# Chapter 14 Potential Energy and Conservation of Energy

# Equation Chapter 8 Section 1 Chapter 14 Potential Energy and Conservation of Energy

> *There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no exception to this law — it is exact as far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity, which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same.* [1]
>
> *Richard Feynman*

So far we have analyzed the motion of point-like objects under the action of forces using Newton's Laws of Motion. We shall now introduce the Principle of Conservation of Energy to study the change in energy of a system between its initial and final states. In particular we shall introduce the concept of potential energy to describe the effect of conservative internal forces acting on the constituent components of a system.

## 14.1 Conservation of Energy

We shall just consider closed systems in which only energy can enter of leave the system. Recall from Chapter 13.1, the principle of conservation of energy. When a system and its surroundings undergo a transition from an initial state to a final state, the change in energy is zero,

$$\Delta E = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0 \,. \tag{14.1}$$

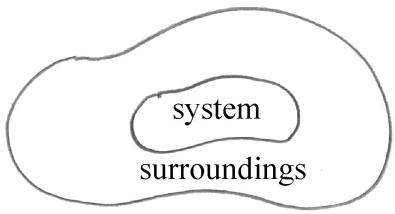


**Figure 14.1** Diagram of a system and its surroundings

[1] Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics,* Vol. 1, p. 4.1.

We shall study types of energy transformations due to interactions both inside and across the boundary of a system.

## 14.2 Conservative and Non-Conservative Forces

Our first type of "energy accounting" involves *mechanical energy.* There are two types of mechanical energy, *kinetic energy* and *potential energy*. Our first task is to define what we mean by the change of the potential energy of a system.

We defined the work done by a force $\vec{\mathbf{F}}$, on an object, which moves along a path from an initial position $\vec{\mathbf{r}}_i$ to a final position $\vec{\mathbf{r}}_f$, as the integral of the component of the force tangent to the path with respect to the displacement of the point of contact of the force and the object,

$$W = \int_{\text{path}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \,. \tag{14.1}$$

Does the work done on the object by the force depend on the path taken by the object?

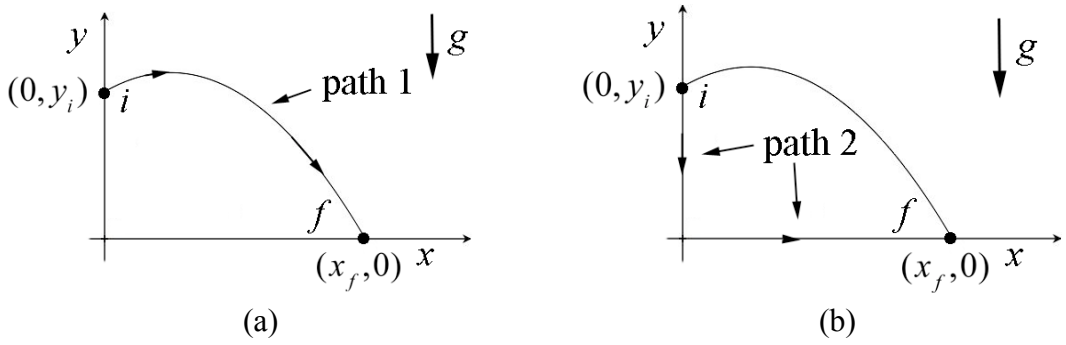


**Figure 14.2 (a) and (b)** Two different paths connecting the same initial and final points

First consider the motion of an object under the influence of a gravitational force near the surface of the earth. Let's consider two paths 1 and 2 shown in Figure 14.2. Both paths begin at the initial point $(x_i, y_i) = (0, y_i)$ and end at the final point $(x_f, y_f) = (x_f, 0)$. The gravitational force always points downward, so with our choice of coordinates, $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$. The infinitesimal displacement along path 1 (Figure 14.2a) is given by $d\vec{\mathbf{r}}_1 = dx_1\,\hat{\mathbf{i}} + dy_1\,\hat{\mathbf{j}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_1 = -mg\,\hat{\mathbf{j}} \cdot (dx_1\,\hat{\mathbf{i}} + dy_1\,\hat{\mathbf{j}}) = -mgdy_1 \,. \tag{14.2}$$

The work done by gravity along path 1 is the integral

$$W_1 = \int_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{(0,y_i)}^{(x_f,0)} -mg\, dy_1 = -mg(0 - y_i) = mgy_i . \tag{14.3}$$

Path 2 consists of two legs (Figure 14.2b), leg A goes from the initial point $(0, y_i)$ to the origin $(0,0)$, and leg B goes from the origin $(0,0)$ to the final point $(x_f, 0)$. We shall calculate the work done along the two legs and then sum them up. The infinitesimal displacement along leg A is given by $d\vec{\mathbf{r}}_A = dy_A\, \hat{\mathbf{j}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_A = -mg\, \hat{\mathbf{j}} \cdot dy_A\, \hat{\mathbf{j}} = -mgdy_A . \tag{14.4}$$

The work done by gravity along leg A is the integral

$$W_A = \int_{\text{leg } A} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_A = \int_{(0,y_i)}^{(0,0)} -mg\, dy_A = -mg(0 - y_i) = mgy_i . \tag{14.5}$$

The infinitesimal displacement along leg B is given by $d\vec{\mathbf{r}}_B = dx_B\, \hat{\mathbf{i}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_B = -mg\, \hat{\mathbf{j}} \cdot dx_B\, \hat{\mathbf{i}} = 0 . \tag{14.6}$$

Therefore the work done by gravity along leg B is zero, $W_B = 0$, which is no surprise because leg B is perpendicular to the direction of the gravitation force. Therefore the work done along path 2 is equal to the work along path 1,

$$W_2 = W_A + W_B = mgy_i = W_1 . \tag{14.7}$$

Now consider the motion of an object on a surface with a kinetic frictional force between the object and the surface and denote the coefficient of kinetic friction by $\mu_k$. Let's compare two paths from an initial point $x_i$ to a final point $x_f$. The first path is a straight-line path. Along this path the work done is just

$$W^f = \int_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path 1}} F_x\, dx = -\mu_k N s_1 = -\mu_k N \Delta x < 0 , \tag{14.8}$$

where the length of the path is equal to the displacement, $s_1 = \Delta x$. Note that the fact that the kinetic frictional force is directed opposite to the displacement, which is reflected in the minus sign in Equation (14.8). The second path goes past $x_f$ some distance and them comes back to $x_f$ (Figure 14.3). Because the force of friction always opposes the motion, the work done by friction is negative,

$$W^f = \int_{\text{path 2}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path 2}} F_x \, dx = -\mu_k N s_2 < 0 . \qquad (14.9)$$

The work depends on the total distance traveled $s_2$, and is greater than the displacement $s_2 > \Delta x$. The magnitude of the work done along the second path is greater than the magnitude of the work done along the first path.

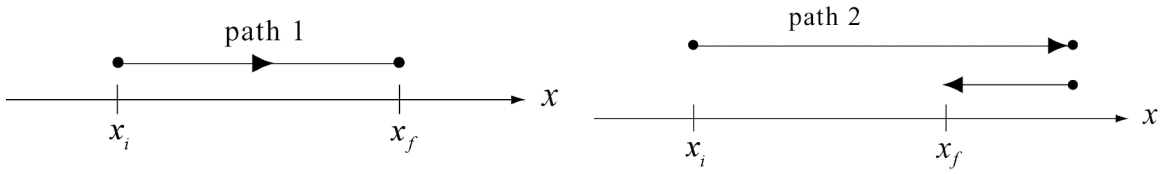


**Figure 14.3** Two different paths from $x_i$ to $x_f$.

These two examples typify two fundamentally different types of forces and their contribution to work. The work done by the gravitational force near the surface of the earth is independent of the path taken between the initial and final points. In the case of sliding friction, the work done depends on the path taken.

> *Whenever the work done by a force in moving an object from an initial point to a final point is independent of the path, the force is called a* ***conservative force****.*

The work done by a conservative force $\vec{\mathbf{F}}_c$ in going around a closed path is zero. Consider the two paths shown in Figure 14.4 that form a closed path starting and ending at the point $A$ with Cartesian coordinates $(1,0)$.

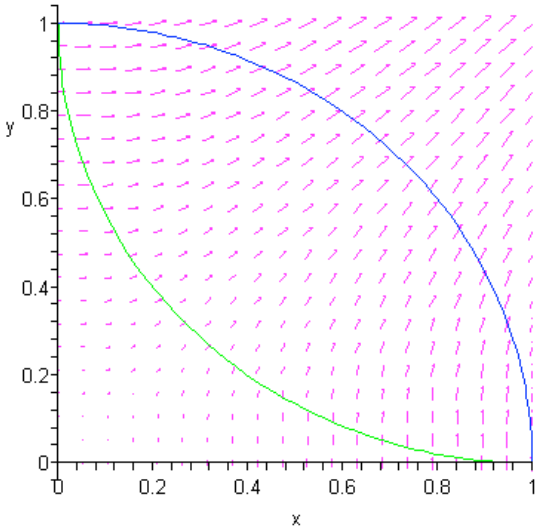


**Figure 14.4** Two paths in the presence of a conservative force.

The work done along path 1 (the upper path in the figure, blue if viewed in color) from point $A$ to point $B$ with coordinates $(0,1)$ is given by

$$W_1 = \int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 \,. \tag{14.10}$$

The work done along path 2 (the lower path, green in color) from $B$ to $A$ is given by

$$W_2 = \int_B^A \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.11}$$

The work done around the closed path is just the sum of the work along paths 1 and 2,

$$W = W_1 + W_2 = \int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 + \int_B^A \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.12}$$

If we reverse the endpoints of path 2, then the integral changes sign,

$$W_2 = \int_B^A \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 = -\int_A^B \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.13}$$

We can then substitute Equation (14.13) into Equation (14.12) to find that the work done around the closed path is

$$W = \int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.14}$$

Since the force is conservative, the work done between the points $A$ to $B$ is independent of the path, so

$$\int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 = \int_A^B \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.15}$$

We now use path independence of work for a conservative force (Equation (14.15) in Equation (14.14)) to conclude that the work done by a conservative force around a closed path is zero,

$$W = \oint_{\substack{\text{closed} \\ \text{path}}} \vec{\mathbf{F}}_c \cdot d\vec{\mathbf{r}} = 0 \,. \tag{14.16}$$

## 14.3 Changes in Potential Energies of a System

Consider an object near the surface of the earth as a system that is initially given a velocity directed upwards. Once the object is released, the gravitation force, acting as an external force, does a negative amount of work on the object, and the kinetic energy decreases until the object reaches its highest point, at which its kinetic energy is zero. The gravitational force then does positive work until the object returns to its initial starting point with a velocity directed downward. If we ignore any effects of air resistance, the descending object will then have the identical kinetic energy as when it was thrown. All the kinetic energy was completely recovered.

Now consider both the earth and the object as a system and assume that there are no other external forces acting on the system. Then the gravitational force is an internal conservative force, and does work on both the object and the earth during the motion. As the object moves upward, the kinetic energy of the system decreases, primarily because the object slows down, but there is also an imperceptible increase in the kinetic energy of the earth. The change in kinetic energy of the earth must also be included because the earth is part of the system. When the object returns to its original height (vertical distance from the surface of the earth), all the kinetic energy in the system is recovered, even though a very small amount has been transferred to the Earth.

If we included the air as part of the system, and the air resistance as a non-conservative internal force, then the kinetic energy lost due to the work done by the air resistance is not recoverable. This lost kinetic energy, which we have called thermal energy, is distributed as random kinetic energy in both the air molecules and the molecules that compose the object (and, to a smaller extent, the earth).

We shall define a new quantity, the change in the internal *potential energy* of the system, which measures the amount of lost kinetic energy that can be recovered during an interaction.

> *When only internal conservative forces act in a closed system, the sum of the changes of the kinetic and potential energies of the system is zero.*

Consider a closed system, $\Delta E_{sys} = 0$, that consists of two objects with masses $m_1$ and $m_2$ respectively. Assume that there is only one conservative force (internal force) that is the source of the interaction between two objects. We denote the force on object 1 due to the interaction with object 2 by $\vec{\mathbf{F}}_{2,1}$ and the force on object 2 due to the interaction with object 1 by $\vec{\mathbf{F}}_{1,2}$. From Newton's Third Law,

$$\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2} . \tag{14.1}$$

The forces acting on the objects are shown in Figure 14.5.

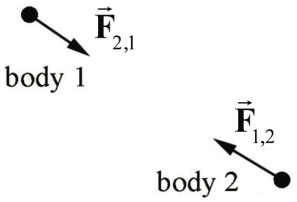


**Figure 14.5** Internal forces acting on two objects

Choose a coordinate system (Figure 14.6) in which the position vector of object 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of object 2 is given by $\vec{\mathbf{r}}_2$. The relative position of object 1 with respect to object 2 is given by $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. During the course of the interaction, object 1 is displaced by $d\vec{\mathbf{r}}_1$ and object 2 is displaced by $d\vec{\mathbf{r}}_2$, so the relative displacement of the two objects during the interaction is given by $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$.

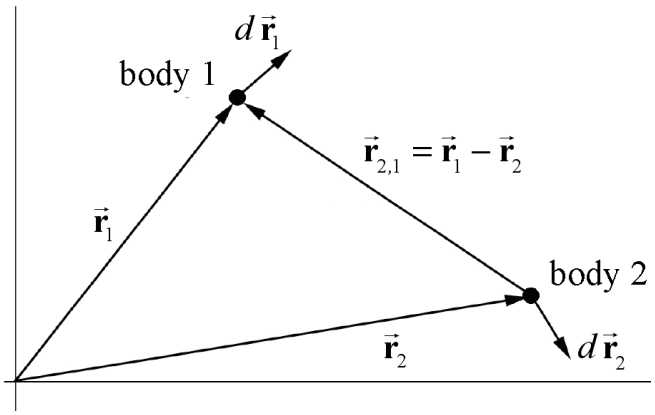


**Figure 14.6** Coordinate system for two objects with relative position vector $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$

Recall that the change in the kinetic energy of an object is equal to the work done by the forces in displacing the object. For two objects displaced from an initial state $A$ to a final state $B$,

$$\Delta K_{\text{sys}} = \Delta K_1 + \Delta K_2 = W_{\text{c}} = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 . \tag{14.2}$$

(In Equation (14.2), the labels "$A$" and "$B$" refer to initial and final states, not paths.)

