Chapter 13 The Concept of Energy and Conservation of Energy

Chapter 13 Energy, Kinetic Energy, and Work

Acceleration of the expansion of the universe is one of the most exciting and significant discoveries in physics, with implications that could revolutionize theories of quantum physics, gravitation, and cosmology. With its revelation that close to the three-quarters of the energy density of the universe, given the name dark energy, is of a new, unknown origin and that its exotic gravitational "repulsion" will govern the fate of the universe, dark energy and the accelerating universe becomes a topic not just of great interest to research physicists but to science students at all levels. [1]

Eric Linder

13.1 The Concept of Energy and Conservation of Energy

The transformation of energy is a powerful concept that enables us to describe a vast number of processes:

Falling water releases stored *gravitational potential energy*, which can become the *kinetic energy* associated with a *coherent motion* of matter. The harnessed *mechanical energy* can be used to spin turbines and alternators, doing *work* to generate *electrical energy*, transmitted to consumers along power lines. When you use any electrical device, the electrical energy is transformed into other forms of energy. In a refrigerator, electrical energy is used to compress a gas into a liquid. During the compression, some of the internal energy of the gas is transferred to the *random motion* of molecules in the outside environment. The liquid flows from a high-pressure region into a low-pressure region where the liquid evaporates. During the evaporation, the liquid absorbs energy from the *random motion* of molecules inside of the refrigerator. The gas returns to the compressor.

"Human beings transform the stored chemical energy of food into various forms necessary for the maintenance of the functions of the various organ system, tissues and cells in the body."[2] A person can do work on their surroundings – for example, by pedaling a bicycle – and transfer energy to the surroundings in the form of increasing random motion of air molecules, by using this catabolic energy.

Burning gasoline in car engines converts *chemical energy*, stored in the molecular bonds of the constituent molecules of gasoline, into coherent (ordered) motion of the molecules that constitute a piston. With the use of gearing and tire/road friction, this motion is converted into kinetic energy of the car; the automobile moves.

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[1] Eric Linder, *Resource Letter: Dark Energy and the Accelerating Universe*, Am.J.Phys.76: 197-204, 2008; p. 197.

[2] George B. Benedek and Felix M.H. Villars, *Physics with Illustrative Examples from Medicine and Biology, Volume 1: Mechanics,* Addison-Wesley, Reading, 1973, p. 115-6.

Stretching or compressing a spring stores *elastic potential energy* that can be released as kinetic energy.

The process of vision begins with stored *atomic energy* released as electromagnetic radiation (light), which is detected by exciting photoreceptors in the eye, releasing chemical energy.

When a proton fuses with deuterium (a hydrogen atom with a neutron and proton for a nucleus), helium-three is formed (with a nucleus of two protons and one neutron) along with radiant energy in the form of photons. The combined *internal energy* of the proton and deuterium are greater than the internal energy of the helium-three. This difference in internal energy is carried away by the photons as light energy.

There are many such processes involving different forms of energy: kinetic energy, gravitational energy, thermal energy, elastic energy, electrical energy, chemical energy, electromagnetic energy, nuclear energy and more. The total energy is always conserved in these processes, although different forms of energy are converted into others.

Any physical process can be characterized by two states, initial and final, between which energy transformations can occur. Each form of energy E_j, where "j" is an arbitrary label identifying one of the N forms of energy, may undergo a change during this transformation,

$$\Delta E_j \equiv E_{\text{final},j} - E_{\text{initial},j}\,. \tag{13.1.1}$$

Conservation of energy means that the sum of these changes is zero,

$$\Delta E_1 + \Delta E_2 + \cdots + \Delta E_N = \sum_{j=1}^{N} \Delta E_j = 0\,. \tag{13.1.2}$$

Two important points emerge from this idea. First, we are interested primarily in changes in energy and so we search for relations that describe how each form of energy changes. Second, we must account for all the ways energy can change. If we observe a process, and the sum of the changes in energy is not zero, either our expressions for energy are incorrect, or there is a new type of change of energy that we had not previously discovered. This is our first example of the importance of conservation laws in describing physical processes, as energy is a key quantity conserved in all physical processes. If we can quantify the changes of different forms of energy, we have a very powerful tool to understand nature.

We will begin our analysis of conservation of energy by considering processes involving only a few forms of changing energy. We will make assumptions that greatly simplify our description of these processes. At first we shall only consider processes acting on bodies in which the atoms move in a coherent fashion, ignoring processes in which energy is transferred into the random motion of atoms. Thus we will initially ignore the effects of friction. We shall then treat processes involving friction between consider rigid bodies. We will later return to processes in which there is an energy transfer resulting in an increase or decrease in random motion when we study the First Law of Thermodynamics.

Energy is always conserved but we often prefer to restrict our attention to a set of objects that we define to be our *system*. The rest of the universe acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 13.1. When we discussed Newton's Laws, an object is called *isolated* if there are no physical interactions between the object and the surroundings. According to Newton's First Law an isolated object will undergo uniform motion. A system is called an *isolated system* if there are no physical interactions between the system and the surroundings. A system is *open* if both energy and matter can enter of leave the system. A system is *closed* if only energy can be transferred to or from the surroundings.

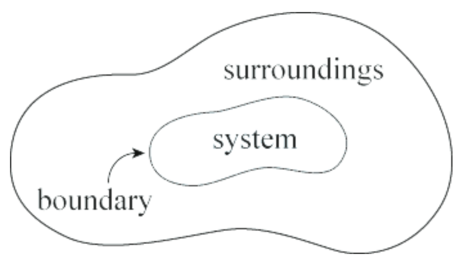

Figure 13.1 A diagram of a system and its surroundings with boundary

We shall just consider closed systems for the purposes of this discussion. Because energy is conserved, any energy that leaves the system must cross through the boundary and enter the surroundings. Consider any physical process in which energy transformations occur between initial and final states. We assert that

when a system and its surroundings undergo a transition from an initial state to a final state, the change in energy is zero,

$$\Delta E = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0 . \tag{13.1.3}$$

Eq. (13.1.3) is called ***conservation of energy*** and is our operating definition for energy. We will sometime refer to Eq. (13.1.3) as the *energy principle*. In any physical application, we first identify our system and surroundings, and then attempt to quantify changes in energy. In order to do this, we need to identify every type of change of energy in every possible physical process. When there is no change in energy in the surroundings then the energy of a closed system is constant.

$$\Delta E_{\text{system}} = 0 . \tag{13.1.4}$$

If we add up all known changes in energy in the system and surroundings and do not arrive at a zero sum, we have an open scientific problem. By searching for the missing changes in energy, we may uncover some new physical phenomenon. Recently, one of the most exciting open problems in cosmology is the apparent acceleration of the expansion of the universe, which has been attributed to *dark energy* that resides in space itself, an energy type without a clearly known source.[3]

13.2 Kinetic Energy

The first form of energy that we will study is an energy associated with the coherent motion of molecules that constitute a body of mass m; this energy is called the *kinetic energy* (from the Greek word *kinetikos* which translates as *moving*). Let us consider a car moving along a straight road (along which we will place the x-axis). For an observer at rest with respect to the ground, the car has velocity $\vec{\mathbf{v}} = v_x \hat{\mathbf{i}}$. The speed of the car is the magnitude of the velocity, $v \equiv |v_x|$.

> *The **kinetic energy** K of a non-rotating body of mass m moving with speed v is defined to be the positive scalar quantity*

$$K \equiv \frac{1}{2} m v^2 \qquad (13.2.1)$$

The kinetic energy is proportional to the square of the speed. The SI units for kinetic energy are $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}]$. This combination of units is defined to be a joule and is denoted by $[\mathrm{J}]$, thus $1\,\mathrm{J} \equiv 1\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$. (The SI unit of energy is named for James Prescott Joule.) The above definition of kinetic energy does not refer to any direction of motion, just the speed of the body.

Let's consider a case in which our car changes velocity. For our initial state, the car moves with an initial velocity $\vec{\mathbf{v}}_i = v_{x,i}\, \hat{\mathbf{i}}$ along the x-axis. For the final state (at some later time), the car has changed its velocity and now moves with a final velocity $\vec{\mathbf{v}}_f = v_{x,f}\, \hat{\mathbf{i}}$. Therefore the change in the kinetic energy is

$$\Delta K = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_i^{\,2}. \qquad (13.2.2)$$

Example 13.1 Change in Kinetic Energy of a Car

Suppose car *A* increases its speed from 10 to 20 mph and car *B* increases its speed from 50 to 60 mph. Both cars have the same mass m. (a) What is the ratio of the change of kinetic energy of car *B* to the change of kinetic energy of car *A*? In particular, which car

[3] http://www-supernova.lbl.gov/~evlinder/sci.html

has a greater change in kinetic energy? (b) What is the ratio of the change in kinetic energy of car *B* to car *A* as seen by an observer moving with the initial velocity of car *A*?

Solution: (a) The ratio of the change in kinetic energy of car *B* to car *A* is

$$\frac{\Delta K_B}{\Delta K_A} = \frac{\frac{1}{2}m(v_{B,f})^2 - \frac{1}{2}m(v_{B,i})^2}{\frac{1}{2}m(v_{A,f})^2 - \frac{1}{2}m(v_{A,i})^2} = \frac{(v_{B,f})^2 - (v_{B,i})^2}{(v_{A,f})^2 - (v_{A,i})^2}$$

$$= \frac{(60\text{ mph})^2 - (50\text{ mph})^2}{(20\text{ mph})^2 - (10\text{ mph})^2} = 11/3.$$

Thus car *B* has a much greater increase in its kinetic energy than car *A*.

(b) In a reference moving with the speed of car *A* , car *A* increases its speed from rest to 10 mph and car *B* increases its speed from 40 to 50 mph. The ratio is now

$$\frac{\Delta K_B}{\Delta K_A} = \frac{\frac{1}{2}m(v_{B,f})^2 - \frac{1}{2}m(v_{B,0})^2}{\frac{1}{2}m(v_{A,f})^2 - \frac{1}{2}m(v_{A,0})^2} = \frac{(v_{B,f})^2 - (v_{B,0})^2}{(v_{A,f})^2 - (v_{A,0})^2}$$

$$= \frac{(50\text{ mph})^2 - (40\text{ mph})^2}{(10\text{ mph})^2} = 9.$$

The ratio is greater than that found in part a). Note that from the new reference frame both car *A* and car *B* have smaller increases in kinetic energy.

