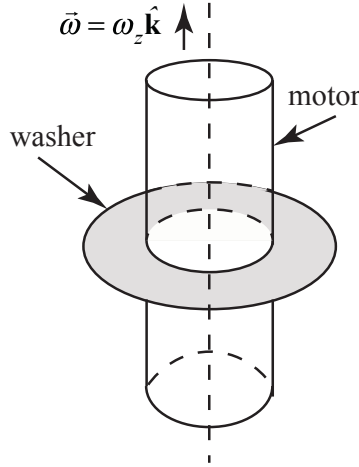
Notice that this is the rotational analog to our statement about impulse and momentum,

$$\vec{\mathbf{I}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\mathbf{F}}^{\text{ext}} dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} dt = \vec{\mathbf{p}}_{\text{sys},f} - \vec{\mathbf{p}}_{\text{sys},i} . \quad (19.5.49)$$

### Example 19.8 Angular Impulse on Steel Washer

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is  $I_0$ . The washer is set into motion. When it reaches an initial angular speed  $\omega_0$ , at  $t = 0$ , the power to the motor is shut off, and the washer slows down until it reaches an angular speed of  $\omega_a$  at time  $t_a$ . At that instant, a second steel washer with a moment of inertia  $I_w$  is dropped on top of the first washer. Assume that the second washer is only in contact

with the first washer. The collision takes place over a time  $\Delta t_{\text{int}} = t_b - t_a$ . Assume the frictional torque on the axle is independent of speed, and remains the same when the second washer is dropped. The two washers continue to slow down during the time interval  $\Delta t_2 = t_f - t_b$  until they stop at time  $t = t_f$ . (a) What is the angular acceleration while the washer and motor are slowing down during the interval  $\Delta t_1 = t_a$ ? (b) Suppose the collision is nearly instantaneous,  $\Delta t_{\text{int}} = (t_b - t_a) \approx 0$ . What is the angular speed  $\omega_b$  of the two washers immediately after the collision is finished (when the washers rotate together)?



**Figure 19.24** Example 19.8

Now suppose the collision is not instantaneous but that the frictional torque is independent of the speed of the rotor. (c) What is the angular impulse during the collision? (d) What is the angular velocity  $\omega_b$  of the two washers immediately after the collision is finished (when the washers rotate together)? (e) What is the angular deceleration  $\alpha_2$  after the collision?

**Solution:** a) The angular acceleration of the motor and washer from the instant when the power is shut off until the second washer was dropped is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{\Delta t_1} < 0. \quad (19.5.50)$$

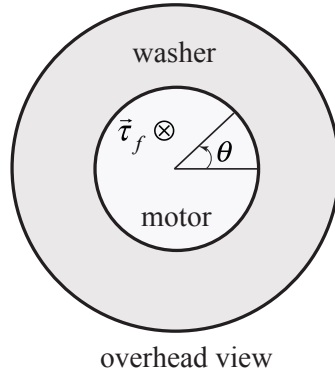
(b) If the collision is nearly instantaneous, then there is no angular impulse and therefore the  $z$ -component of the angular momentum about the rotation axis of the motor remains constant

$$0 = \Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_w)\omega_b - I_0\omega_a. \quad (19.5.51)$$

We can solve Eq. (19.5.51) for the angular speed  $\omega_b$  of the two washers immediately after the collision is finished

$$\omega_b = \frac{I_0}{I_0 + I_w} \omega_a. \quad (19.5.52)$$

(c) The angular acceleration found in part a) is due to the frictional torque in the motor.



**Figure 19.25** Frictional torque in the motor

Let  $\vec{\tau}_f = -\tau_f \hat{\mathbf{k}}$  where  $\tau_f$  is the magnitude of the frictional torque (Figure 19.25) then

$$-\tau_f = I_0 \alpha_1 = \frac{I_0 (\omega_a - \omega_0)}{\Delta t_1}. \quad (19.5.53)$$

During the collision with the second washer, the frictional torque exerts an angular impulse (pointing along the  $z$ -axis in the figure),

$$J_z = -\int_{t_a}^{t_b} \tau_f dt = -\tau_f \Delta t_{\text{int}} = I_0 (\omega_a - \omega_0) \frac{\Delta t_{\text{int}}}{\Delta t_1}. \quad (19.5.54)$$

(d) The  $z$ -component of the angular momentum about the rotation axis of the motor changes during the collision,

$$\Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_w) \omega_b - I_0 \omega_a. \quad (19.5.55)$$

The change in the  $z$ -component of the angular momentum is equal to the  $z$ -component of the angular impulse

$$J_z = \Delta L_z. \quad (19.5.56)$$

Thus, equating the expressions in Equations (19.5.54) and (19.5.55), yields

$$I_0 (\omega_a - \omega_0) \left( \frac{\Delta t_{\text{int}}}{\Delta t_1} \right) = (I_0 + I_w) \omega_b - (I_0) \omega_a. \quad (19.5.57)$$

Solve Equation (19.5.57) for the angular velocity immediately after the collision,

$$\omega_b = \frac{I_0}{(I_0 + I_w)} \left( (\omega_a - \omega_0) \left( \frac{\Delta t_{\text{int}}}{\Delta t_1} \right) + \omega_a \right). \quad (19.5.58)$$

If there were no frictional torque, then the first term in the brackets would vanish, and the second term of Eq. (19.5.58) would be the only contribution to the final angular speed.

(e) The final angular acceleration  $\alpha_2$  is given by

$$\alpha_2 = \frac{0 - \omega_b}{\Delta t_2} = - \frac{I_0}{(I_0 + I_w) \Delta t_2} \left( (\omega_a - \omega_0) \left( \frac{\Delta t_{\text{int}}}{\Delta t_1} \right) + \omega_a \right). \quad (19.5.59)$$