From Newton's Third Law, Equation (14.1), the sum in Equation (14.2) becomes

$$\Delta K_{\text{sys}} = W_c = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot (d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2) = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} \tag{14.3}$$

where $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$ is the relative displacement of the two objects. Note that since $\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}$ and $d\vec{\mathbf{r}}_{2,1} = -d\vec{\mathbf{r}}_{1,2}$, $\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2}$.

*Consider a system consisting of two objects interacting through a conservative force. Let* $\vec{\mathbf{F}}_{2,1}$ *denote the force on object 1 due to the interaction with object 2 and let* $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$ *be the relative displacement of the two objects. The* ***change in internal potential energy of the system*** *is defined to be the negative of the work done by the conservative force when the objects undergo a relative displacement from the initial state* $A$ *to the final state* $B$ *along any displacement that changes the initial state* $A$ *to the final state* $B$*,*

$$\Delta U_{\text{sys}} = -W_{\text{c}} = -\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2}. \qquad (14.4)$$

Our definition of potential energy only holds for conservative forces, because the work done by a conservative force does not depend on the path but only on the initial and final positions. Because the work done by the conservative force is equal to the change in kinetic energy, we have that

$$\Delta U_{\text{sys}} = -\Delta K_{\text{sys}}, \text{ (closed system with no non-conservative forces)}. \qquad (14.5)$$

Recall that the work done by a conservative force in going around a closed path is zero (Equation (14.16)); therefore the change in kinetic energy when a system returns to its initial state is zero. This means that the kinetic energy is completely recoverable.

In the *Appendix 13A: Work Done on a System of Two Particles,* we showed that the work done by an internal force in changing a system of two particles of masses $m_1$ and $m_2$ respectively from an initial state $A$ to a final state $B$ is equal to

$$W = \frac{1}{2}\mu(v_B^2 - v_A^2) = \Delta K_{\text{sys}}, \qquad (14.6)$$

where $v_B^2$ is the square of the relative velocity in state $B$, $v_A^2$ is the square of the relative velocity in state $A$, and $\mu = m_1 m_2 / (m_1 + m_2)$ is a quantity known as the *reduced mass* of the system.

### 14.3.1 Change in Potential Energy for Several Conservative Forces

When there are several internal conservative forces acting on the system we define a separate change in potential energy for the work done by each conservative force,

$$\Delta U_{\text{sys},\,i} = -W_{c,i} = -\int_A^B \vec{\mathbf{F}}_{\text{c},\,i} \cdot d\vec{\mathbf{r}}_i \,. \tag{14.7}$$

where $\vec{\mathbf{F}}_{\text{c},\,i}$ is a conservative internal force and $d\vec{\mathbf{r}}_i$ a change in the relative positions of the objects on which $\vec{\mathbf{F}}_{\text{c},\,i}$ when the system is changed from state $A$ to state $B$. The work done is the sum of the work done by the individual conservative forces,

$$W_{\text{c}} = W_{\text{c},1} + W_{\text{c},2} + \cdots . \tag{14.8}$$

Hence, the sum of the changes in potential energies for the system is the sum

$$\Delta U_{\text{sys}} = \Delta U_{\text{sys},1} + \Delta U_{\text{sys},2} + \cdots . \tag{14.9}$$

Therefore the change in potential energy of the system is equal to the negative of the work done

$$\Delta U_{\text{sys}} = -W_{\text{c}} = -\sum_i \int_A^B \vec{\mathbf{F}}_{\text{c},\,i} \cdot d\vec{\mathbf{r}}_i \,. \tag{14.10}$$

If the system is closed (external forces do no work), and there are no non-conservative internal forces then Eq. (14.5) holds.

## 14.4 Change in Potential Energy and Zero Point for Potential Energy

We already calculated the work done by different conservative forces: constant gravity near the surface of the earth, the spring force, and the universal gravitation force. We chose the system in each case so that the conservative force was an external force. In each case, there was no change of potential energy and the work done was equal to the change of kinetic energy,

$$W_{\text{ext}} = \Delta K_{\text{sys}} \,. \tag{14.1}$$

We now treat each of these conservative forces as internal forces and calculate the change in potential energy of the system according to our definition

$$\Delta U_{\text{sys}} = -W_{\text{c}} = -\int_A^B \vec{\mathbf{F}}_{\text{c}} \cdot d\vec{\mathbf{r}} \,. \tag{14.2}$$

We shall also choose a ***zero reference potential*** for the potential energy of the system, so that we can consider all changes in potential energy relative to this reference potential.

### 14.4.1 Change in Gravitational Potential Energy Near Surface of the Earth

Let's consider the example of an object falling near the surface of the earth. Choose our system to consist of the earth and the object. The gravitational force is now an internal conservative force acting inside the system. The distance separating the object and the center of mass of the earth, and the velocities of the earth and the object specifies the initial and final states.

Let's choose a coordinate system with the origin on the surface of the earth and the $+y$-direction pointing away from the center of the earth. Because the displacement of the earth is negligible, we need only consider the displacement of the object in order to calculate the change in potential energy of the system.

Suppose the object starts at an initial height $y_i$ above the surface of the earth and ends at final height $y_f$. The gravitational force on the object is given by $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{j}}$, the displacement is given by $d\vec{\mathbf{r}} = dy\,\hat{\mathbf{j}}$, and the scalar product is given by $\vec{\mathbf{F}}^g \cdot d\vec{\mathbf{r}} = -mg\,\hat{\mathbf{j}} \cdot dy\hat{\mathbf{j}} = -mgdy$. The work done by the gravitational force on the object is then

$$W^g = \int_{y_i)}^{y_f} \vec{\mathbf{F}}^g \cdot d\vec{\mathbf{r}} = \int_{y_i)}^{y_f} -mg\,dy = -mg(y_f - y_i) \ . \tag{14.3}$$

The change in potential energy is then given by

$$\Delta U^g = -W^g = mg\,\Delta y = mg\,y_f - mg\,y_i \, . \tag{14.4}$$

We introduce a potential energy function $U$ so that

$$\Delta U^g \equiv U_f^g - U_i^g \, . \tag{14.5}$$

Only differences in the function $U^g$ have a physical meaning. We can choose a zero reference point for the potential energy anywhere we like. We have some flexibility to adapt our choice of zero for the potential energy to best fit a particular problem. Because the change in potential energy only depended on the displacement, $\Delta y$. In the above expression for the change of potential energy (Eq. (14.4)), let $y_f = y$ be an arbitrary point and $y_i = 0$ denote the surface of the earth. Choose the zero reference potential for the potential energy to be at the surface of the earth corresponding to our origin $y = 0$, with $U^g(0) = 0$. Then

$$\Delta U^g = U^g(y) - U^g(0) = U^g(y) \, . \tag{14.6}$$

Substitute $y_i = 0$, $y_f = y$ and Eq. (14.6) into Eq. (14.4) yielding a potential energy as a function of the height $y$ above the surface of the earth,

$$U^g(y) = mgy, \text{ with } U^g(y=0) = 0. \tag{14.7}$$

### 14.4.2 Hooke's Law Spring-Object System

Consider a spring-object system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to an object of mass $m$ (Figure 14.7). The spring force is an internal conservative force. The wall exerts an external force on the spring-object system but since the point of contact of the wall with the spring undergoes no displacement, this external force does no work.

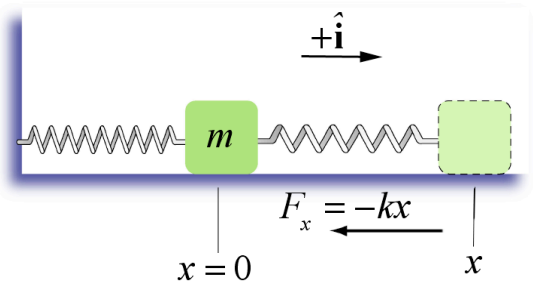


**Figure 14.7** A spring-object system.

Choose the origin at the position of the center of the object when the spring is relaxed (the equilibrium position). Let $x$ be the displacement of the object from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on a mass is then given by $\vec{\mathbf{F}}^s = F_x^s\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}$. The displacement is $d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}}$. The scalar product is $\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = -kx\,\hat{\mathbf{i}}\cdot dx\,\hat{\mathbf{i}} = -kx\,dx$. The work done by the spring force on the mass is

$$W^s = \int_{x=x_i}^{x=x_f} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{x=x_i}^{x=x_f} (-kx)\,dx = -\frac{1}{2}k(x_f^{\,2} - x_i^{\,2}). \tag{14.8}$$

We then define the change in potential energy in the spring-object system in moving the object from an initial position $x_i$ from equilibrium to a final position $x_f$ from equilibrium by

$$\Delta U^s \equiv U^s(x_f) - U^s(x_i) = -W^s = \frac{1}{2}k(x_f^2 - x_i^2). \tag{14.9}$$

Therefore an arbitrary stretch or compression of a spring-object system from equilibrium $x_i = 0$ to a final position $x_f = x$ changes the potential energy by

$$\Delta U^s = U^s(x_f) - U^s(0) = \frac{1}{2} k x^2 . \tag{14.10}$$

For the spring-object system, there is an obvious choice of position where the potential energy is zero, the equilibrium position of the spring- object,

$$U^s(0) \equiv 0 . \tag{14.11}$$

Then with this choice of zero reference potential, the potential energy as a function of the displacement $x$ from the equilibrium position is given by

$$U^s(x) = \frac{1}{2} k x^2, \text{ with } U^s(0) \equiv 0 . \tag{14.12}$$

### 14.4.3 Inverse Square Gravitation Force

Consider a system consisting of two objects of masses $m_1$ and $m_2$ that are separated by a center-to-center distance $r_{2,1}$. A coordinate system is shown in the Figure 14.8. The internal gravitational force on object 1 due to the interaction between the two objects is given by

$$\vec{\mathbf{F}}^G_{2,1} = -\frac{G m_1 m_2}{r^2_{2,1}} \hat{\mathbf{r}}_{2,1} . \tag{14.13}$$

The displacement vector is given by $d\vec{\mathbf{r}}_{2,1} = dr_{2,1}\, \hat{\mathbf{r}}_{2,1}$. So the scalar product is

$$\vec{\mathbf{F}}^G_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\frac{G m_1 m_2}{r^2_{2,1}} \hat{\mathbf{r}}_{2,1} \cdot dr_{2,1}\, \hat{\mathbf{r}}_{2,1} = -\frac{G m_1 m_2}{r^2_{2,1}} dr_{2,1} \quad . \tag{14.14}$$

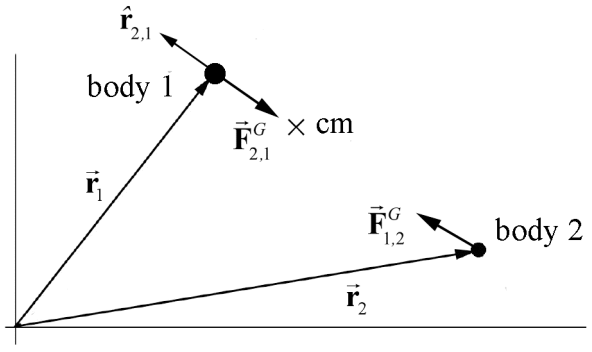


**Figure 14.8** Gravitational interaction

Using our definition of potential energy (Eq. (14.4)), we have that the change in the gravitational potential energy of the system in moving the two objects from an initial position in which the center of mass of the two objects are a distance $r_i$ apart to a final position in which the center of mass of the two objects are a distance $r_f$ apart is given by

$$\Delta U^G = -\int_A^B \vec{\mathbf{F}}^G_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\int_{r_i}^{f} -\frac{G m_1 m_2}{r_{2,1}^2} dr_{2,1} = -\frac{G m_1 m_2}{r_{2,1}}\bigg|_{r_i}^{r_f} = -\frac{G m_1 m_2}{r_f} + \frac{G m_1 m_2}{r_i} . \quad (14.15)$$

We now choose our reference point for the zero of the potential energy to be at infinity, $r_i = \infty$, with the choice that $U^G(\infty) \equiv 0$. By making this choice, the term $1/r$ in the expression for the change in potential energy vanishes when $r_i = \infty$. The gravitational potential energy as a function of the relative distance $r$ between the two objects is given by

$$U^G(r) = -\frac{G m_1 m_2}{r}, \text{ with } U^G(\infty) \equiv 0 . \quad (14.16)$$

## 14.5 Mechanical Energy and Conservation of Mechanical Energy

*The total change in the **mechanical energy** of the system is defined to be the sum of the changes of the kinetic and the potential energies,*

$$\Delta E_m = \Delta K_{\text{sys}} + \Delta U_{\text{sys}} . \quad (14.17)$$

For a closed system with only conservative internal forces, and no energy entering or leaving the system, the total change in the mechanical energy is zero,

$$\Delta E_m = \Delta K_{\text{sys}} + \Delta U_{\text{sys}} = 0 . \quad (14.18)$$

Equation (14.18) is the symbolic statement of what is called ***conservation of mechanical energy.*** Recall that the work done by a conservative force in going around a closed path is zero (Equation (14.16)), therefore both the changes in kinetic energy and potential energy are zero when the closed system with only conservative internal forces returns to its initial state. Throughout the process, the kinetic energy may change into internal potential energy but if the system returns to its initial state, the kinetic energy is completely recoverable. We shall refer to a closed system in which processes take place in which only conservative forces act as ***completely reversible processes***.