13.3 Kinematics and Kinetic Energy in One Dimension

13.3.1 Constant Accelerated Motion

Let's consider a constant accelerated motion of a *rigid body* in one dimension in which we treat the rigid body as a point mass. Suppose at $t = 0$ the body has an initial x-component of the velocity given by $v_{x,i}$. If the acceleration is in the direction of the displacement of the body then the body will increase its speed. If the acceleration is opposite the direction of the displacement then the acceleration will decrease the body's speed. The displacement of the body is given by

$$\Delta x = v_{x,i}\, t + \frac{1}{2} a_x t^2 . \qquad (13.3.1)$$

The product of acceleration and the displacement is

$$a_x \Delta x = a_x (v_{x,i}\ t + \frac{1}{2} a_x t^2)\,. \tag{13.3.2}$$

The acceleration is given by

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{(v_{x,f} - v_{x,i})}{t}\,. \tag{13.3.3}$$

Therefore

$$a_x \Delta x = \frac{(v_{x,f} - v_{x,i})}{t}\left(v_{x,i}\ t + \frac{1}{2}\frac{(v_{x,f} - v_{x,i})}{t} t^2 \right). \tag{13.3.4}$$

Equation (13.3.4) becomes

$$a_x \Delta x = (v_{x,f} - v_{x,i})(v_{x,i}) + \frac{1}{2}(v_{x,f} - v_{x,i})(v_{x,f} - v_{x,i}) = \frac{1}{2} v_{x,f}{}^2 - \frac{1}{2} v_{x,i}^2\,. \tag{13.3.5}$$

If we multiply each side of Equation (13.3.5) by the mass m of the object this kinematical result takes on an interesting interpretation for the motion of the object. We have

$$m\,a_x \Delta x = \frac{1}{2} m v_{x,f}^2 - m\frac{1}{2} v_{x,i}^2 = K_f - K_i\,. \tag{13.3.6}$$

Recall that for one-dimensional motion, Newton's Second Law is $F_x = m\,a_x$, for the motion considered here, Equation (13.3.6) becomes

$$F_x\,\Delta x = K_f - K_i\,. \tag{13.3.7}$$

13.3.2 Non-constant Accelerated Motion

If the acceleration is not constant, then we can divide the displacement into N intervals indexed by $j = 1$ to N. It will be convenient to denote the displacement intervals by Δx_j, the corresponding time intervals by Δt_j and the x-components of the velocities at the beginning and end of each interval as $v_{x,j-1}$ and $v_{x,j}$. Note that the x-component of the velocity at the beginning and end of the first interval $j = 1$ is then $v_{x,1} = v_{x,i}$ and the velocity at the end of the last interval, $j = N$ is $v_{x,N} = v_{x,j}$. Consider the sum of the products of the average acceleration $(a_{x,j})_{ave}$ and displacement Δx_j in each interval,

$$\sum_{j=1}^{j=N} (a_{x,j})_{\text{ave}} \Delta x_j\,. \tag{13.3.8}$$

The average acceleration over each interval is equal to

$$(a_{x,j})_{\text{ave}} = \frac{\Delta v_{x,j}}{\Delta t_j} = \frac{(v_{x,j+1} - v_{x,j})}{\Delta t_j}, \tag{13.3.9}$$

and so the contribution in each integral can be calculated as above and we have that

$$(a_{x,j})_{\text{ave}}\,\Delta x_j = \frac{1}{2}v_{x,j}{}^2 - \frac{1}{2}v_{x,j-1}{}^2. \tag{13.3.10}$$

When we sum over all the terms only the last and first terms survive, all the other terms cancel in pairs, and we have that

$$\sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\,\Delta x_j = \frac{1}{2}v_{x,f}^2 - \frac{1}{2}v_{x,i}^2. \tag{13.3.11}$$

In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j (both conditions must be met!), the limit of the sum is the definition of the definite integral of the acceleration with respect to the position,

$$\lim_{\substack{N\to\infty \\ \Delta x_j \to 0}} \sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\,\Delta x_j \equiv \int_{x=x_i}^{x=x_f} a_x(x)\,dx. \tag{13.3.12}$$

Therefore In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j, with $v_{x,N} \to v_{x,f}$, Eq. (13.3.11) becomes

$$\int_{x=x_i}^{x=x_f} a_x(x)\,dx = \frac{1}{2}(v_{x,f}^2 - v_{x,i}^2) \tag{13.3.13}$$

This integral result is consequence of the definition that $a_x \equiv dv_x / dt$. The integral in Eq. (13.3.13) is an integral with respect to space, while our previous integral

$$\int_{t=t_i}^{t=t_f} a_x(t)\,dt = v_{x,f} - v_{x,i}. \tag{13.3.14}$$

requires integrating acceleration with respect to time. Multiplying both sides of Eq. (13.3.13) by the mass m yields

$$\int_{x=x_i}^{x=x_f} m a_x(x)\,dx = \frac{1}{2}m(v_{x,f}^2 - v_{x,i}^2) = K_f - K_i. \tag{13.3.15}$$

When we introduce Newton's Second Law in the form $F_x = m a_x$, then Eq. (13.3.15) becomes

$$\int_{x=x_i}^{x=x_f} F_x(x)\,dx = K_f - K_i. \qquad (13.3.16)$$

The integral of the x-component of the force with respect to displacement in Eq. (13.3.16) applies to the motion of a point-like object. For extended bodies, Eq. (13.3.16) applies to the center of mass motion because the external force on a rigid body causes the center of mass to accelerate.

13.4 Work done by Constant Forces

We will begin our discussion of the concept of work by analyzing the motion of an object in one dimension acted on by constant forces. Let's consider the following example: push a cup forward with a constant force along a desktop. When the cup changes velocity (and hence kinetic energy), the sum of the forces acting on the cup must be non-zero according to Newton's Second Law. There are three forces involved in this motion: the applied pushing force $\vec{\mathbf{F}}^a$; the contact force $\vec{\mathbf{C}} \equiv \vec{\mathbf{N}} + \vec{\mathbf{f}}_k$; and gravity $\vec{\mathbf{F}}^g = m\vec{\mathbf{g}}$. The force diagram on the cup is shown in Figure 13.2.

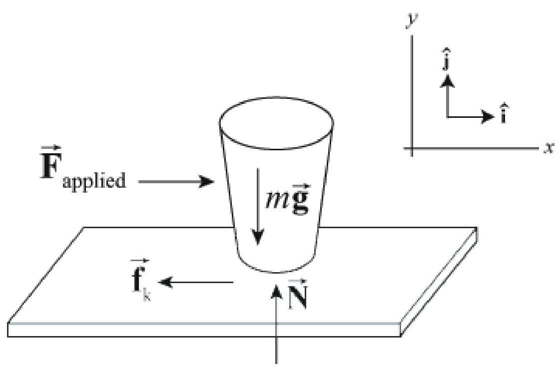

Figure 13.2 Force diagram for cup.

Let's choose our coordinate system so that the $+x$-direction is the direction of the forward motion of the cup. The pushing force can then be described by

$$\vec{\mathbf{F}}^a = F_x^a\,\hat{\mathbf{i}}. \qquad (13.4.1)$$

Suppose a body moves from an initial point x_i to a final point x_f so that the displacement of the point the force acts on is $\Delta x \equiv x_f - x_i$. The ***work done by a constant force*** $\vec{\mathbf{F}}^a = F_x^a\hat{\mathbf{i}}$ acting on the body is the product of the component of the force F_x^a and the displacement Δx,

$$W^a = F_x^a \Delta x\,. \qquad (13.4.2)$$

Work is a scalar quantity; it is not a vector quantity. The SI unit for work is

$$[1\ \text{N}\cdot\text{m}] = [1\ \text{kg}\cdot\text{m}\cdot\text{s}^{-2}][1\ \text{m}] = [1\ \text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}] = [1\ \text{J}]. \qquad (13.4.3)$$

Note that work has the same dimension and the same SI unit as kinetic energy. Because our applied force is along the direction of motion, both $F_x^a > 0$ and $\Delta x > 0$. In this example, the work done is just the product of the magnitude of the applied force and the distance through which that force acts and is positive. In the definition of work done by a force, the force can act at any point on the body. The displacement that appears in Equation (13.4.2) is not the displacement of the body but the *displacement of the point of application of the force*. For point-like objects, the displacement of the point of application of the force is equal to the displacement of the body. However for an extended body, we need to focus on where the force acts and whether or not that point of application undergoes any displacement in the direction of the force as the following example illustrates.

Example 13.2 Work Done by Static Fiction

Suppose you are initially standing and you start walking by pushing against the ground with your feet and your feet do not slip. What is the work done by the static friction force acting on you?

Solution: When you apply a contact force against the ground, the ground applies an equal and opposite contact force on you. The tangential component of this constant force is the force of static friction acting on you. Since your foot is at rest while you are pushing against the ground, there is no displacement of the point of application of this static friction force. Therefore static friction does zero work on you while you are accelerating. You may be surprised by this result but if you think about energy transformation, chemical energy stored in your muscle cells is being transformed into kinetic energy of motion and thermal energy.

When forces are opposing the motion, as in our example of pushing the cup, the kinetic friction force is given by

$$\vec{\mathbf{F}}^f = f_{k,x}\,\hat{\mathbf{i}} = -\mu_k N\,\hat{\mathbf{i}} = -\mu_k mg\,\hat{\mathbf{i}}. \qquad (13.4.4)$$

Here the component of the force is in the opposite direction as the displacement. The work done by the kinetic friction force is negative,

$$W^f = -\mu_k mg\Delta x. \qquad (13.4.5)$$

Since the gravitation force is perpendicular to the motion of the cup, the gravitational force has no component along the line of motion. Therefore the gravitation force does zero work on the cup when the cup is slid forward in the horizontal direction. The normal force is also perpendicular to the motion, and hence does no work.

