Vectors and Linear Combinations

Vectors and linear combinations are fundamental concepts in linear algebra. Below is a detailed explanation of each concept, their properties, and their relationship.

What is a Vector?

A vector is a mathematical object that has both **magnitude** (size) and **direction**. It can be represented in different ways depending on the context:

1. **Geometric Representation**:
   - A vector is often visualized as an arrow pointing from one point to another in space.
   - The length of the arrow represents the magnitude, and its orientation represents the direction[1] [2].
2. **Algebraic Representation**:
   - A vector is represented as an ordered list of numbers (components). For example:

     $$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

     Here,

     $$v_1, v_2, v_3$$

     are the components of the vector[3] [2].
3. **Properties**:
   - Vectors can be added together.
   - They can be scaled (multiplied by a scalar).
   - They exist in a space called a **vector space**, which satisfies specific axioms like associativity, commutativity, and distributivity[4].
4. **Types of Vectors**:
   - **Zero Vector**: A vector with all components equal to zero.
   - **Unit Vector**: A vector with a magnitude of 1.
   - **Position Vector**: A vector originating from the origin to a specific point in space.

Linear Combinations

A linear combination is a way to combine vectors using scalar multiplication and addition.

Definition:

Given vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

in a vector space and scalars

$$c_1, c_2, \ldots, c_n$$

, their linear combination is:

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Here:

- c_i are scalars (real or complex numbers).
- The result $\mathbf{w}$ is another vector in the same vector space[2] [5].

Key Concepts:

1. **Span**:
   - The set of all possible linear combinations of a given set of vectors is called their span.
   - For example, if two vectors span the 2D plane, any point in that plane can be expressed as their linear combination.
2. **Basis and Dimension**:
   - A basis is a set of linearly independent vectors that span the entire vector space.
   - The number of vectors in the basis defines the dimension of the space[2] [4].
3. **Linear Independence**:
   - Vectors are linearly independent if no vector in the set can be written as a linear combination of the others.
   - If they are dependent, at least one vector can be expressed as such a combination.

Applications:

- Representing solutions to systems of linear equations.
- Modeling physical quantities like forces or velocities.
- Constructing transformations in graphics and engineering.

Relationship Between Vectors and Linear Combinations

Vectors form the building blocks for constructing linear combinations. By scaling and adding vectors, you can create new vectors within the same space. This concept underpins many areas of mathematics and physics:

1. In geometry, it allows you to describe points and directions.
2. In physics, it models forces acting on an object.
3. In computer science, it supports data representation (e.g., feature vectors in machine learning).

Visualization

To better understand these concepts:

- Use tools like GeoGebra or Python libraries like Matplotlib for visualizing vectors and their combinations.
- Watch tutorials such as 3Blue1Brown's *Essence of Linear Algebra* series for intuitive explanations[6].

By mastering these ideas, you'll gain insight into more advanced topics like matrix transformations, eigenvalues/eigenvectors, and optimization problems.

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1. https://en.wikipedia.org/wiki/Vector_(mathematics_and_physics)
2. https://web.stanford.edu/class/nbio228-01/handouts/Ch4_Linear_Algebra.pdf
3. https://datahacker.rs/essence-of-linear-algebra-vectors/
4. https://en.wikipedia.org/wiki/Vector_space
5. https://codanics.com/vectors-in-linear-algebra/
6. https://www.youtube.com/watch?v=fNk_zzaMoSs