### 14.5.1 Change in Gravitational potential Energy Near Surface of the Earth

Let's consider the example of an object of mass $m_o$ falling near the surface of the earth (mass $m_e$). Choose our system to consist of the earth and the object. The gravitational

force is now an internal conservative force acting inside the system. The initial and final states are specified by the distance separating the object and the center of mass of the earth, and the velocities of the earth and the object. The change in kinetic energy between the initial and final states for the system is

$$\Delta K_{\text{sys}} = \Delta K_e + \Delta K_o , \tag{14.19}$$

$$\Delta K_{\text{sys}} = \left( \frac{1}{2} m_{\text{e}} (v_{e,f})^2 - \frac{1}{2} m_{\text{e}} (v_{\text{e,i}})^2 \right) + \left( \frac{1}{2} m_o (v_{o,f})^2 - \frac{1}{2} m_o (v_{o,i})^2 \right). \tag{14.20}$$

The change of kinetic energy of the earth due to the gravitational interaction between the earth and the object is negligible. The change in kinetic energy of the system is approximately equal to the change in kinetic energy of the object,

$$\Delta K_{\text{sys}} \cong \Delta K_o = \frac{1}{2} m_o (v_{o,f})^2 - \frac{1}{2} m_o (v_{o,i})^2 . \tag{14.21}$$

We now define the mechanical energy function for the system

$$E_m = K + U^g = \frac{1}{2} m_o (v_b)^2 + m_o g y, \text{ with } U^g(0) = 0 , \tag{14.22}$$

where $K$ is the kinetic energy and $U^g$ is the potential energy. The change in mechanical energy is then

$$\Delta E_m \equiv E_{m,f} - E_{m,i} = (K_f + U_f^g) - (K_i + U_i^g) . \tag{14.23}$$

When the work done by the external forces is zero and there are no internal non-conservative forces, the total mechanical energy of the system is constant,

$$E_{m,f} = E_{m,i} , \tag{14.24}$$

or equivalently

$$(K_f + U_f) = (K_i + U_i) . \tag{14.25}$$

## 14.6 Spring Force Energy Diagram

The spring force on an object is a restoring force $\vec{\mathbf{F}}^s = F_x^s \, \hat{\mathbf{i}} = -k x \, \hat{\mathbf{i}}$ where we choose a coordinate system with the equilibrium position at $x_i = 0$ and $x$ is the amount the spring has been stretched $(x > 0)$ or compressed $(x < 0)$ from its equilibrium position. We calculate the potential energy difference Eq. (14.9) and found that

$$U^s(x) - U^s(x_i) = -\int_{x_i}^{x} F_x^s \, dx = \frac{1}{2} k (x^2 - x_i^2) . \tag{14.1}$$

The first fundamental theorem of calculus states that

$$U(x)-U(x_i)=\int_{x'=x_i}^{x'=x}\frac{dU}{dx'}dx'. \tag{14.2}$$

Comparing Equation (14.1) with Equation (14.2) shows that the force is the negative derivative (with respect to position) of the potential energy,

$$F_x^s=-\frac{dU^s(x)}{dx}. \tag{14.3}$$

Choose the zero reference point for the potential energy to be at the equilibrium position, $U^s(0)\equiv 0$. Then the potential energy function becomes

$$U^s(x)=\frac{1}{2}k\,x^2. \tag{14.4}$$

From this, we obtain the spring force law as

$$F_x^s=-\frac{dU^s(x)}{dx}=-\frac{d}{dx}\left(\frac{1}{2}k\,x^2\right)=-k\,x. \tag{14.5}$$

In Figure 14.9 we plot the potential energy function $U^s(x)$ for the spring force as function of $x$ with $U^s(0)\equiv 0$ (the units are arbitrary).

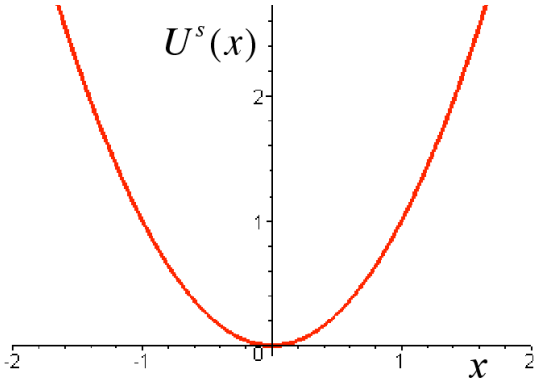


**Figure 14.9** Graph of potential energy function as function of $x$ for the spring.

The minimum of the potential energy function occurs at the point where the first derivative vanishes

$$\frac{dU^s(x)}{dx}=0. \tag{14.6}$$

From Equation (14.4), the minimum occurs at $x = 0$,

$$0 = \frac{dU^s(x)}{dx} = k\,x\,. \tag{14.7}$$

Because the force is the negative derivative of the potential energy, and this derivative vanishes at the minimum, we have that the spring force is zero at the minimum $x = 0$ agreeing with our force law, $F_x^s\Big|_{x=0} = -k\,x\Big|_{x=0} = 0$.

The potential energy function has positive curvature in the neighborhood of a minimum equilibrium point. If the object is extended a small distance $x > 0$ away from equilibrium, the slope of the potential energy function is positive, $dU(x)/dx > 0$, hence the component of the force is negative because $F_x = -dU(x)/dx < 0$. Thus the object experiences a restoring force towards the minimum point of the potential. If the object is compresses with $x < 0$ then $dU(x)/dx < 0$, hence the component of the force is positive, $F_x = -dU(x)/dx > 0$, and the object again experiences a restoring force back towards the minimum of the potential energy as in Figure 14.10.

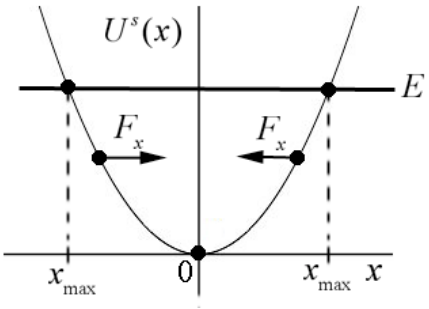


**Figure 14.10** Stability diagram for the spring force.

The mechanical energy at any time is the sum of the kinetic energy $K(x)$ and the potential energy $U^s(x)$

$$E_m = K(x) + U^s(x)\,. \tag{14.8}$$

Suppose our spring-object system has no loss of mechanical energy due to dissipative forces such as friction or air resistance. Both the kinetic energy and the potential energy are functions of the position of the object with respect to equilibrium. The energy is a constant of the motion and with our choice of $U^s(0) \equiv 0$, the energy can be either a positive value or zero. When the energy is zero, the object is at rest at the equilibrium position.

In Figure 14.10, we draw a straight horizontal line corresponding to a non-zero positive value for the energy $E_m$ on the graph of potential energy as a function of $x$. The energy intersects the potential energy function at two points $\{-x_{\max}, x_{\max}\}$ with $x_{\max} > 0$. These points correspond to the maximum compression and maximum extension of the spring, which are called the ***turning points***. The kinetic energy is the difference between the energy and the potential energy,

$$K(x) = E_m - U^s(x). \tag{14.9}$$

At the turning points, where $E_m = U^s(x)$, the kinetic energy is zero. Regions where the kinetic energy is negative, $x < -x_{\max}$ or $x > x_{\max}$ are called the ***classically forbidden regions***, which the object can never reach if subject to the laws of classical mechanics. In quantum mechanics, with similar energy diagrams for quantum systems, there is a very small probability that the quantum object can be found in a classically forbidden region.

**Example 14.1 Energy Diagram**

The potential energy function for a particle of mass $m$, moving in the $x$-direction is given by

$$U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right), \tag{14.10}$$

where $U_1$ and $x_1$ are positive constants and $U(0) = 0$. (a) Sketch $U(x)/U_1$ as a function of $x/x_1$. (b) Find the points where the force on the particle is zero. Classify them as stable or unstable. Calculate the value of $U(x)/U_1$ at these equilibrium points. (c) For energies $E$ that lies in $0 < E < (4/27)U_1$ find an equation whose solution yields the turning points along the x-axis about which the particle will undergo periodic motion. (d) Suppose $E = (4/27)U_1$ and that the particle starts at $x = 0$ with speed $v_0$. Find $v_0$.

**Solution:** a) Figure 14.11 shows a graph of $U(x)$ vs. $x$, with the choice of values $x_1 = 1.5\ \text{m}$, $U_1 = 27/4\ \text{J}$, and $E = 0.2\ \text{J}$.

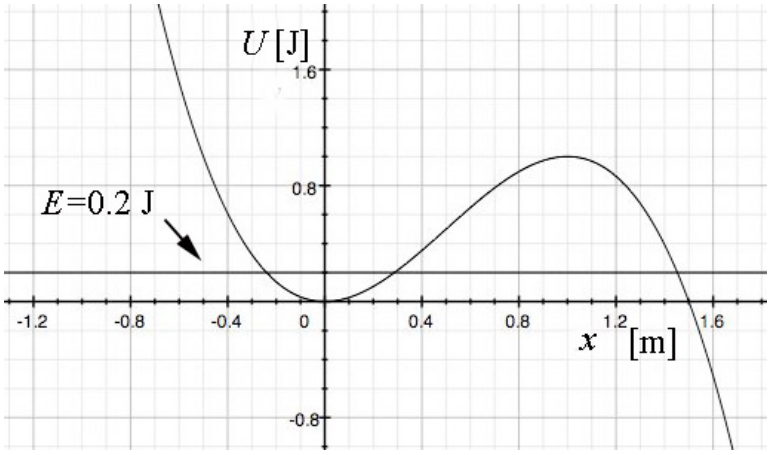


**Figure 14.11** Energy diagram for Example 14.1

b) The force on the particle is zero at the minimum of the potential which occurs at

$$F_x(x) = -\frac{dU}{dx}(x) = U_1\left(\left(\frac{3}{x_1^{\,3}}\right)x^2 - \left(\frac{2}{x_1^{\,2}}\right)x\right) = 0 \qquad (14.11)$$

which becomes

$$x^2 = (2x_1/3)x\,. \qquad (14.12)$$

We can solve Eq. (14.12) for the extrema. This has two solutions

$$x = (2x_1/3) \quad \text{and} \quad x = 0\,. \qquad (14.13)$$

The second derivative is given by

$$\frac{d^2U}{dx^2}(x) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)x - \left(\frac{2}{x_1^{\,2}}\right)\right). \qquad (14.14)$$

Evaluating the second derivative at $x = (2x_1/3)$ yields a negative quantity

$$\frac{d^2U}{dx^2}(x = (2x_1/3)) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)\frac{2x_1}{3} - \left(\frac{2}{x_1^{\,2}}\right)\right) = -\frac{2U_1}{x_1^{\,2}} < 0\,, \qquad (14.15)$$

indicating the solution $x = (2x_1/3)$ represents a local maximum and hence is an unstable point. At $x = (2x_1/3)$, the potential energy is given by the value $U((2x_1/3)) = (4/27)U_1$. Evaluating the second derivative at $x = 0$ yields a positive quantity

$$\frac{d^2U}{dx^2}(x=0) = -U_1\left(\left(\frac{6}{x_1^3}\right)0 - \left(\frac{2}{x_1^2}\right)\right) = \frac{2U_1}{x_1^2} > 0\,, \tag{14.16}$$

indicating the solution $x = 0$ represents a local minimum and is a stable point. At the local minimum $x = 0$, the potential energy $U(0) = 0$.

c) Consider a fixed value of the energy of the particle within the range

$$U(0) = 0 < E < U(2x_1/3) = \frac{4U_1}{27}\ . \tag{14.17}$$

If the particle at any time is found in the region $x_a < x < x_b < 2x_1/3$, where $x_a$ and $x_b$ are the turning points and are solutions to the equation

$$E = U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right)\ . \tag{14.18}$$

then the particle will undergo periodic motion between the values $x_a < x < x_b$ . Within this region $x_a < x < x_b$ , the kinetic energy is always positive because $K(x) = E - U(x)$. There is another solution $x_c$ to Eq. (14.18) somewhere in the region $x_c > 2x_1/3$. If the particle at any time is in the region $x > x_c$ then it at any later time it is restricted to the region $x_c < x < +\infty$.

For $E > U(2x_1/3) = (4/27)U_1$, Eq. (14.18) has only one solution $x_d$. For all values of $x > x_d$, the kinetic energy is positive, which means that the particle can "escape" to infinity but can never enter the region $x < x_d$.

For $E < U(0) = 0$ , the kinetic energy is negative for the range $-\infty < x < x_e$ where $x_e$ satisfies Eq. (14.18) and therefore this region of space is forbidden.

(d) If the particle has speed $v_0$ at $x = 0$ where the potential energy is zero, $U(0) = 0$, the energy of the particle is constant and equal to kinetic energy

$$E = K(0) = \frac{1}{2}mv_0^2\,. \tag{14.19}$$

Therefore

$$(4/27)U_1 = \frac{1}{2}mv_0^2\,, \tag{14.20}$$

which we can solve for the speed

$$v_0 = \sqrt{8U_1 / 27m} \, . \tag{14.21}$$

## 14.7 Change of Mechanical Energy for Closed System with Internal Non-conservative Forces

Consider a closed system that undergoes a transformation from an initial state to a final state by a prescribed set of changes.

> *Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, the force is called a* ***non-conservative force****.*

Suppose the internal forces are both conservative and non-conservative. The work $W$ done by the forces is a sum of the conservative work $W_c$, which is path-independent, and the non-conservative work $W_{nc}$, which is path-dependent,

$$W = W_c + W_{nc} \, . \tag{14.1}$$

The work done by the conservative forces is equal to the negative of the change in the potential energy

$$\Delta U = -W_c \, . \tag{14.2}$$

Substituting Equation (14.2) into Equation (14.1) yields

$$W = -\Delta U + W_{nc} \, . \tag{14.3}$$

The work done is equal to the change in the kinetic energy,

$$W = \Delta K \, . \tag{14.4}$$

Substituting Equation (14.4) into Equation (14.3) yields

$$\Delta K = -\Delta U + W_{nc} \, . \tag{14.5}$$

which we can rearrange as

$$W_{nc} = \Delta K + \Delta U \, . \tag{14.6}$$

We can now substitute Equation (14.4) into our expression for the change in the mechanical energy, Equation (14.17), with the result

$$W_{nc} = \Delta E_m \, . \tag{14.7}$$

The mechanical energy is no longer constant. The total change in energy of the system is zero,

$$\Delta E_{\text{system}} = \Delta E_m - W_{\text{nc}} = 0 . \tag{14.8}$$

Energy is conserved but some mechanical energy has been transferred into non-recoverable energy $W_{\text{nc}}$. We shall refer to processes in which there is non-zero non-recoverable energy as ***irreversible processes***.