We see that the pushing force does positive work, the kinetic friction force does negative work, and the gravitation and normal force does zero work.

Example 13.3 Work Done by Force Applied in the Direction of Displacement

Push a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N for a distance of 0.5 m. The coefficient of friction between the table and the cup is $\mu_k = 0.10$. Calculate the work done by the pushing force and the work done by the friction force.

Solution: The work done by the pushing force is

$$W^a = F_x^a \Delta x = (2.0\ \text{N})(0.5\ \text{m}) = 1.0\ \text{J}. \qquad (13.4.6)$$

The work done by the friction force is

$$W^f = -\mu_k mg\Delta x = -(0.1)(0.2\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(0.5\ \text{m}) = -0.10\ \text{J}. \qquad (13.4.7)$$

Example 13.4 Work Done by Force Applied at an Angle to the Direction of Displacement

Suppose we push the cup in the previous example with a force of the same magnitude but at an angle $\theta = 30^o$ upwards with respect to the table. Calculate the work done by the pushing force. Calculate the work done by the kinetic friction force.

Solution: The force diagram on the cup and coordinate system is shown in Figure 13.3.

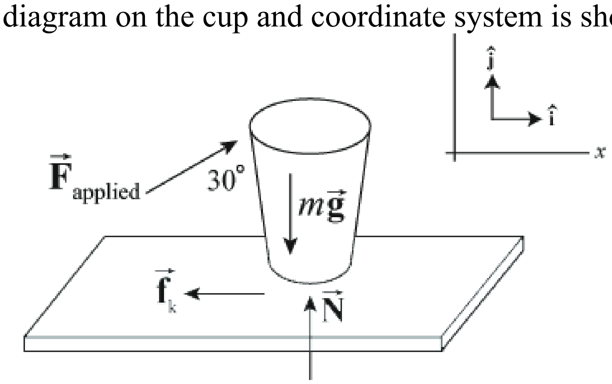

Figure 13.3 Force diagram on cup.

The x-component of the pushing force is now

$$F_x^a = F^a \cos(\theta) = (2.0\text{ N})(\cos(30^\circ)) = 1.7\text{ N}. \tag{13.4.8}$$

The work done by the pushing force is

$$W^a = F_x^a \Delta x = (1.7\text{ N})(0.5\text{ m}) = 8.7\times 10^{-1}\text{ J}. \tag{13.4.9}$$

The kinetic friction force is

$$\vec{\mathbf{F}}^f = -\mu_k N\,\hat{\mathbf{i}}. \tag{13.4.10}$$

In this case, the magnitude of the normal force is not simply the same as the weight of the cup. We need to find the y-component of the applied force,

$$F_y^a = F^a \sin(\theta) = (2.0\text{ N})(\sin(30^o) = 1.0\text{ N}. \tag{13.4.11}$$

To find the normal force, we apply Newton's Second Law in the y-direction,

$$F_y^a + N - mg = 0. \tag{13.4.12}$$

Then the normal force is

$$N = mg - F_y^a = (0.2\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2}) - (1.0\text{ N}) = 9.6\times 10^{-1}\text{ N}. \tag{13.4.13}$$

The work done by the kinetic friction force is

$$W^f = -\mu_k N \Delta x = -(0.1)(9.6\times 10^{-1}\text{ N})(0.5\text{ m}) = 4.8\times 10^{-2}\text{ J}. \tag{13.4.14}$$

Example 13.5 Work done by Gravity Near the Surface of the Earth

Consider a point-like body of mass m near the surface of the earth falling directly towards the center of the earth. The gravitation force between the body and the earth is nearly constant, $\vec{\mathbf{F}}_{grav} = m\vec{\mathbf{g}}$. Let's choose a coordinate system with the origin at the surface of the earth and the $+y$-direction pointing away from the center of the earth Suppose the body starts from an initial point y_i and falls to a final point y_f closer to the earth. How much work does the gravitation force do on the body as it falls?

Solution: The displacement of the body is negative, $\Delta y \equiv y_f - y_i < 0$. The gravitation force is given by

$$\vec{\mathbf{F}}^g = m\vec{\mathbf{g}} = F_y^g\,\hat{\mathbf{j}} = -mg\,\hat{\mathbf{j}}. \tag{13.4.15}$$

The work done on the body is then

$$W^g = F_y^g \Delta y = -mg\Delta y \,. \tag{13.4.16}$$

For a falling body, the displacement of the body is negative, $\Delta y \equiv y_f - y_i < 0$; therefore the work done by gravity is positive, $W^g > 0$. The gravitation force is pointing in the same direction as the displacement of the falling object so the work should be positive.

When an object is rising while under the influence of a gravitation force, $\Delta y \equiv y_f - y_i > 0$. The work done by the gravitation force for a rising body is negative, $W^g < 0$, because the gravitation force is pointing in the opposite direction from that in which the object is displaced.

It's important to note that the choice of the positive direction as being away from the center of the earth ("up") does not make a difference. If the downward direction were chosen positive, the falling body would have a positive displacement and the gravitational force as given in Equation (13.4.15) would have a positive downward component; the product $F_y^g \Delta y$ would still be positive.

13.5 Work done by Non-Constant Forces

Consider a body moving in the x-direction under the influence of a non-constant force in the x-direction, $\vec{\mathbf{F}} = F_x \,\hat{\mathbf{i}}$. The body moves from an initial position x_i to a final position x_f. In order to calculate the work done by a non-constant force, we will divide up the displacement of the point of application of the force into a large number N of small displacements Δx_j where the index j marks the j^{th} displacement and takes integer values from 1 to N. Let $(F_{x,j})_{ave}$ denote the average value of the x-component of the force in the displacement interval $[x_{j-1}, x_j]$. For the j^{th} displacement interval we calculate the contribution to the work

$$W_j = (F_{x,j})_{\text{ave}} \Delta x_j \tag{13.5.1}$$

This contribution is a scalar so we add up these scalar quantities to get the total work

$$W_N = \sum_{j=1}^{j=N} W_j = \sum_{j=1}^{j=N} (F_{x,j})_{\text{ave}} \Delta x_j \,. \tag{13.5.2}$$

The sum in Equation (13.5.2) depends on the number of divisions N and the width of the intervals Δx_j. In order to define a quantity that is independent of the divisions, we take the limit as $N \to \infty$ and $\left|\Delta x_j\right| \to 0$ for all j. The work is then

$$W = \lim_{\substack{N \to \infty \\ |\Delta x_j| \to 0}} \sum_{j=1}^{j=N} (F_{x,j})_{\text{ave}} \Delta x_j = \int_{x=x_i}^{x=x_f} F_x(x)\, dx \tag{13.5.3}$$

This last expression is the definite integral of the x-component of the force with respect to the parameter x. In Figure 13.5 we graph the x-component of the force as a function of the parameter x. The work integral is the area under this curve between $x = x_i$ and $x = x_f$.

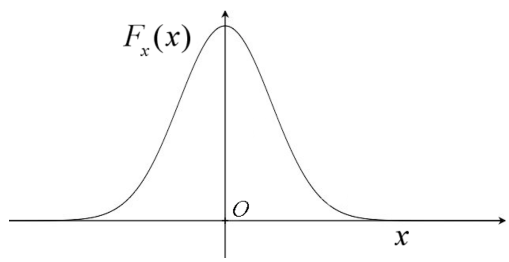

Figure 13.5 Plot of x-component of a sample force $F_x(x)$ as a function of x.

Example 13.6 Work done by the Spring Force

Connect one end of an unstretched spring of length l_0 with spring constant k to an object resting on a smooth frictionless table and fix the other end of the spring to a wall. Choose an origin as shown in the figure. Stretch the spring by an amount x_i and release the object. How much work does the spring do on the object when the spring is stretched by an amount x_f?

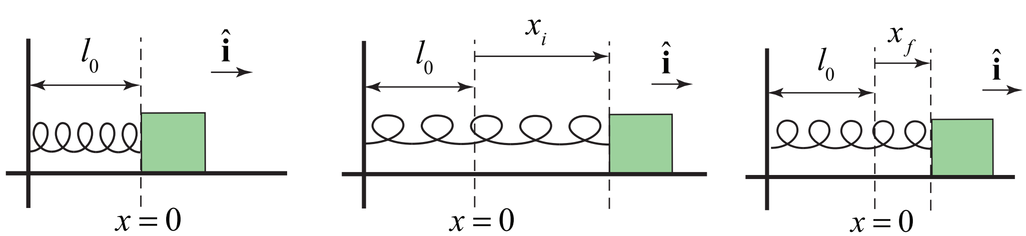

Figure 13.6 Equilibrium, initial and final states for a spring

Solution: We first begin by choosing a coordinate system with our origin located at the position of the object when the spring is unstretched (or uncompressed). We choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched. We choose the coordinate function x to denote the position of the object with respect to the origin. We show the coordinate function and free-body force diagram in the figure below.

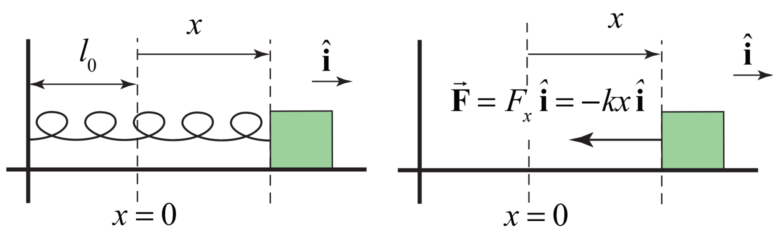

Figure 13.6a Spring force

The spring force on the object is given by (Figure 13.6a)

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -k\,x\,\hat{\mathbf{i}} \qquad (13.5.4)$$

In Figure 13.7 we show the graph of the x-component of the spring force, $F_x(x)$, as a function of x.