### 14.7.1 Change of Mechanical Energy for a Closed System

When energy enters or leaves the closed system, the change in energy of the system is equal to the negative of the change in energy of the surroundings (Eq. (14.1)),

$$\Delta E_{\text{system}} = -\Delta E_{\text{surroundings}} \tag{14.9}$$

The change in energy of the system can be the result of work done by the surroundings on the system (which can be positive or negative), which we shall refer to as *external work* $W_{\text{ext}}$. This work will result in the system undergoing ***coherent motion***. Note that $W_{\text{ext}} > 0$ if work is done on the system ( $\Delta E_{\text{surroundings}} < 0$ ) and $W_{\text{ext}} < 0$ if the system does work on the surroundings ( $\Delta E_{\text{surroundings}} > 0$ ). If the system is in thermal contact with the surroundings, then thermal energy can flow into or out of the system. This energy flow due to thermal contact is often denoted by $Q$ with the convention that $Q > 0$ if the energy flows into the system ( $\Delta E_{\text{surroundings}} < 0$ ) and $Q < 0$ if the energy flows out of the system ( $\Delta E_{\text{surroundings}} > 0$ ). Then Eq. (14.9) can be rewritten as

$$W^{\text{ext}} + Q = \Delta E_{\text{sys}} \tag{14.10}$$

Equation (14.10) is also called ***the first law of thermodynamics.***

This will result in either an increase or decrease in random thermal motion of the molecules inside the system,

There may also be other forms of energy that enter the system, for example *radiative energy*. Several questions naturally arise from this set of definitions and physical concepts. Is it possible to identify all the conservative forces and calculate the associated changes in potential energies? How do we account for non-conservative forces such as friction that act at the boundary of the system?

## 14.8 Dissipative Forces: Friction

Suppose we consider an object moving on a rough surface. As the object slides it slows down and stops. While the sliding occurs both the object and the surface increase in temperature. The increase in temperature is due to the molecules inside the materials increasing their kinetic energy. This random kinetic energy is called *thermal energy*. Kinetic energy associated with the coherent motion of the molecules of the object has been dissipated into kinetic energy associated with random motion of the molecules composing the object and surface.

If we define the system to be just the object, then the friction force acts as an external force on the system and results in the dissipation of energy into both the block and the surface. Without knowing further properties of the material we cannot determine the exact changes in the energy of the system.

Friction introduces a problem in that the point of contact is not well defined because the surface of contact is constantly deforming as the object moves along the surface. If we considered the object and the surface as the system, then the friction force is an internal force, and the decrease in the kinetic energy of the moving object ends up as an increase in the internal random kinetic energy of the constituent parts of the system. When there is dissipation at the boundary of the system, we need an additional model (thermal equation of state) for how the dissipated energy distributes itself among the constituent parts of the system.

### 14.8.1 Source Energy

Consider a person walking. The frictional force between the person and the ground does no work because the point of contact between the person's foot and the ground undergoes no displacement as the person applies a force against the ground, (there may be some slippage but that would be opposite the direction of motion of the person). However the kinetic energy of the object increases. Have we disproved the work-energy theorem? The answer is no! The chemical energy stored in the body tissue is converted to kinetic energy and thermal energy. Because the person-air-ground can be treated as a closed system with no energy entering or leaving, we have that

$$0 = \Delta E_{\text{sys}} = \Delta E_{\text{chemical}} + \Delta E_{\text{thermal}} + \Delta E_{\text{mechanical}} . \tag{14.1}$$

If we assume that there is no change in the potential energy of the system, then $\Delta E_{\text{mechanical}} = \Delta K$. Therefore some of the internal chemical energy has been transformed into thermal energy and the rest has changed into the kinetic energy of the system,

$$-\Delta E_{\text{chemical}} = \Delta E_{\text{thermal}} + \Delta K . \tag{14.2}$$

## 14.9 Worked Examples

### Example 14.2 Escape Velocity of Toro

The asteroid Toro, discovered in 1964, has a radius of about $R = 5.0\,\text{km}$ and a mass of about $m_t = 2.0 \times 10^{15}\,\text{kg}$. Let's assume that Toro is a perfectly uniform sphere. What is the escape velocity for an object of mass $m$ on the surface of Toro? Could a person reach this speed (on earth) by running?

**Solution:** The only potential energy in this problem is the gravitational potential energy. We choose the zero point for the potential energy to be when the object and Toro are an infinite distance apart, $U^G(\infty) \equiv 0$. With this choice, the potential energy when the object and Toro are a finite distance $r$ apart is given by

$$U^G(r) = -\frac{Gm_t\, m}{r} \tag{14.1}$$

with $U^G(\infty) \equiv 0$. The expression ***escape velocity*** refers to the minimum speed necessary for an object to escape the gravitational interaction of the asteroid and move off to an infinite distance away. If the object has a speed less than the escape velocity, it will be unable to escape the gravitational force and must return to Toro. If the object has a speed greater than the escape velocity, it will have a non-zero kinetic energy at infinity. The condition for the escape velocity is that the object will have exactly zero kinetic energy at infinity.

We choose our initial state, at time $t_i$, when the object is at the surface of the asteroid with speed equal to the escape velocity. We choose our final state, at time $t_f$, to occur when the separation distance between the asteroid and the object is infinite.

The initial kinetic energy is $K_i = (1/2)m{v_{\text{esc}}}^2$. The initial potential energy is $U_i = -Gm_t\, m / R$, and so the initial mechanical energy is

$$E_i = K_i + U_i = \frac{1}{2}m v_{\text{esc}}^2 - \frac{Gm_t\, m}{R}. \tag{14.2}$$

The final kinetic energy is $K_f = 0$, because this is the condition that defines the escape velocity. The final potential energy is zero, $U_f = 0$ because we chose the zero point for potential energy at infinity. The final mechanical energy is then

$$E_f = K_f + U_f = 0. \tag{14.3}$$

There is no non-conservative work, so the change in mechanical energy is zero

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i . \tag{14.4}$$

Therefore

$$0 = -\left( \frac{1}{2} m v_{\text{esc}}^2 - \frac{G m_t \, m}{R} \right). \tag{14.5}$$

This can be solved for the escape velocity,

$$\begin{aligned} v_{\text{esc}} &= \sqrt{\frac{2Gm_t}{R}} \\ &= \sqrt{\frac{2(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2})(2.0\times10^{15}\,\text{kg})}{(5.0\times10^{3}\,\text{m})}} = 7.3\ \text{m}\cdot\text{s}^{-1}. \end{aligned} \tag{14.6}$$

Considering that Olympic sprinters typically reach velocities of 12 $\text{m}\cdot\text{s}^{-1}$, this is an easy speed to attain by running on earth. It may be harder on Toro to generate the acceleration necessary to reach this speed by pushing off the ground, since any slight upward force will raise the runner's center of mass and it will take substantially more time than on earth to come back down for another push off the ground.

**Example 14.3 Spring-Block-Loop-the-Loop**

A small block of mass $m$ is pushed against a spring with spring constant $k$ and held in place with a catch. The spring is compressed an unknown distance $x$ (Figure 14.12). When the catch is removed, the block leaves the spring and slides along a frictionless circular loop of radius $r$. When the block reaches the top of the loop, the force of the loop on the block (the normal force) is equal to twice the gravitational force on the mass. (a) Using conservation of energy, find the kinetic energy of the block at the top of the loop. (b) Using Newton's Second Law, derive the equation of motion for the block when it is at the top of the loop. Specifically, find the speed $v_{\text{top}}$ in terms of the gravitation constant $g$ and the loop radius $r$. (c) What distance was the spring compressed?

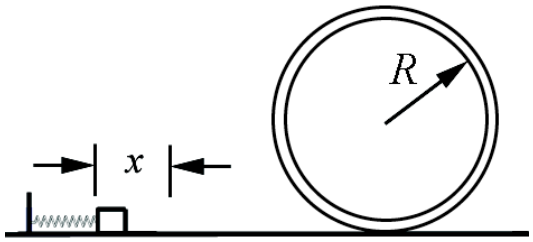


**Figure 14.12** Initial state for spring-block-loop-the-loop system

**Solution:** a) Choose for the initial state the instant before the catch is released. The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = (1/2)k x^2$**.** The initial mechanical energy is then

$$E_i = K_i + U_i = \frac{1}{2} k x^2 . \tag{14.7}$$

Choose for the final state the instant the block is at the top of the loop. The final kinetic energy is $K_f = (1/2)m v_{\text{top}}^2$; the block is in motion with speed $v_{\text{top}}$. The final potential energy is non-zero, $U_f = (mg)(2R)$. The final mechanical energy is then

$$E_f = K_f + U_f = 2mgR + \frac{1}{2} m v_{\text{top}}^2 . \tag{14.8}$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i . \tag{14.9}$$

Mechanical energy is conserved, $E_f = E_i$, therefore

$$2mgR + \frac{1}{2} m v_{\text{top}}^2 = \frac{1}{2} k x^2 . \tag{14.10}$$

From Equation (14.10), the kinetic energy at the top of the loop is

$$\frac{1}{2} m v_{\text{top}}^2 = \frac{1}{2} k x^2 - 2mgR . \tag{14.11}$$

b) At the top of the loop, the forces on the block are the gravitational force of magnitude $mg$ and the normal force of magnitude $N$, both directed down. Newton's Second Law in the radial direction, which is the downward direction, is

$$-mg - N = -\frac{m v_{\text{top}}^2}{R} . \tag{14.12}$$

In this problem, we are given that when the block reaches the top of the loop, the force of the loop on the block (the normal force, *downward* in this case) is equal to twice the weight of the block, $N = 2mg$**.** The Second Law, Eq. (14.12), then becomes

$$3mg = \frac{m v_{\text{top}}^2}{R} . \tag{14.13}$$

We can rewrite Equation (14.13) in terms of the kinetic energy as

$$\frac{3}{2}mg\,R = \frac{1}{2}m v_{\text{top}}^2 . \tag{14.14}$$

The speed at the top is therefore

$$v_{\text{top}} = \sqrt{3mg\,R} . \tag{14.15}$$

c) Combing Equations (14.11) and (14.14) yields

$$\frac{7}{2}mg\,R = \frac{1}{2}k\,x^2 . \tag{14.16}$$

Thus the initial displacement of the spring from equilibrium is

$$x = \sqrt{\frac{7mg\,R}{k}} . \tag{14.17}$$

**Example 14.4 Mass-Spring on a Rough Surface**

A block of mass $m$ slides along a horizontal table with speed $v_0$. At $x = 0$ it hits a spring with spring constant $k$ and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$, where $b$ is a positive constant. Find the loss in mechanical energy when the block first momentarily comes to rest.

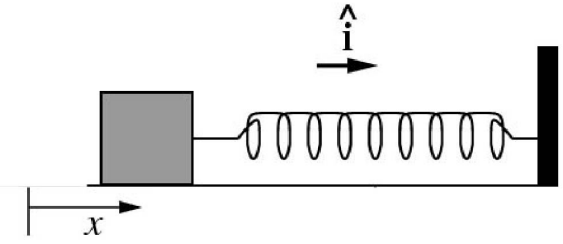


**Figure 14.13** Spring-block system

**Solution:** From the model given for the frictional force, we could find the non-conservative work done, which is the same as the loss of mechanical energy, if we knew the position $x_f$ where the block first comes to rest. The most direct (and easiest) way to find $x_f$ is to use the work-energy theorem. The initial mechanical energy is $E_i = mv_i^2 / 2$ and the final mechanical energy is $E_f = k\,x_f^2 / 2$ (note that there is no potential energy

term in $E_i$ and no kinetic energy term in $E_f$). The difference between these two mechanical energies is the non-conservative work done by the frictional force,

$$\begin{aligned} W_{\text{nc}} &= \int_{x=0}^{x=x_f} F_{\text{nc}}\, dx = \int_{x=0}^{x=x_f} -F_{\text{friction}}\, dx = \int_{x=0}^{x=x_f} -\mu\, N\, dx \\ &= -\int_0^{x_f} b\, x\, mg\, dx = -\frac{1}{2} bmg\, x_f^2. \end{aligned} \tag{14.18}$$

We then have that

$$\begin{aligned} W_{\text{nc}} &= \Delta E_m \\ W_{\text{nc}} &= E_f - E_i \\ -\frac{1}{2} bmg\, x_f^2 &= \frac{1}{2} k\, x_f^2 - \frac{1}{2} m v_i^2. \end{aligned} \tag{14.19}$$

Solving the last of these equations for $x_f^2$ yields

$$x_f^2 = \frac{m v_0^2}{k + bmg}. \tag{14.20}$$

Substitute Eq. (14.20) into Eq. (14.18) gives the result that

$$W_{\text{nc}} = -\frac{bmg}{2} \frac{m v_0^2}{k + bmg} = -\frac{m v_0^2}{2} \left( 1 + \frac{k}{bmg} \right)^{-1}. \tag{14.21}$$

It is worth checking that the above result is dimensionally correct. From the model, the parameter $b$ must have dimensions of inverse length (the coefficient of friction $\mu$ must be dimensionless), and so the product $bmg$ has dimensions of force per length, as does the spring constant $k$; the result is dimensionally consistent.

**Example 14.5 Cart-Spring on an Inclined Plane**

An object of mass $m$ slides down a plane that is inclined at an angle $\theta$ from the horizontal (Figure 14.14). The object starts out at rest. The center of mass of the cart is a distance $d$ from an unstretched spring that lies at the bottom of the plane. Assume the spring is massless, and has a spring constant $k$. Assume the inclined plane to be frictionless. (a) How far will the spring compress when the mass first comes to rest? (b) Now assume that the inclined plane has a coefficient of kinetic friction $\mu_{\text{k}}$. How far will the spring compress when the mass first comes to rest? The friction is primarily between the wheels and the bearings, not between the cart and the plane, but the friction force may be modeled by a coefficient of friction $\mu_{\text{k}}$. (c) In case (b), how much energy has been lost to friction?

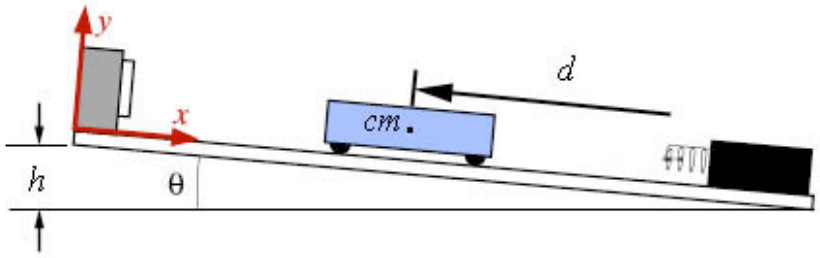


**Figure 14.14** Cart on inclined plane

**Solution:** Let $x$ denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U^g(0)=0$ not at the very bottom of the inclined plane, but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy where the spring is at its equilibrium position, $U^s(0)=0$.

a) Choose for the initial state the instant the object is released (Figure 14.15). The initial kinetic energy is $K_i=0$. The initial potential energy is non-zero, $U_i=mg\,d\sin\theta$. The initial mechanical energy is then

$$E_i=K_i+U_i=mg\,d\sin\theta \tag{14.22}$$

Choose for the final state the instant when the object first comes to rest and the spring is compressed a distance $x$ at the bottom of the inclined plane (Figure 14.16). The final kinetic energy is $K_f=0$ since the mass is not in motion. The final potential energy is non-zero, $U_f=k\,x^2/2-x\,mg\sin\theta$. Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.

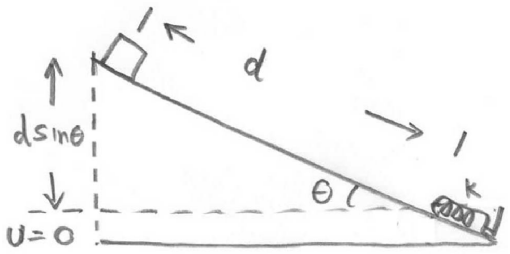


**Figure 14.15** Initial state

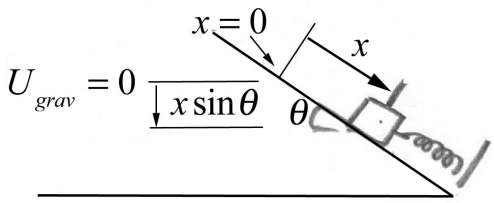


**Figure 14.16** Final state

The final mechanical energy is then

$$E_f=K_f+U_f=\frac{1}{2}k\,x^2-x\,mg\sin\theta\,. \tag{14.23}$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i . \tag{14.24}$$

Therefore

$$d\,mg\sin\theta = \frac{1}{2}k\,x^2 - x\,mg\sin\theta . \tag{14.25}$$

This is a quadratic equation in $x$,

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2d\,mg\sin\theta}{k} = 0 . \tag{14.26}$$

In the quadratic formula, we want the positive choice of square root for the solution to ensure a positive displacement of the spring from equilibrium,

$$\begin{aligned} x &= \frac{mg\sin\theta}{k} + \left( \frac{m^2 g^2 \sin^2\theta}{k^2} + \frac{2d\,mg\sin\theta}{k} \right)^{1/2} \\ &= \frac{mg}{k}(\sin\theta + \sqrt{1 + 2(k\,d\,/\,mg)\sin\theta}). \end{aligned} \tag{14.27}$$

(What would the solution with the negative root represent?)

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object, which has moved a distance, $d + x$, given by

$$W_{\text{nc}} = -f_{\text{k}}(d+x) = -\mu_{\text{k}} N(d+x) = -\mu_{\text{k}} mg\cos\theta(d+x) . \tag{14.28}$$

Note the normal force is found by using Newton's Second Law in the perpendicular direction to the inclined plane,

$$N - mg\cos\theta = 0 . \tag{14.29}$$

The change in mechanical energy is therefore

$$W_{\text{nc}} = \Delta E_m = E_f - E_i , \tag{14.30}$$

which becomes

$$-\mu_{\text{k}} mg\cos\theta(d+x) = \left( \frac{1}{2}k\,x^2 - x\,mg\sin\theta \right) - d\,mg\sin\theta . \tag{14.31}$$

Equation (14.31) simplifies to

$$0 = \left( \frac{1}{2} k x^2 - x\, mg(\sin\theta - \mu_k \cos\theta) \right) - d\, mg(\sin\theta - \mu_k \cos\theta) . \qquad (14.32)$$

This is the same as Equation (14.25) above, but with $\sin\theta \to \sin\theta - \mu_k \cos\theta$. The maximum displacement of the spring is when there is friction is then

$$x = \frac{mg}{k}((\sin\theta - \mu_k \cos\theta) + \sqrt{1 + 2(k\,d / mg)(\sin\theta - \mu_k \cos\theta)}) . \qquad (14.33)$$

.

c) The energy lost to friction is given by $W_{nc} = -\mu_k mg \cos\theta (d + x)$, where $x$ is given in part b).

**Example 14.6 Object Sliding on a Sphere**

A small point like object of mass $m$ rests on top of a sphere of radius $R$. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide (Figure 14.17). Let $g$ denote the gravitation constant. (a) Determine the angle $\theta_1$ with respect to the vertical at which the object will lose contact with the surface of the sphere. (b) What is the speed $v_1$ of the object at the instant it loses contact with the surface of the sphere.

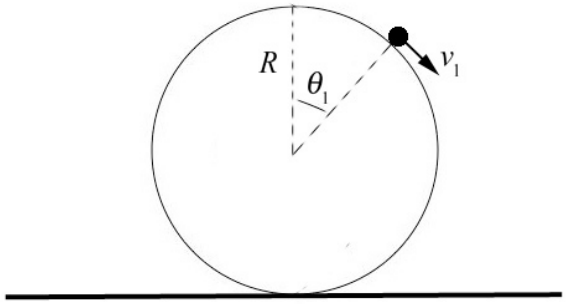


**Figure 14.17** Object sliding on surface of sphere

**Solution:** We begin by identifying the forces acting on the object. There are two forces acting on the object, the gravitation and radial normal force that the sphere exerts on the particle that we denote by $N$. We draw a free-body force diagram for the object while it is sliding on the sphere. We choose polar coordinates as shown in Figure 14.18.

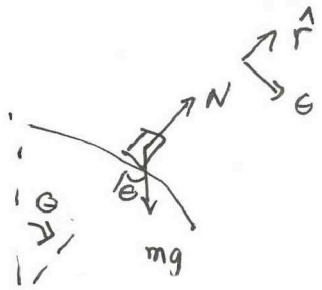


**Figure 14.18** Free-body force diagram on object

The key constraint is that when the particle just leaves the surface the normal force is zero,

$$N(\theta_1) = 0\,, \tag{14.34}$$

where $\theta_1$ denotes the angle with respect to the vertical at which the object will just lose contact with the surface of the sphere. Because the normal force is perpendicular to the displacement of the object, it does no work on the object and hence conservation of energy does not take into account the constraint on the motion imposed by the normal force. In order to analyze the effect of the normal force we must use the radial component of Newton's Second Law,

$$N - mg\cos\theta = -m\frac{v^2}{R}\,. \tag{14.35}$$

Then when the object just loses contact with the surface, Eqs. (14.34) and (14.35) require that

$$mg\cos\theta_1 = m\frac{v_1^{\,2}}{R}\,. \tag{14.36}$$

where $v_1$ denotes the speed of the object at the instant it loses contact with the surface of the sphere. Note that the constrain condition Eq. (14.36) can be rewritten as

$$mgR\cos\theta_1 = mv_1^{\,2}\,. \tag{14.37}$$

We can now apply conservation of energy. Choose the zero reference point $U = 0$ for potential energy to be the midpoint of the sphere.

Identify the initial state as the instant the object is released (Figure 14.19). We can neglect the very small initial kinetic energy needed to move the object away from the top of the sphere and so $K_i = 0$. The initial potential energy is non-zero, $U_i = mgR$. The initial mechanical energy is then

$$E_i = K_i + U_i = mgR\,. \tag{14.38}$$

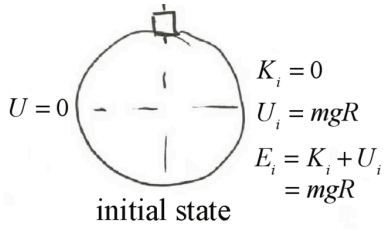


**Figure 14.19** Initial state

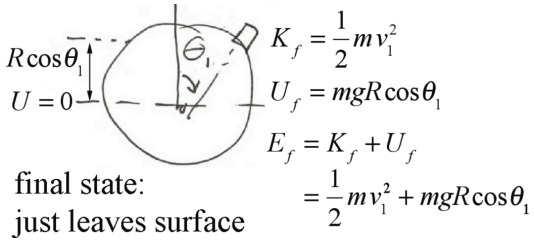


**Figure 14.20** Final state

Choose for the final state the instant the object leaves the sphere (Figure 14.20). The final kinetic energy is $K_f = mv_1^2/2$; the object is in motion with speed $v_1$. The final potential energy is non-zero, $U_f = mgR\cos\theta_1$. The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}mv_1^2 + mgR\cos\theta_1 . \tag{14.39}$$

Because we are assuming the contact surface is frictionless and neglecting air resistance, there is no non-conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i . \tag{14.40}$$

Therefore

$$\frac{1}{2}mv_1^2 + mgR\cos\theta_1 = mgR . \tag{14.41}$$

We now solve the constraint condition Eq. (14.37) into Eq. (14.41) yielding

$$\frac{1}{2}mgR\cos\theta_1 + mgR\cos\theta_1 = mgR . \tag{14.42}$$

We can now solve for the angle at which the object just leaves the surface

$$\theta_1 = \cos^{-1}(2/3) . \tag{14.43}$$

We now substitute this result into Eq. (14.37) and solve for the speed

$$v_1 = \sqrt{2gR/3} . \tag{14.44}$$

# Chapter 14A Thermal Energy

## 14A.1 Thermal Energy, Heat and Temperature

On a cold winter day, suppose you want to warm up by drinking a cup of tea. You start by filling up a kettle with water from the cold water tap (water heaters tend to add unpleasant contaminants and reduce the oxygen level in the water). You place the kettle on the heating element of the stove and allow the water to boil briefly. You let the water cool down slightly to avoid burning the tea leaves or creating bitter flavors and then pour the water into a pre-heated teapot containing a few teaspoons of tea; the tea leaves steep for a few minutes and then you enjoy your drink.

When the kettle is in contact with the heating element of the stove, energy flows from the heating element to the kettle and then to the water. The conduction of energy is due to the contact between the objects. The random motions of the atoms in the heating element are transferred to the kettle and water via collisions. We shall refer to this conduction process as '*energy transferred thermally*'. The term *heat refers to* energy transformed thermally has traditional been called *heat.* The energy associated with the random motions of the water molecules (and also the potential energy associated with the vibrational interactions intrinsic to water molecules and between molecules) is called *thermal energy*.

The thermal unit for heat is the *calorie* and is defined to be the amount of heat required to raise the temperature of one gram of water from $14.5\,^0C$ to $15.5\,^0C$ (where we have yet to properly define the Celsius, a unit of temperature.) Another common unit is the Btu (British Thermal Unit), which is the amount of heat necessary to raise one pound of water from $63^\circ\text{F}$ to $64^\circ\text{F}$. Note that $1\,\text{Btu} = 252\,\text{cal}$.

We can attribute different degrees of "hotness" (based on our experience of inadvertently touching the kettle and the water). Temperature is a measure of the "hotness" of a body. When two isolated objects that are initially at different temperatures are put in contact, the "colder" object heats up while the "hotter" object cools down, until they reach the same temperature, a state we refer to as *thermal equilibrium*. Temperature is that property of a system that determines whether or not a system is in thermal equilibrium with other systems.

### 14A.1.1 Internal Energy

More generally, the *internal energy* $U$ of a physical system is defined to be the sum of all contributions to the total energy of the system in a reference frame in which the center of mass of the system is at rest. For example the internal energy of a gas consist of the kinetic energy of the gas molecules, arising from the center-of-mass motions of the molecules relative to a container that is at rest in the reference frame, and kinetic energy of rotational motion of the molecules. These two motions have no potential energies associated to them. At sufficiently high temperatures, diatomic and polyatomic atoms also have vibrational motions due to interatomic forces, which like a spring have both kinetic and potential energies. Intermolecular forces contribute to the internal energy for solids and liquids, but make negligibly small contributions for gases.

The internal energy also includes contributions due to the rest–mass energy of the constituents, and atomic and nuclear binding energies associated with the structure of the constituents. The internal energy includes does not include potential energies that are due to external interactions, for example the gravitational potential energy due to the interaction between the system and an external body such as Earth. **Thermal energy is the sum of all the internal energies except the binding energies and rest energies.**

### 14A.1.2 Internal Energy of a Solid or Liquid:

Generally, the potential energy of the intermolecular interaction between molecules is repulsive for small $r$ and attractive for large $r$, where $r$ is the separation between molecules. At low temperatures, when the average kinetic energy is small, the molecules can form bound states with negative energy $E_{\text{internal}} < 0$ and condense into liquids or solids. The intermolecular forces act like restoring forces about an equilibrium distance between atoms, a distance at which the potential energy is a minimum. For energies near the potential minimum, the atoms vibrate like springs. For larger (but still negative) energies, the atoms still vibrate but no longer like springs and with larger amplitudes, undergoing thermal expansion. At higher temperatures, due to larger average kinetic energies, the internal energy becomes positive, $E_{\text{internal}} > 0$. In this case, molecules have enough energy to escape intermolecular forces and become a gas.