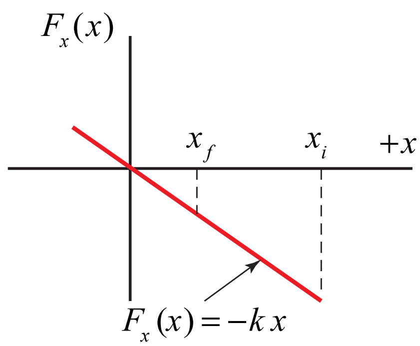

Figure 13.7 Plot of spring force $F_x(x)$ vs. displacement x

The work done is just the area under the curve for the interval x_i to x_f,

$$W = \int_{x'=x_i}^{x'=x_f} F_x(x')\,dx' = \int_{x'=x_i}^{x'=x_f} -kx'\,dx' = -\frac{1}{2}k(x_f^2 - x_i^2) \qquad (13.5.5)$$

This result is independent of the sign of x_i and x_f because both quantities appear as squares. If the spring is less stretched or compressed in the final state than in the initial state, then the absolute value, $|x_f| < |x_i|$, and the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of "greater tension" to a state of "lesser tension."

13.6 Work-Kinetic Energy Theorem

There is a direct connection between the work done on a point-like object and the change in kinetic energy the point-like object undergoes. If the work done on the object is non-zero, this implies that an unbalanced force has acted on the object, and the object will have undergone acceleration. For an object undergoing one-dimensional motion the left hand side of Equation (13.3.16) is the work done on the object by the component of the sum of the forces in the direction of displacement,

$$W = \int_{x=x_i}^{x=x_f} F_x \, dx = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_i^{\,2} = K_f - K_i = \Delta K \qquad (13.6.1)$$

When the work done on an object is positive, the object will increase its speed, and negative work done on an object causes a decrease in speed. When the work done is zero, the object will maintain a constant speed. In fact, the work-energy relationship is quite precise; the work done by the applied force on an object is identically equal to the change in kinetic energy of the object.

Example 13.7 Gravity and the Work-Energy Theorem

Suppose a ball of mass $m = 0.2\ \text{kg}$ starts from rest at a height $y_0 = 15\ \text{m}$ above the surface of the earth and falls down to a height $y_f = 5.0\ \text{m}$ above the surface of the earth. What is the change in the kinetic energy? Find the final velocity using the work-energy theorem.

Solution: As only one force acts on the ball, the change in kinetic energy is the work done by gravity,

$$\begin{aligned} W^g &= -mg(y_f - y_0) \\ &= (-2.0\times10^{-1}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(5\ \text{m} - 15\ \text{m}) = 2.0\times10^{1}\ \text{J}. \end{aligned} \qquad (13.6.2)$$

The ball started from rest, $v_{y,0} = 0$. So the change in kinetic energy is

$$\Delta K = \frac{1}{2} m v_{y,f}^{\;2} - \frac{1}{2} m v_{y,0}^{\;2} = \frac{1}{2} m v_{y,f}^{\;2}. \qquad (13.6.3)$$

We can solve Equation (13.6.3) for the final velocity using Equation (13.6.2)

$$v_{y,f} = \sqrt{\frac{2\Delta K}{m}} = \sqrt{\frac{2W^g}{m}} = \sqrt{\frac{2(2.0\times10^{1}\ \text{J})}{0.2\ \text{kg}}} = 1.4\times10^{1}\ \text{m}\cdot\text{s}^{-1}. \qquad (13.6.4)$$

For the falling ball in a constant gravitation field, the positive work of the gravitation force on the body corresponds to an increasing kinetic energy and speed. For a rising body in the same field, the kinetic energy and hence the speed decrease since the work done is negative.

Example 13.7 Final Kinetic Energy of Moving Cup

A person pushes a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N at an angle of 30° with respect to the horizontal for a distance of 0.5 m as in Example 13.4. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. If the cup was initially at rest, what is the final kinetic energy of the cup after being pushed 0.5 m? What is the final speed of the cup?

Solution: The total work done on the cup is the sum of the work done by the pushing force and the work done by the friction force, as given in Equations (13.4.9) and (13.4.14),

$$\begin{aligned} W &= W^a + W^f = (F_x^a - \mu_k N)(x_f - x_i) \\ &= (1.7\ \text{N} - 9.6\times 10^{-2}\ \text{N})(0.5\ \text{m}) = 8.0\times 10^{-1}\ \text{J} \end{aligned} \quad (13.6.5)$$

The initial velocity is zero so the change in kinetic energy is just

$$\Delta K = \frac{1}{2} m v_{y,f}{}^2 - \frac{1}{2} m v_{y,0}{}^2 = \frac{1}{2} m v_{y,f}{}^2 . \quad (13.6.6)$$

Thus the work-kinetic energy theorem, Eq.(13.6.1)), enables us to solve for the final kinetic energy,

$$K_f = \frac{1}{2} m v_f{}^2 = \Delta K = W = 8.0\times 10^{-1}\ \text{J} . \quad (13.6.7)$$

We can solve for the final speed,

$$v_{y,f} = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(8.0\times 10^{-1}\ \text{J})}{0.2\ \text{kg}}} = 2.9\ \text{m}\cdot\text{s}^{-1} . \quad (13.6.8)$$

13.7 Power Applied by a Constant Force

Suppose that an applied force $\vec{\mathbf{F}}^a$ acts on a body during a time interval Δt, and the displacement of the point of application of the force is in the x-direction by an amount Δx. The work done, ΔW^a, during this interval is

$$\Delta W^a = F_x^a\, \Delta x . \quad (13.7.1)$$

where F_x^a is the x-component of the applied force. (Equation (13.7.1) is the same as Equation (13.4.2).)

The ***average power*** of an applied force is defined to be the rate at which work is done,

$$P_{ave}^{a} = \frac{\Delta W^{a}}{\Delta t} = \frac{F_x^a \, \Delta x}{\Delta t} = F_x^a v_{\text{ave},x} \,. \qquad (13.7.2)$$

The average power delivered to the body is equal to the component of the force in the direction of motion times the component of the average velocity of the body. Power is a scalar quantity and can be positive, zero, or negative depending on the sign of work. The SI units of power are called watts $[\text{W}]$ and $[1\ \text{W}] = [1\ \text{J}\cdot\text{s}^{-1}]$.

The ***instantaneous power*** at time t is defined to be the limit of the average power as the time interval $[t, t+\Delta t]$ approaches zero,

$$P^{a} = \lim_{\Delta t \to 0} \frac{\Delta W^{a}}{\Delta t} = \lim_{\Delta t \to 0} \frac{F_x^a \, \Delta x}{\Delta t} = F_x^a \left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \right) = F_x^a v_x \,. \qquad (13.7.3)$$

The instantaneous power of a constant applied force is the product of the component of the force in the direction of motion and the instantaneous velocity of the moving object.

Example 13.8 Gravitational Power for a Falling Object

Suppose a ball of mass $m = 0.2\ \text{kg}$ starts from rest at a height $y_0 = 15\ \text{m}$ above the surface of the earth and falls down to a height $y_f = 5.0\ \text{m}$ above the surface of the earth. What is the average power exerted by the gravitation force? What is the instantaneous power when the ball is at a height $y_f = 5.0\ \text{m}$ above the surface of the Earth? Make a graph of power vs. time. You may ignore the effects of air resistance.

Solution: There are two ways to solve this problem. Both approaches require calculating the time interval Δt for the ball to fall. Set $t_0 = 0$ for the time the ball was released. We can solve for the time interval $\Delta t = t_f$ that it takes the ball to fall using the equation for a freely falling object that starts from rest,

$$y_f = y_0 - \frac{1}{2} g t_f^{\,2} \,. \qquad (13.7.4)$$

Thus the time interval for falling is

$$t_f = \sqrt{\frac{2}{g}(y_0 - y_f)} = \sqrt{\frac{2}{9.8\ \text{m}\cdot\text{s}^{-2}}(15\ \text{m} - 5\ \text{m})} = 1.4\ \text{s} \,. \qquad (13.7.5)$$

First approach: we can calculate the work done by gravity,

$$\begin{aligned} W^g &= -mg(y_f - y_0) \\ &= (-2.0\times 10^{-1}\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})(5\text{ m} - 15\text{ m}) = 2.0\times 10^{1}\text{ J.} \end{aligned} \tag{13.7.6}$$

Then the average power is

$$P^g_{\text{ave}} = \frac{\Delta W}{\Delta t} = \frac{2.0\times 10^{1}\text{ J}}{1.4\text{ s}} = 1.4\times 10^{1}\text{ W}. \tag{13.7.7}$$

Second Approach. We calculate the gravitation force and the average velocity. The gravitation force is

$$F^g_y = -mg = -(2.0\times 10^{-1}\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2}) = -2.0\text{ N}. \tag{13.7.8}$$

The average velocity is

$$v_{\text{ave},y} = \frac{\Delta y}{\Delta t} = \frac{5\text{ m} - 15\text{ m}}{1.4\text{ s}} = -7.0\text{ m}\cdot\text{s}^{-1}. \tag{13.7.9}$$

The average power is therefore

$$\begin{aligned} P^g_{\text{ave}} &= F^g_y\, v_{\text{ave},y} = (-mg)v_{\text{ave},y} \\ &= (-2.0\text{ N})(-7.0\text{ m}\cdot\text{s}^{-1}) = 1.4\times 10^{1}\text{ W.} \end{aligned} \tag{13.7.10}$$

In order to find the instantaneous power at any time, we need to find the instantaneous velocity at that time. The ball takes a time $t_f = 1.4\text{ s}$ to reach the height $y_f = 5.0\text{ m}$. The velocity at that height is given by

$$v_y = -gt_f = -(9.8\text{ m}\cdot\text{s}^{-2})(1.4\text{ s}) = -1.4\times 10^{1}\text{ m}\cdot\text{s}^{-1}. \tag{13.7.11}$$

So the instantaneous power at time $t_f = 1.4\text{ s}$ is

$$\begin{aligned} P^g &= F^g_y\, v_y = (-mg)(-gt_f) = mg^2 t_f \\ &= (0.2\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})^2(1.4\text{ s}) = 2.7\times 10^{1}\text{ W} \end{aligned} \tag{13.7.12}$$

If this problem were done symbolically, the answers given in Equation (13.7.11) and Equation (13.7.12) would differ by a factor of two; the answers have been rounded to two significant figures.