## 14A.2 Zeroth Law of Thermodynamics

**Temperature is a measure of the thermal energy of a system.** At absolute zero temperature, the thermal energy of a gas is zero even though the internal energy is still a positive constant due the binding energies and rest energies.

Consider two systems A and B that are separated from each other by an adiabatic boundary (*adiabatic = no heat passes through*) that does not allow any thermal contact. Both A and B are placed in thermal contact with a third system C until thermal equilibrium is reached. If the adiabatic boundary is then removed between A and B, no energy will transfer thermally between A and B. Thus

> *Zeroth Law of Thermodynamics*: Two systems in thermal equilibrium with a third system are in thermal equilibrium with each other.

Temperature $T$ is that property of a system that determines whether or not a system is in thermal equilibrium with other systems.

## 14A.3 Gas

We begin our analysis of energy transformations by considering a vessel containing a *gas*; a system consisting of a very large number of particles (typically $10^{24}$ or many orders of magnitude more) occupying a volume of space that is very large compared to

the size ($10^{-10}$ m) of any typical atom or molecule. The state of the gas can be described by a few macroscopically measurable quantities that completely determine the system. The volume of the gas in a container can be measured by the size the container. The pressure of a gas can be measured using a pressure gauge. The temperature can be measured with a thermometer. The mass, or number of moles or number of molecules, is a measure of the quantity of matter.

### 14A.3.1 Macroscopic vs. Atomistic Description of a Gas

How can we use the laws of mechanics that describe the motions and interactions of individual atomic particles to predict macroscopic properties of the system such as pressure, volume, and temperature? In principle, each point-like atomic particle can be specified by its position and velocity (neglecting any internal structure). We cannot know exactly where and with what velocities all the particles are moving so we must take averages. In addition, we need quantum mechanical laws to describe how particles interact. In fact, the inability of classical mechanics to predict how the heat capacity of a gas varies with temperature was the first experimental suggestion that a new set of principles (quantum mechanics) operates at the scale of the size of atoms. However, as a starting point we shall make some simplifying assumptions about the properties of a gas, a model which we shall refer to as an *ideal gas*.

### 14A.3.2 Ideal Gas

Consider a gas consisting of a large number of molecules inside a rigid container. We shall assume that the volume occupied by the molecules is small compared to the volume occupied by the gas, that is, the volume of the container (dilute gas assumption). We also assume that the molecules move randomly and satisfy Newton's Laws of Motion. The gas molecules collide with each other and the walls of the container. We shall assume that all the collisions are instantaneous and any energy converted to potential energy during the collision is recoverable as kinetic energy after the collision is finished. Thus the collisions are elastic and have the effect of altering the direction of the velocities of the molecules but not their speeds. We also assume that the intermolecular interactions contribute negligibly to the internal energy.

An ideal monatomic gas atom has no internal structure, so we treat it as point particle. Therefore there are no possible rotational degrees of freedom or internal degrees of freedom; the ideal gas has only three degrees of freedom, and the internal energy of the ideal gas is

$$E_{\text{internal}} = N\frac{3}{2}kT\,. \tag{14A.1}$$

Eq. (14A.1) is called the *thermal equation of state* of a monatomic ideal gas. The average kinetic energy of each ideal gas atom is then

$$\frac{1}{2}m(v^2)_{\text{ave}} = \frac{3}{2}kT \tag{14A.2}$$

where $(v^2)_{\text{ave}}$ is the average of the square of the speeds and is given by

$$(v^2)_{\text{ave}} = \frac{3kT}{m} \ . \tag{14A.3}$$

The temperature of this ideal gas is proportional to the average kinetic of the ideal gas molecule. It is an incorrect inference to say that temperature is defined as the mean kinetic energy of gas. At low temperatures or non-dilute densities, the kinetic energy is no longer proportional to the temperature. For some gases, the kinetic energy depends on number density and a more complicated dependence on temperature than that given in Eq. (14A.2).

### 14A.3.3 Pressure of an Ideal Gas

Consider an ideal gas consisting of a large number $N$ of identical gas molecules, each of of mass $m$, inside a container of volume $V$ and pressure $P$. The number of gas molecules per unit volume is then $n = N/V$. The density of the gas is $\rho = nm$. The gas molecules collide elastically with each other and the walls of the container. The pressure that the gas exerts on the container is due to the elastic collisions of the gas molecules with the walls of the container. We shall now use concepts of energy and momentum to model collisions between the gas molecules and the walls of the container in order to determine the pressure of the gas in terms of the volume $V$, particle number $N$ and Kelvin temperature $T$.

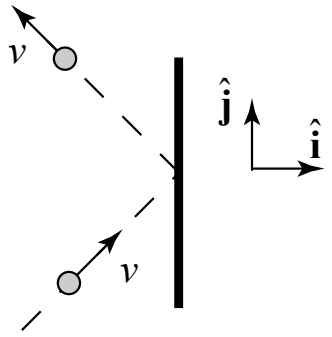


**Figure 14A.1** Collision of a gas molecule with a wall of a container

We begin by considering the collision of one molecule with one of the walls of the container, oriented with a unit normal vector pointing out of the container in the positive $\hat{\mathbf{i}}$-direction (Figure 14A.1). Suppose the molecule has mass $m$ and is moving with velocity $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}$. Because the collision with the wall is elastic, the $y$-and $z$-components of the velocity of the molecule remain constant and the $x$-component of the

velocity changes sign (Figure 29.2), resulting in a change of momentum of the gas molecule;

$$\Delta\vec{\mathbf{p}}_m = \vec{\mathbf{p}}_{m,f} - \vec{\mathbf{p}}_{m,i} = -2mv_x\,\hat{\mathbf{i}}. \tag{14.1.4}$$

Therefore the momentum transferred by the gas molecule to the wall is

$$\Delta\vec{\mathbf{p}}_w = 2mv_x\,\hat{\mathbf{i}}. \tag{14.1.5}$$

Now, let's consider the effect of the collisions of a large number of randomly moving molecules. For our purposes, "random" will be taken to mean that any direction of motion is possible, and the distribution of velocity components is the same for each direction.

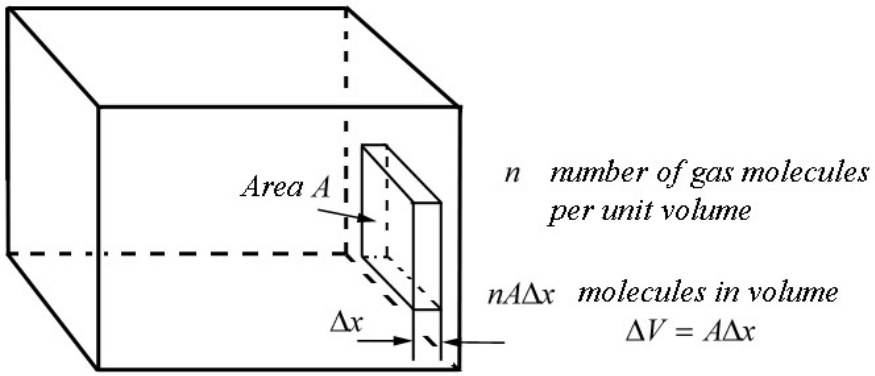


**Figure 14A.2** Small volume adjacent to the wall of container

Consider a small rectangular volume $\Delta V = A\Delta x$ of gas adjacent to one of the walls of the container as shown in Figure 14A.2. There are $nA\Delta x$ gas molecules in this small volume. Let each group have the same $x$-component of the velocity. Let $n_j$ denote the number of gas molecules in the $j^{th}$ group with $x$-component of the velocity $v_{x,j}$. Because the gas molecules are moving randomly, only half of the gas molecules in each group will be moving towards the wall in the positive $x$-direction. Therefore in a time interval $\Delta t_j = \Delta x / v_{x,j}$, the number of gas molecules that strike the wall with $x$-component of the velocity $v_{x,j}$ is given by

$$\Delta n_j = \frac{1}{2} n_j A \Delta x. \tag{14.1.6}$$

(During this time interval some gas molecules may leave the edges of the box, but because the number that cross the area per second is proportional to the area, in the limit as $\Delta x \to 0$, the number leaving the edges also approaches zero.) The number of gas molecules per second is then

$$\frac{\Delta n_j}{\Delta t_j} = \frac{1}{2} n_j A \frac{\Delta x}{\Delta t_j} = \frac{1}{2} n_j A v_{x,j} . \tag{14.1.7}$$

The momentum per second that the gas molecules in this group deliver to the wall is

$$\frac{\Delta \vec{\mathbf{p}}_j}{\Delta t_j} = \frac{\Delta n_j}{\Delta t_j} 2mv_{x,j} \hat{\mathbf{i}} = n_j mA v_{x,j}^2 \hat{\mathbf{i}} . \tag{14.1.8}$$

By Newton's Second Law, the average force on the wall due to this group of molecules is equal to the momentum per second delivered by the gas molecules to the wall;

$$(\vec{\mathbf{F}}_{j,w})_{\text{ave}} = \frac{\Delta \vec{\mathbf{p}}_j}{\Delta t_j} = n_j mA v_{x,j}^2 \hat{\mathbf{i}} . \tag{14.1.9}$$

The pressure contributed by this group of gas molecules is then

$$P_j = \frac{\left|(\vec{\mathbf{F}}_{j,w})_{\text{ave}}\right|}{A} = n_j m v_{x,j}^2 . \tag{14.1.10}$$

The pressure exerted by all the groups of gas molecules is the sum

$$P = \sum_{j=1}^{j=N_g} (P_j)_{\text{ave}} = m \sum_{j=1}^{j=N_g} n_j v_{x,j}^2 . \tag{14.1.11}$$

The average of the square of the $x$-component of the velocity is given by

$$(v_x^2)_{\text{ave}} = \frac{1}{n} \sum_{j=1}^{j=N_g} n_j v_{x,j}^2 , \tag{14.1.12}$$

where $n$ is the number of gas molecules per unit volume in the container. Therefore we can rewrite Eq. (14.1.11) as

$$P = mn(v_x^2)_{\text{ave}} = \rho(v_x^2)_{\text{ave}} , \tag{14.1.13}$$

where $\rho$ is the density of the gas. Because we assumed that the gas molecules are moving randomly, the average of the square of the $x$-, $y$- and $z$-components of the velocity of the gas molecules are equal,

$$(v_x^2)_{\text{ave}} = (v_y^2)_{\text{ave}} = (v_z^2)_{\text{ave}} . \tag{14.1.14}$$

The average of the square of the speed $(v^2)_{\text{ave}}$ is equal to the sum of the average of the squares of the components of the velocity,

$$(v^2)_{\text{ave}} = (v_x^2)_{\text{ave}} + (v_y^2)_{\text{ave}} + (v_z^2)_{\text{ave}} . \tag{14.1.15}$$

Therefore

$$(v^2)_{\text{ave}} = 3(v_x^2)_{\text{ave}} . \tag{14.1.16}$$

Substituting Eq. (14.1.16) into Eq. (14.1.13) for the pressure of the gas yields

$$P = \frac{1}{3}\rho(v^2)_{\text{ave}} . \tag{14.1.17}$$

The square root of $(v^2)_{\text{ave}}$ is called the **root-mean-square** ("rms") speed of the molecules. Substituting Eq. (14A.3) into Eq. (14.1.17) yields

$$P = \frac{\rho kT}{m} . \tag{14.1.18}$$

Recall that the density of the gas

$$\rho = \frac{M}{V} = \frac{Nm}{V} . \tag{14.1.19}$$

Therefore Eq. (14.1.18) can be rewritten as

$$P = \frac{NkT}{V} . \tag{14.1.20}$$

Eq. (14.1.20) can be re-expressed as

$$PV = N\,kT . \tag{14.1.21}$$

Eq. (14.1.21) is known as the *ideal gas equation of state* also known as the *Perfect Gas Law* or *Ideal Gas Law*.

The total number of molecules in the gas $N = n_{\text{m}} N_A$ where $n_{\text{m}}$ is the number of moles and $N_A$ is the Avogadro constant. The ideal gas law becomes

$$PV = n_{\text{m}} N_A kT . \tag{14.1.22}$$

The *universal gas constant* is $R = k\,N_A = 8.31\,\text{J}\cdot\text{K}^{-1}\cdot\text{mol}^{-1}$. The ideal gas law can be re-expressed as

$$PV = n_{\text{m}}\,RT . \tag{14.1.23}$$

Although we started with atomistic description of the collisions of individual gas molecules satisfying the principles of conservation of energy and momentum, we ended

up with a relationship between the macroscopic variables pressure, volume, number of moles, and temperature that are measurable properties of the system.

One important consequence of the Ideal Gas Law is that equal volumes of different ideal gases at the same temperature and pressure must contain the same number of molecules,

$$N = \frac{1}{k}\frac{PV}{T}. \tag{14.1.24}$$

When gases combine in chemical reactions at constant temperature and pressure, the numbers of each type of gas molecule combine in simple integral proportions. This implies that the volumes of the gases must always be in simple integral proportions. Avogadro used this last observation about gas reactions to define one mole of a gas as a unit for large numbers of particles.

### 14A.3.3 Atoms, Moles, and Avogadro's Number

The *Avogadro number* was originally defined as the number of molecules in one gram of hydrogen. The number was then redefined to be the number of atoms in 12 grams of the carbon isotope carbon-12. Now the Avogadro number is the fixed numerical value of the *Avogadro constant* $N_A$ when expressed in the unit $\text{mol}^{-1}$

$$N_A = 6.022\,140\,76 \times 10^{23}\ \text{mol}^{-1}. \tag{14A.25}$$

Recall that the mole is a base unit in the SI system of units for an amount of substance with symbol [mol]. Based on the new definition of Avogadro constant $N_A$, one mole contains $6.022\,140\,76 \times 10^{23}$ elementary entities:

$$1\,\text{mol} = \frac{N_A}{6.02214076 \times 10^{23}} \tag{14A.26}$$

## 14A.4 Degrees of Freedom

An individual gas molecule can translate in any spatial direction. Multi-atomic gas molecules may undergo rotational motions associated with the structure of the molecule. Additionally, there may be intermolecular vibrational motion between nearby gas particles, and vibrational motion arising from intramolecular forces between atoms that form the molecules. Each of these independent contributions to the internal energy motions are called *degree of freedoms*.