The instantaneous power grows linearly with time. The graph of power vs. time is shown in Figure 13.8. From the figure, it should be seen that the instantaneous power at any time is twice the average power between $t = 0$ and that time.

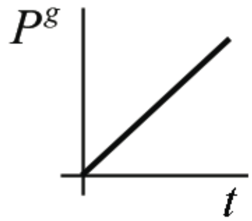

Figure 13.8 Graph of power vs. time

Example 13.9 Power Pushing a Cup

A person pushes a cup of mass $0.2\ \text{kg}$ along a horizontal table with a force of magnitude $2.0\ \text{N}$ at an angle of 30^o with respect to the horizontal for a distance of $0.5\ \text{m}$, as in Example 13.4. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. What is the average power of the pushing force? What is the average power of the kinetic friction force?

Solution: We will use the results from Examples 13.4 and 13.7 but keeping extra significant figures in the intermediate calculations. The work done by the pushing force is

$$W^a = F_x^a (x_f - x_0) = (1.732\ \text{N})(0.50\ \text{m}) = 8.660 \times 10^{-1}\ \text{J}. \qquad (13.7.13)$$

The final speed of the cup is $v_{x,f} = 2.860\ \text{m} \cdot \text{s}^{-1}$. Assuming constant acceleration, the time during which the cup was pushed is

$$t_f = \frac{2(x_f - x_0)}{v_{x,f}} = 0.3496\ \text{s}. \qquad (13.7.14)$$

The average power of the pushing force is then, with $\Delta t = t_f$,

$$P^a_{\text{ave}} = \frac{\Delta W^a}{\Delta t} = \frac{8.660 \times 10^{-1}\ \text{J}}{0.3496\ \text{s}} = 2.340\ \text{W}, \qquad (13.7.15)$$

or $2.3\ \text{W}$ to two significant figures. The work done by the friction force is

$$\begin{aligned} W^f &= f_{\text{k}}(x_f - x_0) \\ &= -\mu_{\text{k}} N(x_f - x_0) = -(9.6 \times 10^{-2}\ \text{N})(0.50\ \text{m}) = -(4.8 \times 10^{-2}\ \text{J}). \end{aligned} \qquad (13.7.16)$$

The average power of kinetic friction is

$$P^f_{\text{ave}} = \frac{\Delta W^f}{\Delta t} = \frac{-4.8\times 10^{-2}\text{ J}}{0.3496\text{ s}} = -1.4\times 10^{-1}\text{ W}. \tag{13.7.17}$$

The time rate of change of the kinetic energy for a body of mass m moving in the x-direction is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}m v_x^{\,2}\right) = m\frac{dv_x}{dt}v_x = m\,a_x v_x. \tag{13.7.18}$$

By Newton's Second Law, $F_x = m\,a_x$, and so Equation (13.7.18) becomes

$$\frac{dK}{dt} = F_x v_x = P. \tag{13.7.19}$$

The instantaneous power delivered to the body is equal to the time rate of change of the kinetic energy of the body.

13.8 Work and the Scalar Product

We shall introduce a vector operation, called the **scalar product** or "dot product" that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the scalar product between the force and the displacement vectors.

13.8.1 Scalar Product

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Because any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 13.9. Note that θ can vary from 0 to π.

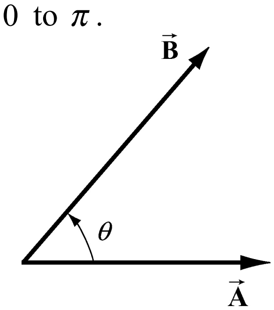

Figure 13.9 Scalar product geometry.

*The **scalar product** $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:*

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos(\theta)\,, \qquad (13.8.1)$$

where $A = |\vec{\mathbf{A}}|$ and $B = |\vec{\mathbf{B}}|$ represent the magnitude of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively. The scalar product can be positive, zero, or negative, depending on the value of $\cos\theta$. The scalar product is always a scalar quantity.

The angle formed by two vectors is therefore

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|}\right). \qquad (13.8.2)$$

The magnitude of a vector $\vec{\mathbf{A}}$ is given by the square root of the scalar product of the vector $\vec{\mathbf{A}}$ with itself.

$$|\vec{\mathbf{A}}| = (\vec{\mathbf{A}} \cdot \vec{\mathbf{A}})^{1/2}. \qquad (13.8.3)$$

We can give a geometric interpretation to the scalar product by writing the definition as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A\cos(\theta))\,B\,. \qquad (13.8.4)$$

In this formulation, the term $A\cos\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{B}}$. This projection is shown in Figure 13.10a. So the scalar product is the product of the projection of the length of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$ with the length of $\vec{\mathbf{B}}$. Note that we could also write the scalar product as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A(B\cos(\theta))\,. \qquad (13.8.5)$$

Now the term $B\cos(\theta)$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{A}}$ as shown in Figure 13.10b. From this perspective, the scalar product is the product of the projection of the length of $\vec{\mathbf{B}}$ in the direction of $\vec{\mathbf{A}}$ with the length of $\vec{\mathbf{A}}$.

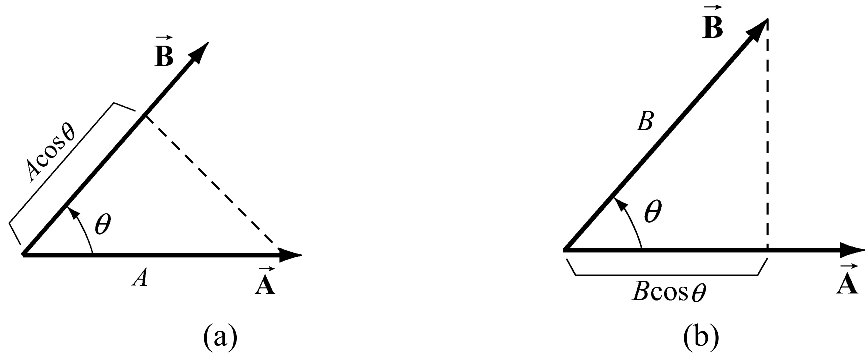

Figure 13.10 (a) and (b) Projection of vectors and the scalar product

From our definition of the scalar product we see that the scalar product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi/2$ and $\cos(\pi/2) = 0$.

We can calculate the scalar product between two vectors in a Cartesian coordinates system as follows. Consider two vectors $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}$. Recall that

$$\begin{aligned} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} &= \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} &= \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = 0. \end{aligned} \tag{13.8.6}$$

The scalar product between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is then

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = A_x\,B_x + A_y\,B_y + A_z\,B_z\,. \tag{13.8.7}$$

The time derivative of the scalar product of two vectors is given by

$$\begin{aligned} &\frac{d}{dt}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) = \frac{d}{dt}(A_x\,B_x + A_y\,B_y + A_z\,B_z) \\ &= \frac{d}{dt}(A_x)B_x + \frac{d}{dt}(A_y)B_y + \frac{d}{dt}(A_z)B_z + A_x\frac{d}{dt}(B_x) + A_y\frac{d}{dt}(B_y) + A_z\frac{d}{dt}(B_z) \\ &= \left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{B}} + \vec{\mathbf{A}}\cdot\left(\frac{d}{dt}\vec{\mathbf{B}}\right). \end{aligned} \tag{13.8.8}$$

In particular when $\vec{\mathbf{A}} = \vec{\mathbf{B}}$, then the time derivative of the square of the magnitude of the vector $\vec{\mathbf{A}}$ is given by

$$\frac{d}{dt}A^2 = \frac{d}{dt}(\vec{\mathbf{A}}\cdot\vec{\mathbf{A}}) = \left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{A}} + \vec{\mathbf{A}}\cdot\left(\frac{d}{dt}\vec{\mathbf{A}}\right) = 2\left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{A}}\,. \tag{13.8.9}$$

13.8.2 Kinetic Energy and the Scalar Product

For an object undergoing three-dimensional motion, the velocity of the object in Cartesian components is given by $\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$. Recall that the magnitude of a vector is given by the square root of the scalar product of the vector with itself,

$$A \equiv \left|\vec{\mathbf{A}}\right| \equiv (\vec{\mathbf{A}}\cdot\vec{\mathbf{A}})^{1/2} = (A_x^2 + A_y^2 + A_z^2)^{1/2}\,. \tag{13.8.10}$$

Therefore the square of the magnitude of the velocity is given by the expression

$$v^2 \equiv (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = v_x^2 + v_y^2 + v_z^2 . \qquad (13.8.11)$$

Hence the kinetic energy of the object is given by

$$K = \frac{1}{2} m(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \frac{1}{2} m(v_x^2 + v_y^2 + v_z^2) . \qquad (13.8.12)$$

13.8.2 Work and the Scalar Product

Work is an important physical example of the mathematical operation of taking the scalar product between two vectors. Recall that when a constant force acts on a body and the point of application of the force undergoes a displacement along the x-axis, only the component of the force along that direction contributes to the work,

$$W = F_x \Delta x . \qquad (13.8.13)$$

Suppose we are pulling a body along a horizontal surface with a force $\vec{\mathbf{F}}$. Choose coordinates such that horizontal direction is the x-axis and the force $\vec{\mathbf{F}}$ forms an angle β with the positive x-direction. In Figure 13.11 we show the force vector $\vec{\mathbf{F}} = F_x \,\hat{\mathbf{i}} + F_y \,\hat{\mathbf{j}}$ and the displacement vector of the point of application of the force $\Delta\vec{\mathbf{x}} = \Delta x \,\hat{\mathbf{i}}$. Note that $\Delta\vec{\mathbf{x}} = \Delta x \,\hat{\mathbf{i}}$ is the component of the displacement and hence can be greater, equal, or less than zero (but is shown as greater than zero in the figure for clarity). The scalar product between the force vector $\vec{\mathbf{F}}$ and the displacement vector $\Delta\vec{\mathbf{x}}$ is

$$\vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F_x \,\hat{\mathbf{i}} + F_y \,\hat{\mathbf{j}}) \cdot (\Delta x \,\hat{\mathbf{i}}) = F_x \,\Delta x . \qquad (13.8.14)$$

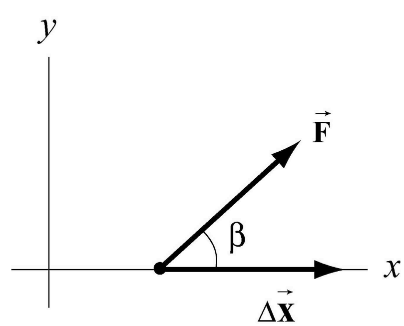

Figure 13.11 Force and displacement vectors

The work done by the force is then

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} . \qquad (13.8.15)$$

In general, the angle β takes values within the range $-\pi \le \beta \le \pi$ (in Figure 13.11, $0 \le \beta \le \pi / 2$). Because the x-component of the force is $F_x = F\cos(\beta)$ where $F = |\vec{\mathbf{F}}|$ denotes the magnitude of $\vec{\mathbf{F}}$, the work done by the force is

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F\cos(\beta))\Delta x \,. \tag{13.8.16}$$

Example 13.10 Object Sliding Down an Inclined Plane

An object of mass $m = 4.0\,kg$, starting from rest, slides down an inclined plane of length $l = 3.0\,m$. The plane is inclined by an angle of $\theta = 30^0$ to the ground. The coefficient of kinetic friction is $\mu_k = 0.2$. (a) What is the work done by each of the three forces while the object is sliding down the inclined plane? (b) For each force, is the work done by the force positive or negative? (c) What is the sum of the work done by the three forces? Is this positive or negative?

Solution: (a) and (b) Choose a coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, and the positive y-direction pointing towards the upper right as shown in Figure 13.12. While the object is sliding down the inclined plane, three uniform forces act on the object, the gravitational force which points downward and has magnitude $F_g = mg$, the normal force N which is perpendicular to the surface of the inclined plane, and the friction force which opposes the motion and is equal in magnitude to $f_k = \mu_k N$. A force diagram on the object is shown in Figure 13.13.

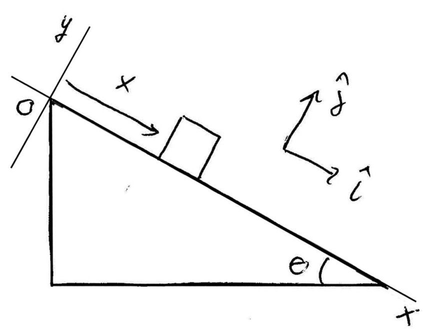

Figure 13.12 Coordinate system for object sliding down inclined plane

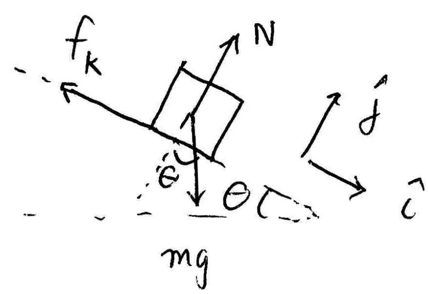

Figure 13.13 Free-body force diagram for object

In order to calculate the work we need to determine which forces have a component in the direction of the displacement. Only the component of the gravitational force along the positive x-direction $F_{gx} = mg\sin\theta$ and the friction force are directed along the displacement and therefore contribute to the work. We need to use Newton's Second Law to determine the magnitudes of the normal force. Because the object is constrained to move along the positive x-direction, $a_y = 0$, Newton's Second Law in the $\hat{\mathbf{j}}$-direction $N - mg\cos\theta = 0$. Therefore $N = mg\cos\theta$ and the magnitude of the friction force is $f_k = \mu_k mg\cos\theta$.

With our choice of coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, the displacement of the object is given by the vector $\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\mathbf{i}}$ (Figure 13.14).

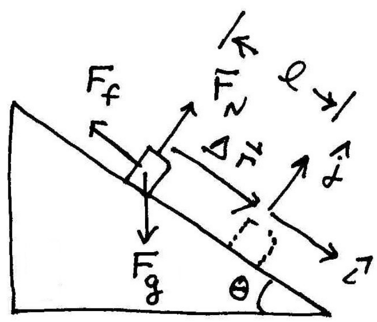

Figure 13.14 Force vectors and displacement vector for object

The vector decomposition of the three forces are $\vec{\mathbf{F}}^g = mg\sin\theta\,\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}}$, $\vec{\mathbf{F}}^f = -\mu_k mg\cos\theta\hat{\mathbf{i}}$, and $\vec{\mathbf{F}}^N = mg\cos\theta\hat{\mathbf{j}}$. The work done by the normal force is zero because the normal force is perpendicular the displacement

$$W^N = \vec{\mathbf{F}}^N \cdot \Delta\vec{\mathbf{r}} = mg\cos\theta\hat{\mathbf{j}} \cdot l\,\hat{\mathbf{i}} = 0\,.$$

Then the work done by the friction force is negative and given by

$$W^f = \vec{\mathbf{F}}^f \cdot \Delta\vec{\mathbf{r}} = -\mu_k mg\cos\theta\hat{\mathbf{i}} \cdot l\,\hat{\mathbf{i}} = -\mu_k mg\cos\theta l < 0\,.$$

Substituting in the appropriate values yields

$$W^f = -\mu_k mg\cos\theta l = -(0.2)(4.0\,\text{kg})(9.8\text{m}\cdot\text{s}^{-2})(3.0\,\text{m})(\cos(30^o)(3.0\,\text{m}) = -20.4\text{ J}\,.$$

The work done by the gravitational force is positive and given by

$$W^g = \vec{\mathbf{F}}^g \cdot \Delta\vec{\mathbf{r}} = (mg\sin\theta\,\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}}) \cdot l\,\hat{\mathbf{i}} = mgl\sin\theta > 0\,.$$

Substituting in the appropriate values yields

$$W^g = mgl\sin\theta = (4.0\,\text{kg})(9.8\text{ m}\cdot\text{s}^{-2})(3.0\,\text{m})(\sin(30^o) = 58.8\text{ J}\,.$$

(c) The scalar sum of the work done by the three forces is then

$$W = W^g + W^f = mgl(\sin\theta - \mu_k \cos\theta)$$
$$W = (4.0\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2})(3.0\,\text{m})(\sin(30^o) - (0.2)(\cos(30^o)) = 38.4\ \text{J}.$$

13.9 Work done by a Non-Constant Force Along an Arbitrary Path

Suppose that a non-constant force $\vec{\mathbf{F}}$ acts on a point-like body of mass m while the body is moving on a three dimensional curved path. The position vector of the body at time t with respect to a choice of origin is $\vec{\mathbf{r}}(t)$. In Figure 13.15 we show the orbit of the body for a time interval $[t_i, t_f]$ moving from an initial position $\vec{\mathbf{r}}_i \equiv \vec{\mathbf{r}}(t = t_i)$ at time $t = t_i$ to a final position $\vec{\mathbf{r}}_f \equiv \vec{\mathbf{r}}(t = t_f)$ at time $t = t_f$.

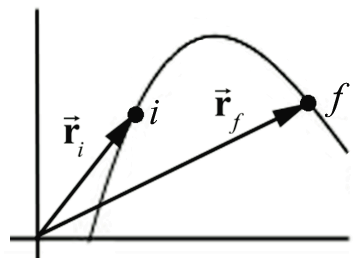

Figure 13.15 Path traced by the motion of a body.

We divide the time interval $[t_i, t_f]$ into N smaller intervals with $[t_{j-1}, t_j]$, $j = 1, \cdots, N$ with $t_N = t_f$. Consider two position vectors $\vec{\mathbf{r}}_j \equiv \vec{\mathbf{r}}(t = t_j)$ and $\vec{\mathbf{r}}_{j-1} \equiv \vec{\mathbf{r}}(t = t_{j-1})$ the displacement vector during the corresponding time interval as $\Delta\vec{\mathbf{r}}_j = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_{j-1}$. Let $\vec{\mathbf{F}}$ denote the force acting on the body during the interval $[t_{j-1}, t_j]$. The average force in this interval is $(\vec{\mathbf{F}}_j)_{\text{ave}}$ and the average work ΔW_j done by the force during the time interval $[t_{j-1}, t_j]$ is the scalar product between the average force vector and the displacement vector,

$$\Delta W_j = (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j. \qquad (13.8.17)$$

The force and the displacement vectors for the time interval $[t_{j-1}, t_j]$ are shown in Figure 13.16 (note that the subscript "ave" on $(\vec{\mathbf{F}}_j)_{\text{ave}}$ has been suppressed).

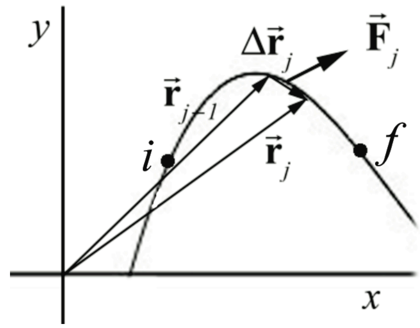

Figure 13.16 An infinitesimal work element.

We calculate the work by adding these scalar contributions to the work for each interval $[t_{j-1}, t_j]$, for $j = 1$ to N,

$$W_N = \sum_{j=1}^{j=N} \Delta W_j = \sum_{j=1}^{j=N} (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j \,. \tag{13.8.18}$$

We would like to define work in a manner that is independent of the way we divide the interval, so we take the limit as $N \to \infty$ and $\left|\Delta\vec{\mathbf{r}}_j\right| \to 0$ for all j. In this limit, as the intervals become smaller and smaller, the distinction between the average force and the actual force vanishes. Thus if this limit exists and is well defined, then the work done by the force is

$$W = \lim_{\substack{N\to\infty \\ \left|\Delta\vec{\mathbf{r}}_j\right|\to 0}} \sum_{j=1}^{j=N} (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \,. \tag{13.8.19}$$

Notice that this summation involves adding scalar quantities. This limit is called the ***line integral*** of the force $\vec{\mathbf{F}}$. The symbol $d\vec{\mathbf{r}}$ is called the ***infinitesimal vector line element***. At time t, $d\vec{\mathbf{r}}$ is tangent to the orbit of the body and is the limit of the displacement vector $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)$ as Δt approaches zero. In this limit, the parameter t does not appear in the expression in Equation (13.8.19).