For gas molecules, there are three translational degrees of freedom associated with the center of mass motion in each direction. For diatomic gases like oxygen molecule ($O_2$) or carbon monoxide ($CO$), there are additional degrees of freedom: two rotational

degrees of freedom corresponding to independent rotations about axes that are perpendicular to the line connecting the centers of the two atoms, and two degrees of freedom corresponding to the kinetic and potential energies associated with vibrational motion about the center of mass, resulting in a total of seven degrees of freedom. Note the extra factor of two for the vibrational modes can be understood by modeling the vibrational motion of the molecules as an oscillating spring in one-dimension with two contributions to the internal energy, $E_{vibrational} = (1/2)mv^2 + (1/2)kx^2$. For linear triatomic linear molecules like carbon dioxide $(CO_2)$: there are also two rotational degrees of freedom, and six degrees of freedom associated with the three vibrational modes of the molecule, totaling eleven degrees of freedom. For non-linear triatomic molecules, there is an extra rotational degree of freedom compared to the linear case, hence twelve degrees of freedom. For polyatomic molecules, there are many vibrational modes, so the number of degrees of freedom is greater than twelve.

## 14A.5 Equipartition of Energy

We shall make our first assumption about how the internal energy distributes itself among $N$ gas molecules, as follows:

*Each independent degree of freedom has an equal amount of energy equal to* $(1/2)kT$,

where the constant $k$ is called the **Boltzmann constant** and is defined by

$$k = 1.380649 \times 10^{-23}\ \mathrm{J \cdot K^{-1}}\ .$$

The total internal energy $U$ of the ideal gas is then

$$U = N(\#\text{ of degrees of freedom})\frac{1}{2}kT\ . \qquad (14A.27)$$

This equal division of the energy is called the ***equipartition of the energy.***

### 14A.5.1 Boltzamnn constant

Recall that the Boltzmann constant is now one of the seven defining constants that determine the SI units and along with the constants $h$, $c$ *and* $\Delta\nu_{Cs}$ determine the unit kelvin.

> *The kelvin, symbol* $\mathrm{K}$*, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant* $k$ *to be* $1.380649 \times 10^{-23}$ *when expressed in the unit* $\mathrm{J \cdot K^{-1}}$*, which is equal to* $\mathrm{kg \cdot m^2 \cdot s^{-2} K^{-1}}$*, where the kilogram, meter and second are defined in terms of* $h$, $c$ *and* $\Delta\nu_{Cs}$.

*This definition implies the exact relation* $k = 1.380649 \times 10^{-23}\,\mathrm{J \cdot K^{-1}}$. *Its effect is that one kelvin is equal to the change of thermodynamic temperature* $T$ *that results in a change of thermal energy* $kT$ *by* $1.380649 \times 10^{-23}\,\mathrm{J}$. [2]

Therefore the unit kelvin is defined as

$$1\,\mathrm{K} = \frac{1.380649 \times 10^{-23}}{k} \cdot \mathrm{kg \cdot m^2 \cdot s^{-2}} \tag{14A.28}$$

Using the SI definitions for the kilogram, meter and second, Eq. (14A.28) becomes

$$1\,\mathrm{K} = \left( \frac{1.380649 \times 10^{-23}}{k} \right) \cdot \left( 1.4755214 \times 10^{40}\ \frac{h\Delta\nu_{\mathrm{Cs}}}{c^2} \right) \cdot \left( 30.663\,314\,9\ \frac{c}{\Delta\nu_{\mathrm{Cs}}} \right)^2 \cdot \left( \frac{9\,192\,631\,770}{\Delta\nu_{\mathrm{Cs}}} \right)^{-2} \tag{14A.29}$$

One kelvin is then equal to

$$1\,\mathrm{K} = 2.266\,665\,265\ \frac{h\Delta\nu_{\mathrm{Cs}}}{k} \tag{14A.30}$$

### 14A.5.2 Freezing out Degrees of Freedom

Vibrational modes generally do not occur at room temperature. At higher temperatures, a diatomic gas molecule has a potential energy associated with the interaction between the two molecules. This potential energy acts like a spring between the two atoms contributing to a vibrational mode along the -axis. Analogous to a spring connected two objects, there are two degrees of freedom associated with a vibrational mode, the potential energy stored in the interaction and the kinetic energy associated with the vibration. Thus there are seven total degrees of freedom for the energy to partition among. So in principle of all of these energy modes are accessible, then the total internal energy $U$ for a diatomic gas consisting $N$ molecules is

$$U = N(\#\ of\ deg)((1/2)kT) = N(7/2)kT \tag{14A.31}$$

According to our classical theory of the gas, all these modes should be equally occupied at all temperatures but in fact they are not! This important deviation from classical physics was the first place that a more detailed model of the atom is needed to correctly describe experimental observations.

[2] https://www.bipm.org/en/measurement-units/base-units.html

### 14A.5.3 Example Diatomic Nitrogen Gas

What is the internal energy of the diatomic $N_2$ gas at room temperature?

**Solution:** At room temperature, the internal energy is due to only the five degrees of freedom associated with the three translational and two rotational degrees of freedom,

$$U = \frac{5}{2} NkT \,. \tag{14A.32}$$

As discussed above, at temperatures well above room temperature, but low enough for nitrogen to form diatomic molecules, there is are two additional vibrational degree of freedoms. Therefore there are seven degrees of freedom and so the internal energy is

$$U = N(\#\text{ of degrees of freedom})\frac{1}{2} kT = \frac{7}{2} N\,kT \,. \tag{14A.33}$$

## 14A.6 Temperature, Scales, and Thermometers

### 14A.6.1 Temperature

In our discussion so far, we have not defined precisely how we can measure temperature. In particular, we have not determined how the flow of thermal energy into a system raises its temperature. We begin with a macroscopic characterization of the temperature of a body.

To measure the temperature of a system, we need to measure a *thermometric* property of the system, one that varies with its hotness or coldness. There are many such properties; for example, electrical resistance of a filament, pressure of a gas, thermal electromotive force, radiant emittance, or magnetic susceptibility. Let $X$ be any thermometric property of a material. Then we define the temperature scale so that the temperature $\theta$ is linear proportional to $X$

$$\theta(X) = aX \,. \tag{14A.34}$$

where $a$ is a constant of proportionality. By this linearity, the ratio of temperatures between any two states of the system is then the ratio of the thermometric properties of those states,

$$\frac{\theta_1}{\theta_2} = \frac{X_1}{X_2} \,. \tag{14A.35}$$

Traditionally, to determine temperature for any state, we need to define temperature for a standard state. The standard fixed state for thermometry is the *triple point of water*. This is the state in which ice, water, and water vapor coexist.

This occurs at $0.01^{\circ}\mathrm{C}$ and at a water-vapor pressure of $610\ \mathrm{Pa}$. For historical and scientific reasons to be explained, we define the temperature of the triple point of water to be $273.16\ \mathrm{K}$ on the Kelvin scale, which fixes the constant $a$ as follows. Let $X_{TP}$ be the value of the thermometric property $X$ at the triple point. Then

$$a = \frac{273.16\ \mathrm{K}}{X_{TP}}. \tag{14A.36}$$

Hence the temperature at any value of $X$ is then

$$\theta(X) = aX = a\frac{273.16\ K}{X_{TP}}\ . \tag{14A.37}$$

**14A.6.2 Temperature Scales**

We use the Kelvin scale as a measure of absolute temperature. The commonly used Celsius scale employs the same size for each degree as the Kelvin scale, but the zero point is shifted by $273.15$ degrees so that the triple point of water has a Celsius temperature of $0.01^{\circ}\mathrm{C}$,

$$T(^{\circ}\mathrm{C}) = \theta(\mathrm{K}) - 273.15\ ^{\circ}\mathrm{C}\ , \tag{14A.38}$$

and the freezing point of water at standard atmospheric pressure to be $0^{\circ}\mathrm{C}$. The Fahrenheit scale is related to the Celsius scale by

$$T(^{\circ}\mathrm{F}) = \frac{9}{5}T(^{\circ}\mathrm{C}) + 32\ ^{\circ}\mathrm{F}\ . \tag{14A.39}$$

The freezing point of pure water at standard atmospheric pressure occurs at $0^{\circ}\mathrm{C}$ and $32\ ^{\circ}\mathrm{F}$. The boiling point of pure water at standard atmospheric pressure occurs at $100^{\circ}\mathrm{C}$ and $212\ ^{\circ}\mathrm{F}$.

**14A.6.3 Example Gas Thermometer**

The gas thermometer measures temperature based on the pressure of a gas at constant volume and is used as the standard thermometer, because the variations between different gases can be greatly reduced when low pressures are used. A schematic device of a gas thermometer is shown in Figure 14A.3. The volume of the gas is kept constant by raising or lowering the mercury reservoir so that the mercury level on the left arm in Figure 14A.3 just reaches the point $I$. When the bulb is placed in thermal equilibrium with a system whose temperature is to be measured, the difference in height between the mercury levels in the left and right arms is measured. The bulb pressure is atmospheric pressure plus the pressure in mercury a distance $h$ below the surface (Pascal's Law). A thermometer needs to have two scale points, for example the height of the column of

mercury (the height is a function of the pressure of the gas) when the bulb is placed in thermal equilibrium with ice water and in thermal equilibrium with standard steam.

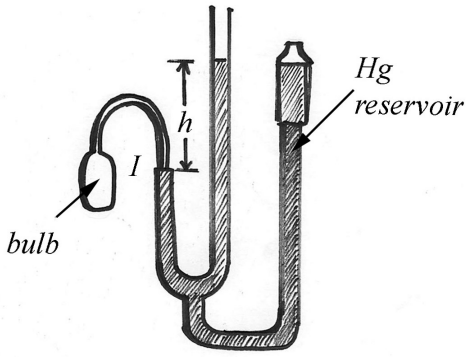


**Figure 14A.3** Constant volume gas thermometer

At constant volume, and at ordinary temperatures, the pressure of gases is proportional to the temperature,

$$T \propto P . \tag{14A.40}$$

We define a linear scale for temperature based on the pressure in the bulb by

$$T = a P \tag{14A.41}$$

where $a$ is a positive constant. In order to fix the constant $a$ in Eq. (14A.41), a standard state must be chosen as a reference point. The standard fixed state for thermometry is the *triple point of water*, the state in which ice, water, and water vapor coexist. This state occurs at only one definite value of temperature and pressure. By convention, the temperature of the triple point of water is chosen to be exactly $273.16\ \mathrm{K}$ on the *Kelvin scale*, at a water-vapor pressure of $610\ \mathrm{Pa}$. Let $P_{\mathrm{TP}}$ be the value of the pressure $P$ at the triple point in the gas thermometer. Set the constant $a$ according to

$$a = \frac{273.16\ \mathrm{K}}{P_{\mathrm{TP}}} . \tag{14A.42}$$

Hence the temperature at any value of $P$ is then

$$T(P) = a P = \frac{273.16\ \mathrm{K}}{P_{\mathrm{TP}}} P . \tag{14A.43}$$

The ratio of temperatures between any two states of a system is then measured by the ratio of the pressures of those states,

$$\frac{T_1}{T_2} = \frac{P_1}{P_2} . \tag{14A.44}$$

## 14A.7 Conservation of Energy and Energy Transformations

When we do work on a system, for example we can increase the speed of an object by pushing it, and the kinetic energy increases. We can do work compressing a spring, and the potential energy increases. We can also do work on a system in such a way that the mechanical energy stays constant, but we generate thermal energy. For example, we can slide an object along a surface at constant speed. If we consider the object and the surface as our system, then we do work on the sliding object, and increase the thermal energy of the system.

We can also decrease both the kinetic energy and potential energy of a system, and increase the thermal energy. Consider the interaction between water falling over a waterfall and the earth. Between the top and bottom of a waterfall, there is a net loss in mechanical energy. As the water falls, it accelerates, an amount of gravitational potential energy transforms into kinetic energy. When the falling water strikes the surface, much of that kinetic energy is lost from the mechanical system. However the temperature of the water at the bottom of the fall will be higher than the temperature at the top. We can also increase or decrease the energy of a system by heating or cooling as we observed with warming a kettle of water. We shall study types of energy transformations due to interactions both inside and across the boundary of a system.

### 14A.7.1 System, Boundary and Surroundings

Recall in Chapter 13, when we specify a system, we also specified the surroundings (everything else) and a boundary between the system and the surroundings. The boundaries are interfaces through which energy can be transferred. The above examples suggest that we can change the energy of the system by doing work on the system, or by the flow of "heat" into the system.

Recall that a system is *open* if both energy and matter can enter of leave the system. A system is *closed* if only energy can be transferred to or from the surroundings. A closed system in which energy is constant may not be isolated. For example consider a compressed spring in which one end is attached to a cart and the other end attached to a wall. The cart is held in place on a frictionless air track. Choose as the system the cart and spring. Then release the cart. While the spring is expanding and the cart is accelerating, there is an external force of the wall on the spring so the system is not isolated but there is no transfer of energy to or from the system. The potential energy of the spring is transformed into kinetic energy of the cart. So the energy of the system is constant but it is not isolated.