In general this line integral depends on the particular path the body takes between the initial position $\vec{\mathbf{r}}_i$ and the final position $\vec{\mathbf{r}}_f$, which matters when the force $\vec{\mathbf{F}}$ is non-constant in space, and when the contribution to the work can vary over different paths in space. We can represent the integral in Equation (13.8.19) explicitly in a coordinate system by specifying the infinitesimal vector line element $d\vec{\mathbf{r}}$ and then explicitly computing the scalar product.

13.9.1 Work Integral in Cartesian Coordinates

In Cartesian coordinates the line element is

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}\,, \tag{13.8.20}$$

where dx, dy, and dz represent arbitrary displacements in the $\hat{\mathbf{i}}$-, $\hat{\mathbf{j}}$-, and $\hat{\mathbf{k}}$-directions respectively as seen in Figure 13.17.

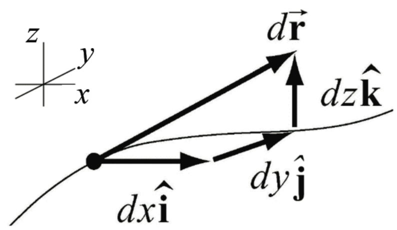

Figure 13.17 A line element in Cartesian coordinates.

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}\,. \tag{13.8.21}$$

The infinitesimal work is the sum of the work done by the component of the force times the component of the displacement in each direction,

$$dW = F_x dx + F_y dy + F_z dz\,. \tag{13.8.22}$$

Eq. (13.8.22) is just the scalar product

$$\begin{aligned} dW &= \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}})\cdot(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}) \\ &= F_x dx + F_y dy + F_z dz \end{aligned}\,, \tag{13.8.23}$$

The work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} (F_x dx + F_y dy + F_z dz) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_x dx + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_y dy + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz\,. \tag{13.8.24}$$

13.9.2 Work Integral in Cylindrical Coordinates

In cylindrical coordinates the line element is

$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{k}}\,, \tag{13.8.25}$$

where dr, $rd\theta$, and dz represent arbitrary displacements in the $\hat{\mathbf{r}}$-, $\hat{\boldsymbol{\theta}}$-, and $\hat{\mathbf{k}}$-directions respectively as seen in Figure 13.18.

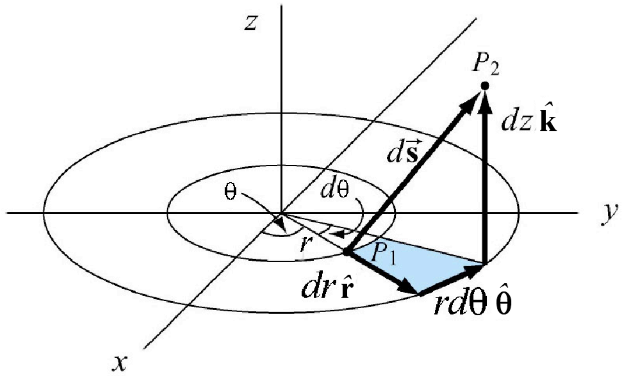

Figure 13.18 Displacement vector $d\,\vec{\mathbf{s}}$ between two points

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_z\,\hat{\mathbf{k}}\,. \tag{13.8.26}$$

The infinitesimal work is the scalar product

$$\begin{aligned} dW &= \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = (F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_z\,\hat{\mathbf{k}})\cdot(dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{k}}) \\ &= F_r dr + F_\theta r d\theta + F_z dz. \end{aligned} \tag{13.8.27}$$

The work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} (F_r dr + F_\theta r d\theta + F_z dz) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_r dr + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_\theta r d\theta + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz\,. \tag{13.8.28}$$

13.10 Worked Examples

Example 13.11 Work Done in a Constant Gravitation Field

The work done in a uniform gravitation field is a fairly straightforward calculation when the body moves in the direction of the field. Suppose the body is moving under the influence of gravity, $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$ along a parabolic curve. The body begins at the point (x_0, y_0) and ends at the point (x_f, y_f). What is the work done by the gravitation force on the body?

Solution: The infinitesimal line element $d\vec{\mathbf{r}}$ is therefore

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}\,. \tag{13.9.1}$$

The scalar product that appears in the line integral can now be calculated,

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -mg\,\hat{\mathbf{j}} \cdot [dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}] = -mgdy\,. \qquad (13.9.2)$$

This result is not surprising since the force is only in the y-direction. Therefore the only non-zero contribution to the work integral is in the y-direction, with the result that

$$W = \int_{\mathrm{r}_0}^{\mathrm{r}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{y=y_0}^{y=y_f} F_y dy = \int_{y=y_0}^{y=y_f} -mgdy = -mg(y_f - y_0)\,. \qquad (13.9.3)$$

In this case of a constant force, the work integral is independent of path.

Example 13.12 Hooke's Law Spring-Body System

Consider a spring-body system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to a body of mass m (Figure 13.19). Calculate the work done by the spring force on body as the body moves from some initial position to some final position.

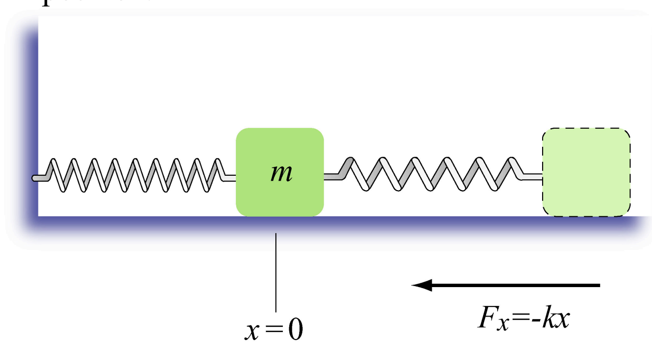

Figure 13.19 A spring-body system.

Solution: Choose the origin at the position of the center of the body when the spring is relaxed (the equilibrium position). Let x be the displacement of the body from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the body moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on the body is then given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}\,. \qquad (13.9.4)$$

The work done by the spring force on the mass is

$$W_{\text{spring}} = \int_{x=x_0}^{x=x_f} (-kx)\,dx = -\frac{1}{2}k(x_f^{\,2} - x_0^{\,2})\,. \qquad (13.9.5)$$

Example 13.13 Work done by the Inverse Square Gravitation Force

Consider a body of mass m in moving in a fixed orbital plane about the sun. The mass of the sun is m_s. How much work does the gravitation interaction between the sun and the body done on the body during this motion?

Solution: Let's assume that the sun is fixed and choose a polar coordinate system with the origin at the center of the sun. Initially the body is at a distance r_0 from the center of the sun. In the final configuration the body has moved to a distance $r_f < r_0$ from the center of the sun. The infinitesimal displacement of the body is given by $d\vec{\mathbf{r}} = dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}}$. The gravitation force between the sun and the body is given by

$$\vec{\mathbf{F}}_{grav} = F_{grav}\ \hat{\mathbf{r}} = -\frac{Gm_s m}{r^2}\hat{\mathbf{r}}. \tag{13.9.6}$$

The infinitesimal work done work done by this gravitation force on the body is given by

$$dW = \vec{\mathbf{F}}_{grav} \cdot d\vec{\mathbf{r}} = (F_{grav,r}\ \hat{\mathbf{r}}) \cdot (dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}}) = F_{grav,r} dr\ . \tag{13.9.7}$$

Therefore the work done on the object as the object moves from r_i to r_f is given by the integral

$$W = \int_{r_i}^{r_f} \vec{\mathbf{F}}_{grav} \cdot d\vec{\mathbf{r}} = \int_{r_i}^{r_f} F_{grav,r}\ dr = \int_{r_i}^{r_f} \left(-\frac{Gm_{\text{sun}} m}{r^2} \right) dr\ . \tag{13.9.8}$$

Upon evaluation of this integral, we have for the work

$$W = \int_{r_i}^{r_f} \left(-\frac{Gm_{\text{sun}} m}{r^2} \right) dr = \left. \frac{Gm_{\text{sun}} m}{r} \right|_{r_i}^{r_f} = Gm_{\text{sun}} m \left(\frac{1}{r_f} - \frac{1}{r_i} \right). \tag{13.9.9}$$

Because the body has moved closer to the sun, $r_f < r_i$, hence $1/r_f > 1/r_i$. Thus the work done by gravitation force between the sun and the body, on the body is positive,

$$W = Gm_{\text{sun}} m \left(\frac{1}{r_f} - \frac{1}{r_i} \right) > 0 \tag{13.9.10}$$

We expect this result because the gravitation force points along the inward radial direction, so the scalar product and hence work of the force and the displacement is positive when the body moves closer to the sun. Also we expect that the sign of the work is the same for a body moving closer to the sun as a body falling towards the earth in a constant gravitation field, as seen in Example 4.7.1 above.

Example 13.14 Work Done by the Inverse Square Electrical Force

Let's consider two point-like bodies, body 1 and body 2, with charges q_1 and q_2 respectively interacting via the electric force alone. Body 1 is fixed in place while body 2 is free to move in an orbital plane. How much work does the electric force do on the body 2 during this motion?

Solution: The calculation in nearly identical to the calculation of work done by the gravitational inverse square force in Example 13.13. The most significant difference is that the electric force can be either attractive or repulsive while the gravitation force is always attractive. Once again we choose polar coordinates centered on body 2 in the plane of the orbit. Initially a distance r_0 separates the bodies and in the final state a distance r_f separates the bodies. The electric force between the bodies is given by

$$\vec{\mathbf{F}}_{\text{elec}} = F_{\text{elec}}\,\hat{\mathbf{r}} = F_{elec,r}\,\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}}\,. \tag{13.9.11}$$

The work done by this electric force on the body 2 is given by the integral

$$W = \int_{r_i}^{r_f} \vec{\mathbf{F}}_{elec}\cdot d\vec{\mathbf{r}} = \int_{r_i}^{r_f} F_{elec,r}\,dr = \frac{1}{4\pi\varepsilon_0}\int_{r_i}^{r_f}\frac{q_1 q_2}{r^2}\,dr\,. \tag{13.9.12}$$

Evaluating this integral, we have for the work done by the electric force

$$W = \int_{r_i}^{r_f}\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,dr = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\Bigg|_{r_i}^{r_f} = -\frac{1}{4\pi\varepsilon_0}q_1 q_2\left(\frac{1}{r_f}-\frac{1}{r_i}\right). \tag{13.9.13}$$

If the charges have opposite signs, $q_1 q_2 < 0$, we expect that the body 2 will move closer to body 1 so $r_f < r_i$, and $1/r_f > 1/r_i$. From our result for the work, the work done by electrical force in moving body 2 is positive,

$$W = -\frac{1}{4\pi\varepsilon_0}q_1 q_2\left(\frac{1}{r_f}-\frac{1}{r_i}\right) > 0\,. \tag{13.9.14}$$

Once again we see that bodies under the influence of electric forces only will naturally move in the directions in which the force does positive work. If the charges have the same sign, then $q_1 q_2 > 0$. They will repel with $r_f > r_i$ and $1/r_f < 1/r_i$. Thus the work is once again positive:

$$W = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \left(\frac{1}{r_f} - \frac{1}{r_i} \right) > 0 . \qquad (13.9.15)$$

13.11 Work-Kinetic Energy Theorem in Three Dimensions

Recall our mathematical result that for one-dimensional motion

$$m\int_i^f a_x \, dx = m\int_i^f \frac{dv_x}{dt} dx = m\int_i^f dv_x \frac{dx}{dt} = m\int_i^f v_x \, dv_x = \frac{1}{2} m v_{x,f}^2 - \frac{1}{2} m v_{x,i}^2 . \qquad (13.11.1)$$

Using Newton's Second Law in the form $F_x = m a_x$, we concluded that

$$\int_i^f F_x \, dx = \frac{1}{2} m v_{x,f}^2 - \frac{1}{2} m v_{x,i}^2 . \qquad (13.11.2)$$

Eq. (13.11.2) generalizes to the y- and z-directions:

$$\int_i^f F_y \, dy = \frac{1}{2} m v_{y,f}^2 - \frac{1}{2} m v_{y,i}^2 , \qquad (13.11.3)$$

$$\int_i^f F_z \, dz = \frac{1}{2} m v_{z,f}^2 - \frac{1}{2} m v_{z,i}^2 . \qquad (13.11.4)$$

Adding Eqs. (13.11.2), (13.11.3), and (13.11.4) yields

$$\int_i^f (F_x \, dx + F_y \, dy + F_z \, dz) = \frac{1}{2} m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2) . \qquad (13.11.5)$$

Recall (Eq. (13.8.24)) that the left hand side of Eq. (13.11.5) is the work done by the force $\vec{\mathbf{F}}$ on the object

$$W = \int_i^f dW = \int_i^f (F_x \, dx + F_y \, dy + F_z \, dz) = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad (13.11.6)$$

The right hand side of Eq. (13.11.5) is the change in kinetic energy of the object

$$\Delta K \equiv K_f - K_i = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_0^{\,2} = \frac{1}{2} m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2) . \qquad (13.11.7)$$

Therefore Eq. (13.11.5) is the three dimensional generalization of the work-kinetic energy theorem

$$\int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = K_f - K_i. \tag{13.11.8}$$

When the work done on an object is positive, the object will increase its speed, and negative work done on an object causes a decrease in speed. When the work done is zero, the object will maintain a constant speed.

13.11.1 Instantaneous Power Applied by a Non-Constant Force for Three Dimensional Motion

Recall that for one-dimensional motion, the *instantaneous power* at time t is defined to be the limit of the average power as the time interval $[t, t + \Delta t]$ approaches zero,

$$P(t) = F_x^a(t) v_x(t). \tag{13.11.9}$$

A more general result for the instantaneous power is found by using the expression for dW as given in Equation (13.8.23),

$$P = \frac{dW}{dt} = \frac{\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}. \tag{13.11.10}$$

The time rate of change of the kinetic energy for a body of mass m is equal to the power,

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \right) = m \frac{d\vec{\mathbf{v}}}{dt} \cdot \vec{\mathbf{v}} = m \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = P. \tag{13.11.11}$$

where the we used Eq. (13.8.9), Newton's Second Law and Eq. (13.11.10).

Appendix 13A Work Done on a System of Two Particles

We shall show that the work done by an internal force in changing a system of two particles of masses m_1 and m_2 respectively from an initial state A to a final state B is equal to

$$W_{\mathrm{c}} = \frac{1}{2}\mu(v_B^2 - v_A^2) \tag{13.1.1}$$

where v_B^2 is the square of the relative velocity in state B, v_A^2 is the square of the relative velocity in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$.

Consider two bodies 1 and 2 and an interaction pair of forces shown in Figure 13A.1.

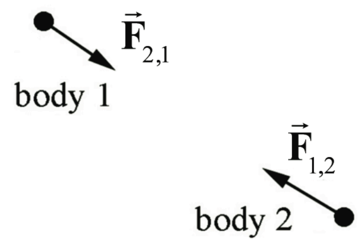

Figure 13A.1 System of two bodies interacting

We choose a coordinate system shown in Figure 13A.2.

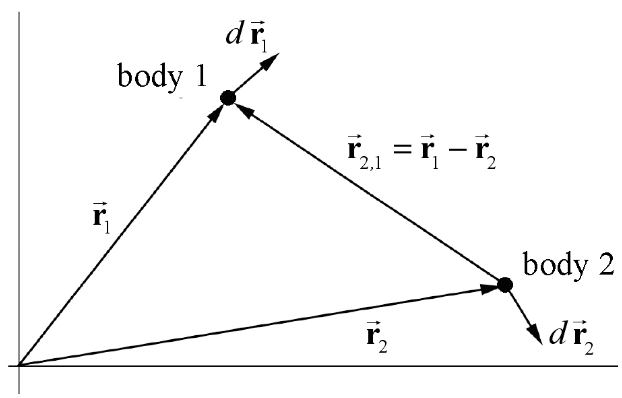

Figure 13A.2 Coordinate system for two-body interaction

Newton's Second Law applied to body 1 is

$$\vec{\mathbf{F}}_{2,1} = m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2} \tag{13.1.2}$$

and applied to body 2 is

$$\vec{\mathbf{F}}_{1,2} = m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2} . \qquad (13.1.3)$$

Divide each side of Equation (13.1.2) by m_1,

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} \qquad (13.1.4)$$

and divide each side of Equation (13.1.3) by m_2,

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2} . \qquad (13.1.5)$$

Subtract Equation (13.1.5) from Equation (13.1.4) yielding

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} - \frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} , \qquad (13.1.6)$$

where $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. Use Newton's Third Law, $\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}$ on the left hand side of Equation (13.1.6) to obtain

$$\vec{\mathbf{F}}_{2,1}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} . \qquad (13.1.7)$$

The quantity $d^2\vec{\mathbf{r}}_{1,2} / dt^2$ is the *relative acceleration* of body 1 with respect to body 2. Define

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2} . \qquad (13.1.8)$$

The quantity μ is known as the ***reduced mass*** of the system. Equation (13.1.7) now takes the form

$$\vec{\mathbf{F}}_{2,1} = \mu \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} . \qquad (13.1.9)$$

The work done in the system in displacing the two masses from an initial state A to a final state B is given by

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 . \qquad (13.1.10)$$

Recall by the work energy theorem that the LHS is the work done on the system,

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 = \Delta K . \qquad (13.1.11)$$

From Newton's Third Law, the sum in Equation (13.1.10) becomes

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot (d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2) = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} , \qquad (13.1.12)$$

where $d\vec{\mathbf{r}}_{2,1}$ is the relative displacement of the two bodies. We can now substitute Newton's Second Law, Equation (13.1.9), for the relative acceleration into Equation (13.1.12),

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = \int_A^B \mu \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} \cdot d\vec{\mathbf{r}}_{2,1} = \mu \int_A^B \left(\frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) dt , \qquad (13.1.13)$$

where we have used the relation between the differential elements $d\vec{\mathbf{r}}_{2,1} = \frac{d\vec{\mathbf{r}}_{2,1}}{dt} dt$. The product rule for derivatives of the scalar product of a vector with itself is given for this case by

$$\frac{1}{2} \frac{d}{dt} \left(\frac{d\vec{\mathbf{r}}_{2,1}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) = \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} . \qquad (13.1.14)$$

Substitute Equation (13.1.14) into Equation (13.1.13), which then becomes

$$W = \mu \int_A^B \frac{1}{2} \frac{d}{dt} \left(\frac{d\vec{\mathbf{r}}_{2,1}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) dt . \qquad (13.1.15)$$

Equation (13.1.15) is now the integral of an exact derivative, yielding

$$W = \frac{1}{2} \mu \left(\frac{d\vec{\mathbf{r}}_{2,1}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) \Bigg|_A^B = \frac{1}{2} \mu (\vec{\mathbf{v}}_{2,1} \cdot \vec{\mathbf{v}}_{2,1}) \Big|_A^B = \frac{1}{2} \mu (v_B^2 - v_A^2) , \qquad (13.1.16)$$

where $\vec{\mathbf{v}}_{2,1}$ is the *relative velocity* between the two bodies. It's important to note that in the above derivation had we exchanged the roles of body 1 and 2 i.e. $1 \to 2$ and $2 \to 1$, we would have obtained the identical result because

$$\begin{aligned}
\vec{\mathbf{F}}_{1,2} &= -\vec{\mathbf{F}}_{2,1} \\
\vec{\mathbf{r}}_{1,2} &= \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = -\vec{\mathbf{r}}_{2,1} \\
d\,\vec{\mathbf{r}}_{1,2} &= d(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) = -d\,\vec{\mathbf{r}}_{2,1} \\
\vec{\mathbf{v}}_{1,2} &= -\vec{\mathbf{v}}_{2,1}.
\end{aligned} \tag{13.1.17}$$

Equation (13.1.16) implies that the work done is the change in the kinetic energy of the system, which we can write in terms of the reduced mass and the change in the square of relative speed of the two objects

$$\Delta K = \frac{1}{2}\mu(v_B^2 - v_A^2)\,. \tag{13.1.18}$$