For a closed system, the change in energy of the system and the surroundings sum to zero,

$$\Delta E^{\text{total}} = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0 \qquad (14A.45)$$

When a system undergoes a change from state $A$, to state $B$, the sum of the kinetic and potential energy (mechanical energy $\Delta E_{\text{mechanical}}$) may change, and the kinetic energy associated with random motions, (thermal energy $\Delta E_{\text{thermal}}$), may also change. In addition there may be other forms of energy $\Delta E_{\text{other}}$ that may change as well. So the total change in energy of the system is the sum of these changes

$$\Delta E_{\text{system}} = \Delta E_{\text{mechanical}} + \Delta E_{\text{thermal}} + \Delta E_{\text{other}} . \tag{14A.46}$$

If the total energy of the system changes, then the total energy of the surrounding must change by the opposite amount,

$$\Delta E_{\text{system}} = -\Delta E_{\text{surroundings}} \tag{14A.47}$$

If the energy is a system changes, then energy must flow across the boundary. We shall study two different types of energy flows across the boundaries of a system. The first type is when the surroundings do work on the system (or the system does work on the surroundings). Consider a closed cylinder of gas with a piston at one end. Identify the gas as the system. If an external force pushes the piston inwards, then the surroundings do (external) work on the system, and the total energy of the system changes,

$$\Delta E^{(1)}_{\text{system}} = W_{\text{external}} \ . \tag{14A.48}$$

A second type of energy flow through the boundary involves the flow of thermal energy, which we denote by $Q$. We adapt the convention that $Q > 0$ means that a positive amount of heat flows into the system, the energy of the system increases, therefore

$$\Delta E^{(2)}_{\text{system}} = Q \ . \tag{14A.49}$$

## 14A.8 First Law of Thermodynamics

The energy of a closed system can increase or decrease either through external work done on or by the system, and by the flow of thermal energy into or out of the system.

Let $Q > 0$ represent a positive amount of thermal energy that flows into a system. If $Q < 0$, then thermal energy flows from the system to the surroundings.

Let $W_{ext} > 0$ denote the work done by the surroundings on a system. (If $W_{ext} < 0$ then the system is doing work on the surroundings.)

In what follows we shall denote the total change in internal energy of a system by $\Delta U$. Then the *first law of thermodynamics* describes the sum of the change in energy due to heating and the work done by the surroundings on the system,

$$\Delta U = W_{ext} + Q \ . \tag{14A.50}$$

Whenever a closed system is divided into a system $(1)$ and surroundings $(2)$ that are in thermal contact, and there is a thermal energy flow from (1) to (2), then the thermal energy lost from (1) is gained by $(2)$,

$$Q_1 = -Q_2 \tag{14A.51}$$

### 14A.8.1 Mechanical Equivalent of Heat

We have already used the joule as the unit for mechanical energy; we would like to determine the constant of proportionality $k$ between the rate of loss of mechanical energy as measured in watts and the rate of the flow of thermal energy as measured in calories per sec

$$\frac{dE_{mech}}{dt} = -k\frac{dQ}{dt} \ . \tag{14A.52}$$

James Joule in $1847$ first measured this connection between mechanical energy and heat and found that $4.2\,\mathrm{J} = 1\,\mathrm{cal}$. The modern result at $15^{\circ}\mathrm{C}$ is $4.186\,\mathrm{J} = 1\,\mathrm{cal}$.

When a mass slides along the table, work done by the contact friction generates thermal energy that is absorbed by both the mass and the table. So we must include the mass and the table as part of our system. In fact, the gas molecules near the table absorb some thermal energy, so strictly speaking they must also be included in the closed system. Is mechanical energy really 'lost' in a system? If we examine the individual molecules in our system, and we discover that their average kinetic energy increases along with an increase in the potential energy associated with their molecular interactions. Microscopically, energy is conserved!

If we immerse a light bulb in water, the electrical power delivered to the light bulb is dissipated into the water causing the thermal energy of the water to increase. Macroscopically we could measure this increased thermal energy by measuring the rise in the temperature of the water.

## 14A.9 States of Matter

### 14A.9.1 Heat Capacity and Specific Heat

When thermal energy $Q > 0$ flows into a system, the temperature of the system may or may not undergo a change. When the temperature does rise by an amount $\Delta T > 0$, as we can observe by heating water with a light bulb, the *average heat capacity* of the system is defined to be the total amount of thermal energy that flows into the system divided by the rise in temperature,

$$C = \frac{Q}{\Delta T} \tag{14A.53}$$

If we were to divide our system precisely in half, the same flow of thermal energy will induce double the temperature change, hence halving the heat capacity. If we divide heat capacity by the amount of mass present, then we have a property of the system that will not change when we halve the system. Thus we define the *average specific heat* as the heat capacity per mass,

$$c = \frac{Q}{m\Delta T} \ . \tag{14A.54}$$

The units for specific heat are $\left[\mathrm{J \cdot kg^{-1} \cdot K^{-1}}\right]$. For water, the specific heat varies as a function of temperature. Figure 14A.4 shows the specific heat of water plotted as a function of temperature for the range $\left[0\ ^{\circ}\mathrm{C},\ 100^{\circ}\mathrm{C}\right]$.

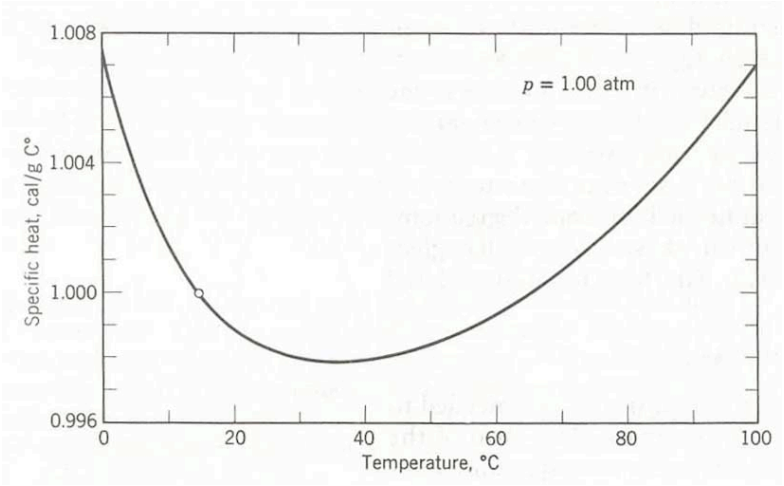


**Figure 14A.4** Specific heat of water as a function of temperature.

For the range 14.5 $^{0}$C to 15.5 $^{0}$C, the value is

$$c_{\mathrm{H_2O}} = 4.1860 \times 10^{3}\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}} \ . \tag{14A.55}$$

### 14A.9.2 Specific Heats of an Ideal Gas

The specific heat of a substance is the amount of heat required per unit mass per unit temperature change. When the unit of mass is the mole, the specific heat is called the molar heat capacity. A gas can have two types of molar heat capacities: at constant pressure, $C_P$, or at constant volume, $C_V$.

Consider $n_m$ moles of an ideal gas that is in thermal contact with a reservoir of thermal energy. The temperature of the reservoir is slowly raised. The volume of the gas does not change during the process. Therefore no work is done on the gas. From the first law of thermodynamics, the change in the internal energy of the gas is due entirely to the flow of thermal energy into the gas,

$$\Delta U = Q \ . \tag{14A.56}$$

The amount of thermal energy required is equal to

$$Q = n_m C_V \Delta T \ . \tag{14A.57}$$

Therefore the change in internal energy is given by

$$\Delta U = n_m C_V \Delta T \ . \tag{14A.58}$$

For an ideal gas, the change in internal energy only depends on the temperature change

$$\Delta U = \frac{3}{2} n_m R \Delta T \ . \tag{14A.59}$$

Therefore comparing these expressions shows that the molar heat capacity at constant volume is

$$C_V = \frac{3}{2} R \ . \tag{14A.60}$$

For a gas molecule with $D$ degrees of freedom, the change in internal energy is

$$\Delta U = \frac{D}{2} n_m R \Delta T \tag{14A.61}$$

and so the heat capacity is

$$C_V = \frac{D}{2} R \ . \tag{14A.62}$$

Table 14A.1 shows the molar specific heats for different gases at $15^{\circ}\mathrm{C}$ and $1\,\mathrm{atm}$ pressure..

**Table 14A.1** Constant Volume Molar Specific Heats for Different Gases.

| Gas | Molar Specific Heat $C_V$ | $C_V / R$ |
|---|---|---|
| Ar | 12.5 | 1.50 |
| He | 12.5 | 1.50 |
| CO | 20.7 | 2.49 |
| $H_2$ | 20.4 | 2.45 |
| HCl | 21.4 | 2.57 |
| $N_2$ | 20.6 | 2.49 |
| NO | 20.9 | 2.51 |
| $O_2$ | 21.1 | 2.54 |
| $Cl_2$ | 24.8 | 2.98 |
| $CO_2$ | 28.2 | 3.40 |
| $CS_2$ | 40.9 | 4.92 |
| $H_2S$ | 25.4 | 3.06 |
| $N_2O$ | 28.5 | 3.42 |
| $SO_2$ | 31.3 | 3.76 |

Figure 14A.5 shows the variation in the hydrogen molar specific heat at constant volume as a function of the temperature. The temperature scale is logarithmic. The classical theory does not agree with experiment!

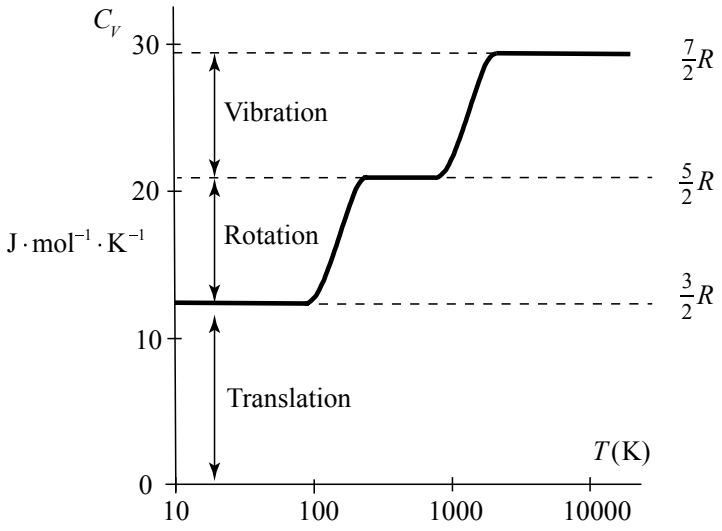


**Figure 14A.5** Hydrogen molar specific heat at constant volume as a function of temperature.

### 14A.9.3 Example Molar heat capacities of an ideal gas

Consider an ideal gas undergoing a constant pressure expansion, with a temperature change $\Delta T$. The gas does work on the surroundings, $W = P\Delta V$, and hence the surroundings do negative work on the gas. From the ideal gas law, the external work done is then

$$W_{ext} = -P\Delta V = -n_m R\Delta T \ . \tag{14A.63}$$

According to the first law of thermodynamics, the change in the internal energy of the gas is

$$\Delta U = Q + W_{ext} = Q - P\Delta V \ . \tag{14A.64}$$

The amount of thermal energy that flowed into the gas is then

$$Q = n_m C_P \Delta T \ . \tag{14A.65}$$

So the change in internal energy for the constant pressure expansion is

$$\Delta U = n_m C_P \Delta T - n_m R\Delta T \ . \tag{14A.66}$$

For an ideal gas, the change in internal energy only depends on the temperature change

$$\Delta U = \frac{3}{2} n_m R\Delta T \ . \tag{14A.67}$$

Therefore the change in internal energy is,

$$\Delta U = n_m C_V \Delta T = n_m C_P \Delta T - n_m R\Delta T \ . \tag{14A.68}$$

The two molar heat capacities are related by solving this equation to yield

$$C_P = C_V + R \ . \tag{14A.69}$$

Using the fact that $C_V = (3/2)R$, we find that the molar heat capacity at constant pressure for an ideal gas is then

$$C_P = C_V + R = \frac{3}{2}R + R = \frac{5}{2}R \ . \tag{14A.70}$$

Let $\gamma = C_P / C_V$ denote the ratio of the heat capacities. Then for the ideal gas

$$\gamma = C_P / C_V = 5/3 \ . \tag{14A.71}$$

The ideal gas law establishes a proportional of the product of the pressure with the volume to the temperature according to

$$PV = n_m RT \ . \tag{14A.72}$$

The internal energy of the gas is

$$U = (3/2) n_m RT \ . \tag{14A.73}$$

.

Thus combining these equations yields

$$PV = \frac{2}{3} U \ . \tag{14A.74}$$

We can rewrite this last equation is terms of the ratio of the heat capacities,

$$PV = (\gamma - 1) U \tag{14A.75}$$

**14A.9.4 Example Adiabatic compression of an ideal gas**

Suppose we compress the gas by an amount $dV < 0$ so that there is no loss of energy as heat through the container, adiabatic compression. The work that is done on compressing the gas,

$$dW = -PdV = \Delta U > 0 \ , \tag{14A.76}$$

will increase the internal energy. Note that the minus sign ensures that the work done on the gas is positive. Since the volume of the gas is decreasing, the pressure of the gas must increase. So using Eq. (14A.75), the differential rate of change of the internal energy of the gas is given by

$$(\gamma - 1)\Delta U = dPV + PdV \tag{14A.77}$$

Thus substituting Eq. (14A.76) into Eq. (14A.77), yields

$$(\gamma - 1)(-PdV) = dPV + PdV \ . \tag{14A.78}$$

Collecting terms yields

$$-\gamma PdV = dPV \tag{14A.79}$$

This equation is separable

$$-\gamma \frac{dV}{V} = \frac{dP}{P} \ . \tag{14A.80}$$

.

which can then be integrated

$$-\gamma \int_{V_i}^{V_f} \frac{dV}{V} = \int_{P_i}^{P_f} \frac{dP}{P} \tag{14A.81}$$

resulting in

$$\gamma \ln(V_i / V_f) = \ln(P_f / P_i) . \qquad (14A.82)$$

Exponentiating both sides then yields

$$(V_i / V_f)^{\gamma} = (P_f / P_i) \ . \qquad (14A.83)$$

Thus the product of the pressure and the volume raised to the power $\gamma$ is a constant for the adiabatic compression of the gas,

$$PV_i^{\gamma} = P_f V_f^{\gamma} \qquad (14A.84)$$